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ESTIMATION OF OVERLAPPING MEASURES USING NUMERICAL APPROXIMATIONS: WEIBULL DISTRIBUTIONS

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Abstract. This paper deals with the estimation problem of the two overlapping (OVL) measures, namely; Matusita ρ and Morisita λ measures when two independent random variables X and Y follow Weibull distribution. The two measures ρ *and* λ have been studied in the literature in the case of two Weibull distributions under the assumption that the two shape parameters are equal. In this work, a new general expression for each measure is provided under the Weibull distribution without using any assumptions about the distribution parameters. The numerical integration methods known as trapezoidal, Simpson 1/3 and Simpson 3/8 rules that facilitate making inference on these measures are utilized. The relative bias (RB) and relative mean square error (RMSE) of the resulting proposed estimators were investigated and compared with some existing estimators via Monte-Carlo simulation technique. The results demonstrated clearly the superiority of the proposed estimators over the existing one in almost all considered cases.

1. Introduction

The overlapping coefficients(OVL) indicates to the similarity of two probability distributions, measured by their intersection area of graphs of two or more probability density functions, and it is another a simple method to identify the closeness between samples or populations that are usually described in terms of their distribution functions (Weitzman, 1970). Pastore and Calcagni (2019) mentioned that the main reason of using such overlapping measures is that they improve the interpretability and conclusions reliability of data analysis. In the literature, there are

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many overlap coefficients such as Matusita (1955), Morisita (1959), Weitzman(1970) and Pianka (Chaubey *et al.*, 2008) measures and others.

In this paper, we will focus on Matusita's and Morisita's measures of OVL. Let $f_X(x)$ and $f_Y(x)$ be two continuous probability density functions for the two random variables X and Y respectively, then the OVL are defined as follows:

(1) Matusita's measure (1955) is given by:

$$\rho = \int \sqrt{f_X(x) f_Y(x)} dx$$

(2) Morsita's measure(1959) is given by:

$$\lambda = \frac{2 \int f_X(x) f_Y(x) \; dx}{\int [f_X(x)]^2 dx \; + \int [f_Y(x)]^2 dx}$$

If X and Y are discrete random variables then ρ and λ are defined the same as the above formulas but any integral in their expressions should be replaced by summation (Inman and Bradly, 1989).

The value of each of the above formula of overlap measures of two densities are measured on the scale of 0 to 1. If the value is close to 0 then it indicates that there is no common area between two densities (i.e. $f_X(x) \neq f_Y(x), \forall\, x$) and if the overlap value is 1 then it indicates a perfect agreement between two densities , which is equivalent to say, $f_X(x) = f_Y(x)$, $\forall\, x$.

The overlapping is used in many applications to measure the similarity of data sets. Thus, it is applied to data sets arising in different fields such as ecology (Pianka, 1973 and Hurlbert, 1978), geography, social behavior, niche structure and organism morphology (LU, Smith, and Good 1989). Sneath (1977) used the OVL coefficient as a measure of disjunction and other authors pointed out to the application of OVL measure on income differentials (see Mulekar and Mishra, 1994, Mulekar and Fukasawa, 2010 and Inman and and Bradley, 1989) and genetics (Federer *et al.* 1963). Recently, Alodat *et al.* (2021) derived the asymptotic sampling distribution of the kernel estimator of the Matusita measure and they showed the importance of OVL in goodness-of-fit test for two independent populations. Samawi *et al.* (2011) introduced a new nonparametric test of symmetry based on OVL measure ρ using kernel density estimation.

Most studies have considered the estimation of ρ *and* λ under some specific pair distributions. Mulekar and Mishra (2000) suggested the use of Jackknife and Bootstrap methods to construct the confidence intervals for ρ and λ in the case of two normal distributions with common mean. Chaubey *et al.* (2008) studied the properties and addressed point estimation for ρ when the two populations are assumed to be described by the inverse Gaussian distributions (Wald distributions) with equal means. Eidous and Daradkeh (2022) proposed a new technique for estimating the Matusita coefficient ρ under pair-normal distributions, and Eidous and Al-Shourman (2022) used another method based on the numerical approximation method to estimate ρ under two normal distributions.

The case of exponential distributions can be considered as a special case of Weibull distributions by taking $\beta = 1$. The case of two exponential distributions was studied by Madhuri *et al.* (2001) and Samawi and Al-Saleh (2008), who also studied the effect of sampling scheme on ρ. Helu and Samawi (2011) investigated the OVL coefficients for two Lomax distributions with different sampling procedures. Parallel to the work of Helu and Samawi (2011), Dhaker *et al.* (2021) considered the case of two inverse Lomax distributions to study the measures ρ.

Let X and Y are independent Weibull distributions random variable with the same shape ($\beta_1 = \beta_2 = \beta$) but different scale parameter α_1 *and* α_2 respectively, then the pdf of X is:

$$f_X(x;\alpha_1,\beta) = \frac{\beta}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta-1} e^{-(x/\alpha_1)^\beta}, \quad x > 0, \quad \alpha_1,\ \beta > 0.$$

We will denote it by, $X \sim We(\alpha_1, \ \beta)$. Thus, if $Y \sim We(\alpha_2, \ \beta)$ then the *pdf* of Y is,

$$f_Y(y;\alpha_2,\beta) = \frac{\beta}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta-1} e^{-(y/\alpha_2)^\beta}, \quad y > 0, \quad \alpha_2,\ \beta > 0.$$

Under the assumption that X and Y are independent, Al-Saidy *et al.* (2005) (See also, Eidous and Al-Hayjaa, 2023) considered the case of two Weibull distributions with the same shape and different scale parameters. Let $K = \alpha_1/\alpha_2$ and $Q = (2\beta - 1)/\beta$ ($\alpha_1, \ \alpha_2$ are scale parameters and β is the shape parameter). Al-Saidy *et al.* (2005)

derived the formulas of ρ, which are given by

$$\rho = \frac{2\sqrt{K^{\beta}}}{1+K^{\beta}}$$

and

$$\lambda = \frac{2^{Q+1}K^{\beta}}{(1+K)(1+K^{\beta})^{Q}} \ ,$$

Most of the previous studied were accomplished by using some restrictions on the distributions parameters. For example, without using the assumption $\beta_1 = \beta_2 = \beta$ (Al-Saidy *et al.*, 2005), the above formula of ρ and λ are no longer true. The main objective of this paper is to estimate the OVL ρ and λ by adopting the numerical integration approximation methods; trapezoidal and Simpson rules. Then, for each numerical integration rule, a new maximum likelihood (ML) estimator of ρ and λ are obtained. The various rules are introduced in this paper, and their performances are assessed throughout the simulation technique. Accordingly, this paper has been organized as follows:

The maximum likelihood estimators for the parameters of the Weibull distributions were derived in Section 2. Section 3 gives some explanations and introduces the formulas of the trapezoidal and Simpson rules. The approximation formulas for ρ and λ based on numerical integration methods have been given in Section 4 and Section 5, respectively. In Section 6, three new estimators are proposed and developed for each of ρ and λ. Section 7 discusses some concepts for applying the proposed estimators of ρ and λ in practice. Section 8 gives a Monte-Carlo simulation study to investigate the finite properties of the proposed estimators and to compare their performance with some other estimators found in the literature. The final conclusions are stated in Section 9.

2. Maximum Likelihood Estimators of Weibull Distribution Parameters

Let $X_1, X_2, \ldots, X_{n_1}$ be a random sample of size n_1 from $We(\alpha_1, \ \beta_1)$ and let $Y_1, Y_2, \ldots, Y_{n_2}$ be another random sample of size n_2 from $We(\alpha_2, \ \beta_2)$, where the two samples are independent. The log-likelihood function is,

$$lnL\,(\alpha_1, \beta_1, \ \alpha_2, \beta_2) = n_1 ln\beta_1 + n_2 ln\beta_2 - (n_1\beta_1 ln\alpha_1 + n_2\beta_2 ln\alpha_2) +$$

$$+(\beta_1-1)\sum_{i=1}^{n_1} lnx_i+(\beta_2-1)\sum_{i=1}^{n_2} lny_i-\frac{1}{{\alpha_1}^{\beta_1}}\sum_{i=1}^{n_1} x_i^{\beta_1}-\frac{1}{{\alpha_2}^{\beta_2}}\sum_{i=1}^{n_2} y_i^{\beta_2}$$

The ML estimators of $\alpha_1, \beta_1,\ \alpha_2$ and β_2 are obtained by solving the following equations simultaneously,

$$\frac{1}{\beta_1}+\frac{1}{n_1}\sum_{i=1}^{n_1} lnx_i\ -\frac{\sum_{i=1}^{n_1}\left[x_i^{\beta_1} lnx_i\ \right]}{\sum_{i=1}^{n_1} x_i^{\beta_1}}=0$$

$$\frac{1}{\beta_2}+\frac{1}{n_2}\sum_{i=1}^{n_2} lny_i\ -\frac{\sum_{i=1}^{n_2}\left[y_i^{\beta_2} lny_i\ \right]}{\sum_{i=1}^{n_2} y_i^{\beta_2}}=0$$

$$\alpha_1=\left(\frac{\sum_{i=1}^{n_1} x_i^{\beta_1}}{n_1}\right)^{\frac{1}{\beta_1}}$$

and

$$\alpha_2=\left(\frac{\sum_{i=1}^{n_2} y_i^{\beta_2}}{n_2}\right)^{\frac{1}{\beta_2}}$$

If $\widehat{\alpha}_1,\ \widehat{\alpha}_2,\ \widehat{\beta}_1$ and $\widehat{\beta}_2$ are the ML estimators of $\alpha_1, \alpha_2, \beta_1$and β_2 respectively then the ML estimators of $f_X\left(x;\alpha_1,\beta_1\right)$ and $f_Y\left(x;\alpha_2,\beta_2\right)$ are $f_X\left(x;\widehat{\alpha}_1,\widehat{\beta}_1\right)$ and $f_Y\left(x;\widehat{\alpha}_2,\widehat{\beta}_2\right)$ respectively.

3. Numerical Integration Methods

Many definite integrals are not simple to find or when definite integral approximation is required. Wherefore, we can use numerical approximation methods to quickly and easily solve problems with a satisfactory higher absolute error (see Chapra and Canale, 2014). Trapezoidal, Simpson 1/3 and Simpson 3/8 rules to approximate the interested integral are adopted in this paper. Let $h\left(t\right)$ be a continuous function on $[a,b]$ and let $\Delta t=\frac{b-a}{k}$, a and b are finite real numbers. Suppose that the interval $[a,b]$ is divided into k subintervals each of length Δt as follows,

$$a=t_0<\ t_1<\ t_2<\ \cdots<t_k=b,$$

where $t_i=a+i\Delta t$, $i=0,\ 1,\ldots,\ \ k-1$. The three interested numerical integration rules to approximate $\int_a^b h\left(t\right)dt$ are (Chapra and Canale, 2014),

(1) The trapezoidal rule is given by,

$$\int_a^b h\left(t\right)dt=\ \frac{\Delta t}{2}\left[h\left(a\right)+2h\left(t_1\right)+2h\left(t_2\right)\cdots+2h\left(t_{k-1}\right)+h\left(b\right)\right],$$

(2) The Simpson 1/3 rule is given by,

$$\int_a^b h(t)\,dt = \frac{\Delta t}{3}\left[h(a) + 4h(t_1) + 2h(t_2) + \cdots + 4h(t_{k-1}) + h(b)\right],$$

(3) The Simpson 3/8 rule is given by,

$$\int_a^b h(t)\,dt = \frac{3\Delta t}{8}[h(a) + 3h(t_1) + 3h(t_2) + 2h(t_3) + 3h(t_4) + 3h(t_5)$$
$$+2h(t_6) + \cdots + 2h(t_{k-3}) + 3h(t_{k-2}) + 3h(t_{k-1}) + h(b)]$$

4. Approximation of $\rho(X, Y)$

The formula of Matusita measure $\rho(X,\ Y)$ between X and Y is,

$$\rho(X,Y) = \int_0^\infty \sqrt{f_X(x;\alpha_1,\beta_1)\, f_Y(x;\alpha_2,\beta_2)}dx.$$

Under the assumption $\beta_1 = \beta_2$ $(= \beta$, say), Al-Saidy *et al.* (2005) computed the value of the above integral, which is given by,

$$\rho(X,Y) = \frac{2\sqrt{(\alpha_1/\alpha_2)^\beta}}{1+(\alpha_1/\alpha_2)^\beta}\ .$$

However, it is not an easy job to evaluate the above integral without using the assumption $\beta_1 = \beta_2$. Therefore, our main interest is to approximate the above integral by using the trapezoidal and Simpson numerical integration methods without using any restrictions on the parameters of Weibull distributions. Since these methods require the integral limits (bounds) to be finite then as a first step, we have to find a proper transformation that enables us to apply the trapezoidal and Simpson rules.

As a general case, consider the transformation $t = W(x)$, where $W(x)$ is a continuous increasing (or decreasing) function in x such that $W(0) = c$ and $W(\infty) = d$, where c and d are two real numbers (If W is strictly increasing function then $c < d$, while $c > d$ for strictly decreasing W). Then $x = W^{-1}(t)$ and $dx = \frac{dW^{-1}(t)}{dt}$. To simplify the notations, let $V(t) = W^{-1}(t)$ then $x = V(t)$ and $dx = V'(t)\,dt$. Therefore,

$$\rho(X,Y) = \int_c^d \sqrt{f_X(V(t);\alpha_1,\beta_1)\, f_Y(V(t);\alpha_2,\beta_2)}V'(t)dt. \tag{1}$$

Theorem 4.1. Let W be a continuous increasing (or decreasing) function then,

$$\rho(W(X)\ ,W(Y)) = \rho(X,\ Y)\,.$$

Which indicates that ρ is invariant measure with respect to any continuous increasing or decreasing function.

Proof. Let $T_1 = W(X)$ be a continuous increasing (or decreasing) function of a random variable X. The *pdf* of T_1 is,

$$f_{T_1}(t_1) = f_X(V(t_1);\alpha_1,\beta_1)\,|V'(t_1)|$$

and the *pdf* of $T_2 = W(Y)$ is,

$$f_{T_2}(t_2) = f_Y(V(t_2);\alpha_2,\beta_2)\,|V'(t_2)|$$

Now, if

$$\begin{aligned}\rho(W(X)\ ,W(Y)) &= \rho(T_1\ ,\ T_2)\\ &= \int_c^d \sqrt{f_{T_1}(t_1)\,f_{T_2}(t_1)}dt_1\\ &= \int_c^d \sqrt{f_{T_1}(t_1)\,f_{T_2}(t_1)\,|V'(t_1)|\,f_Y(V(t_1);\alpha_2,\beta_2)\,|V'(t_2)|}\,dt_1\\ &= \int_c^d \sqrt{f_{T_1}(t_1)\,f_{T_2}(t_1)\,f_Y(V(t_1);\alpha_2,\beta_2)}\ \ |V'(t_2)|\,dt_1\\ &= \rho(X,\ Y).\end{aligned}$$

□

The last step is obtained based on Eq. 1. This completes the proof.

Now, suppose that the selected transformation W gives finite numbers for c and d then formula of ρ that is given by Eq. 1 can be approximated using trapezoidal, Simpson 1/3 and Simpson 3/8 rules as follows:

Let $h(t) = \sqrt{f_X(V(t);\alpha_1,\beta_1)\,f_Y(V(t);\alpha_2,\beta_2)}V'(t)$ then,

$$\begin{aligned}\rho &= \int_c^d \sqrt{f_X(V(t);\alpha_1,\beta_1)\,f_Y(V(t);\alpha_2,\beta_2)}V'(t)dt\\ &= \int_c^d h(t)\,dt.\end{aligned}$$

The approximations of $\rho = \int_c^d h(t)\,dt$ by using numerical integral methods are given as follows:

(1) **Trapezoidal Approximation:** The approximation of ρ using trapezoidal rule is,

$$\rho_1 \cong \frac{d-c}{2k}\left[h(c)+2h(u_1)+2h(u_2)\cdots+2h(u_{k-1})+h(d)\right].$$

(2) **Simpson 1/3 Approximation:** The approximation of ρ using Simpson 1/3 rule is,

$$\rho_2 \cong \frac{d-c}{3k}\left[h(c)+4h(u_1)+2h(u_2)+\cdots+4h(u_{k-1})+h(d)\right].$$

(3) **Simpson 3/8 Approximation:** The approximation of ρ using Simpson 3/8 rule is,

$$\rho_3 \cong \frac{3(d-c)}{8k}\{h(c)+3h(u_1)+3h(u_2)+2h(u_3)+3h(u_4)+3h(u_5)+2h(u_6)+\cdots+2h(u_{k\cdot}$$

5. Approximation of $\boldsymbol{\lambda}$

The Morisita measure λ between the two independent random variables X and Y is,

$$\lambda = \frac{2\int_0^\infty f_X(x;\alpha_1,\beta_1)\, f_Y(x;\alpha_2,\beta_2)\,dx}{\int_0^\infty (f_X(x;\alpha_1,\beta_1))^2 dx + \int_0^\infty (f_Y(x;\alpha_2,\beta_2))^2\, dx}$$

$$= \frac{2\varphi_{XY}}{\varphi_X + \varphi_Y},$$

where,

$$\varphi_{XY} = \int_0^\infty f_X(x;\alpha_1,\beta_1)\, f_Y(x;\alpha_2,\beta_2)\,dx$$

$\varphi_X = \int_0^\infty (f_X(x;\alpha_1,\beta_1))^2 dx$,and

$$\varphi_Y = \int_0^\infty (f_Y(x;\alpha_2,\beta_2))^2\, dx\ .$$

To approximate λ, we need to approximate φ_{XY}, φ_X and φ_Y, that require their integrals limits to be finite. This can be achieved by selecting a proper transformation. The same technique used the approximation in the previous in section 4 .

By using the same transformation $t = W(x)$ with inverse tramsformation $x = W^{-1}(t) = V(t)$ and the same notations of previous subsection, we obtain,

$$\varphi_{XY} = \int_c^d f_X(V(t);\alpha_1,\beta_1)\, f_Y(V(t);\alpha_2,\beta_2)\, V'(t)dt$$

$\varphi_X = \int_c^d (f_X(V(t);\alpha_1,\beta_1))^2 V'(t)dt$,and

$$\varphi_Y = \int_c^d (f_Y(V(t);\alpha_2,\beta_2))^2\ V'(t)dt\ .$$

Thus,

$$\lambda = \frac{2\int_c^d f_X(V(t);\alpha_1,\beta_1)\, f_Y(V(t);\alpha_2,\beta_2)\, V'(t)dt}{\int_c^d (f_X(V(t);\alpha_1,\beta_1))^2 V'(t)dt\ +\int_c^d (f_Y(V(t);\alpha_2,\beta_2))^2\ V'(t)dt}\ .$$

To simplify the notations, let

$$h_1(t) = f_X(V(t);\alpha_1,\beta_1)\, f_Y(V(t);\alpha_2,\beta_2)\, V'(t)$$

$$h_2(t) = (f_X(V(t);\alpha_1,\beta_1))^2 V'(t)$$

and

$$h_3(t) = (f_Y(V(t);\alpha_2,\beta_2))^2\ V'(t)\,.$$

Then $\varphi_{XY} = \int_c^d h_1(t)\,dt$, $\varphi_X = \int_c^d h_2(t)\,dt$, $\varphi_Y = \int_c^d h_3(t)\,dt$ and

$$\lambda = \frac{2\int_c^d h_1(t)\,dt}{\int_c^d h_2(t)\,dt\ +\int_c^d h_3(t)\,dt}\ .$$

Then the approximations of λ by using numerical integral methods are given as follows:

(1) **Trapezoidal Approximation:** The approximation of λ using trapezoidal rule is,

$$\lambda_1 \cong \frac{2\left(h_1(c) + 2\sum_{j=1}^{k-1} h_1(u_j) + h_1(d)\right)}{h_2(c) + 2\sum_{j=1}^{k-1} h_2(u_j) + h_2(d) + h_3(c) + 2\sum_{j=1}^{k-1} h_3(u_j) + h_3(d)}$$

(2) **Simpson 1/3 Approximation:** The approximation of λ using Simpson 1/3 rule is (k is an integer positive number and a multiple of 2),

$\lambda_2 = \frac{2(h_1(c)+B_1+h_1(d))}{h_2(c)+B_2+h_2(d)+h_3(c)+B_3+h_3(d)}$where,

$$B_i = 4\sum_{\substack{j=1\\ j\neq 2m}}^{k/2} h_i(u_{2j-1}) + 2\sum_{j=1}^{k/2-1} h_i(u_{2j}),\quad i=1,\ 2,\ 3,\quad m\in N_0.$$

(1) **Simpson 3/8 Approximation:** The approximation of λ by using Simpson 3/8 rule is (k is an integer positive number and a multiple of 3),

$$\lambda_3 \cong \frac{2(h_1(c) + A_1 + h_1(d))}{h_2(c) + A_2 + h_2(d) + h_3(c) + A_3 + h_3(d)}$$

where,

$$A_i = 3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1} h_i\left(u_j\right) + 2\sum_{j=1}^{k/3-1} h_i\left(u_{3j}\right),\ i = 1,\ 2,\ 3, m\in N_0$$

6. Estimation of Overlapping Coefficients

Let $\widehat{\alpha}_1,\ \widehat{\alpha}_2,\ \widehat{\beta}_1$ and $\widehat{\beta}_2$ be the ML estimators of $\alpha_1, \alpha_2,\ \ \beta_1$ and β_2 respectively then the ML estimators of $f_X\left(x;\alpha_1,\beta_1\right)$ and $f_Y\left(x;\alpha_2,\beta_2\right)$ are $f_X\left(x;\widehat{\alpha}_1,\widehat{\beta}_1\right)$ and $f_Y\left(x;\widehat{\alpha}_2,\widehat{\beta}_2\right)$ respectively. Eidous and Al-Maqableh (2022) suggested and developed an estimator of $\rho\left(X,Y\right) and\ \lambda(X,Y)$. Their work depends entirely on writing the formula of $\rho\ and\ \lambda$ as an expected value of some function(s), and then they estimated the resulting expected value(s) by using the method of moments (See also Eidous and Al-Talafhah, 2020). The two estimators of ρ and λ are respectively,

$$\widehat{\rho}_{EM} = \frac{1}{2}\left[\frac{1}{n_1}\sum_{k=1}^{n_1}\left(\frac{f_Y\left(X_k;\widehat{\alpha}_2,\widehat{\beta}_2\right)}{f_X\left(X_k;\widehat{\alpha}_1,\widehat{\beta}_1\right)}\right)^{1/2} + \frac{1}{n_2}\sum_{k=1}^{n_2}\left(\frac{f_X\left(Y_k;\widehat{\alpha}_1,\widehat{\beta}_1\right)}{f_Y\left(Y_k;\widehat{\alpha}_2,\widehat{\beta}_2\right)}\right)^{1/2}\right]$$

$$\widehat{\lambda}_{EM} = \frac{\frac{1}{n_2}\sum_{k=1}^{n_2} f_X\left(Y_k;\widehat{\alpha}_1,\widehat{\beta}_1\right) + \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y\left(X_k;\widehat{\alpha}_2,\widehat{\beta}_2\right)}{\frac{1}{n_1}\sum_{k=1}^{n_1} f_X\left(X_k;\widehat{\alpha}_1,\widehat{\beta}_1\right) + \frac{1}{n_2}\sum_{k=1}^{n_2} f_Y\left(Y_k;\widehat{\alpha}_2,\widehat{\beta}_2\right)}$$

6.1 Estimation of ρ

The proposed estimators of ρ can be obtained simply by substituting the ML estimators $f_X\left(.;\widehat{\alpha}_1,\widehat{\beta}_1\right)$ and $f_Y\left(.;\widehat{\alpha}_2,\widehat{\beta}_2\right)$ of $f_X\left(.;\alpha_1,\beta_1\right)$ and $f_Y\left(.;\alpha_2,\beta_2\right)$ back into $h\left(u\right)$ to obtain the corresponding estimators of ρ_{Trap}, ρ_{Simp1} and ρ_{Simp2}. Briefly, the proposed estimators of $\rho\left(X,\ Y\right)$ are given as follows:

Let $\hat{h}\left(t\right) = \sqrt{f_X\left(V\left(t\right);\widehat{\alpha}_1,\widehat{\beta}_1\right) f_Y\left(V\left(t\right);\widehat{\alpha}_2,\widehat{\beta}_2\right)}V'(t)$ then

(1) The proposed estimator of ρ using the trapezoidal approximation is,

$$\widehat{\rho}_{Trap} = \frac{d-c}{2k}\left[\hat{h}\left(c\right) + 2\sum_{j=1}^{k-1}\hat{h}\left(u_j\right) + \hat{h}\left(d\right)\right].$$

(2) The proposed estimator of ρ using Simpson 1/3 approximation is,

$$\widehat{\rho}_{Simp1} = \frac{d-c}{3k}\left[\hat{h}\left(c\right) + 4\sum_{j=1}^{k/2}\hat{h}\left(u_{2j-1}\right) + 2\sum_{j=1}^{k/2-1}\hat{h}\left(u_{2j}\right) + \hat{h}\left(d\right)\right].$$

(3) The proposed estimator of ρ using Simpson 3/8 approximation is,

$$\widehat{\rho}_{simp2} = \frac{3(d-c)}{8k}\left\{\hat{h}\left(c\right) + 3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1}\hat{h}\left(u_j\right) + 2\sum_{j=1}^{k/3-1}\hat{h}\left(u_{3j}\right) + \hat{h}\left(d\right)\right\},\ m \in N_0.$$

6.2 Estimation of $\boldsymbol{\lambda}$

Let

$$\hat{h}_1\left(t\right) = f_X\left(V\left(t\right); \widehat{\alpha}_1, \widehat{\beta}_1\right) f_Y\left(V\left(t\right); \widehat{\alpha}_2, \widehat{\beta}_2\right) V'\left(t\right),$$

$$\hat{h}_2\left(t\right) = \left(f_X\left(V\left(t\right); \widehat{\alpha}_1, \widehat{\beta}_1\right)\right)^2 V'(t)$$

and

$$\hat{h}_3\left(t\right) = \left(f_Y\left(V\left(t\right); \widehat{\alpha}_2, \widehat{\beta}_2\right)\right)^2 V'\left(t\right).$$

(1) The proposed estimators of λ using the trapezoidal approximation is,

$$\widehat{\lambda}_{Trap} = \frac{2\left(\hat{h}_1\left(c\right) + 2\sum_{j=1}^{k-1}\hat{h}_1\left(u_j\right) + \hat{h}_1\left(d\right)\right)}{\hat{h}_2\left(c\right) + \hat{h}_3\left(c\right) + 2\sum_{j=1}^{k-1}\hat{h}_2\left(u_j\right) + 2\sum_{j=1}^{k-1}\hat{h}_3\left(u_j\right) + \hat{h}_2\left(d\right) + \hat{h}_3\left(d\right)}$$

(2) The proposed estimators of λ using Simpson 1/3 approximation is,

$\widehat{\lambda}_{Simp1} = \frac{2(\hat{h}_1(c)+\hat{B}_1+h_1(d))}{\hat{h}_2(c)+\hat{B}_2+\hat{h}_2(d)+\hat{h}_3(c)+\hat{B}_3+\hat{h}_3(d)}$where,

$$\hat{B}_i = 4\sum_{\substack{j=1\\ j\neq 2m}}^{k/2}\hat{h}_i\left(u_{2j-1}\right) + 2\sum_{j=1}^{k/2-1}\hat{h}_i\left(u_{2j}\right), \quad i = 1,\ 2,\ 3, \quad m \in N_0.$$

(1) The proposed estimators of λ using Simpson 3/8 approximation is,

$$\widehat{\lambda}_{Simp2} = \frac{2\left(\hat{h}_1\left(c\right) + \hat{A}_1 + \hat{h}_1\left(d\right)\right)}{\hat{h}_2\left(c\right) + \hat{A}_2 + \hat{h}_2\left(d\right) + \hat{h}_3\left(c\right) + \hat{A}_3 + \hat{h}_3\left(d\right)}$$

where,

$$\hat{A}_i = 3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1}\hat{h}_i\left(u_j\right) + 2\sum_{j=1}^{k/3-1}\hat{h}_i\left(u_{3j}\right), \quad i = 1,\ 2,\ 3, \quad m \in N_0.$$

7. Number of Partitions and Transformation

To use the various estimators proposed for $(\rho \ and \ \lambda)$ in the practical application stage, there are two quantities to be determined. The first quantity is the transformation function W. Let Z be a continuous random variable with cumulative distribution function $F_Z(z)$, $z \geq 0$ then our special interest is to take $W(x) = F_Z(x)$. In this case, $a = W(0) = 0$ and $b = W(\infty) = 1$. More specifically, we consider the following transformation in our simulation study in the next section,

$$F_Z(x) = 1 - e^{-x} , \quad 0 \leq x < \infty.$$

That is, $Z \sim \exp(1) = Weib(1,1)$. In addition, and to study the effect of the selected transformation on the estimation process we also take $Z \sim Weib(1, \ 2)$. In general, let $Z \sim Weib(1,\theta)$ then,

$$F_Z(x) = 1 - e^{-x^{\theta}} , \quad 0 \leq x < \infty.$$

In this case, $t = W(x) = F_Z(x) = 1 - e^{-x^{\theta}}$. The inverse transformation is $x = V(t) = (-ln(1-t))^{1/\theta}$ and $dx = V'(t)\,dt = \frac{(-\ln(1-t)\ )^{\frac{1}{\theta}-1}}{\theta(1-t)}dt.$

The second quantity that need to be determine is the number of partitions (subintervals) k. In this study, we suggest to take its value to be $k = min\{n_1, \ n_2\}$. It is well known that the maximum absolute error of the different numerical integral approximation decreases as the number of partitions k increases (Chapra and Canale, 2014). However, a preliminary simulation study was performed for different values of k greater than $min\{n_1, \ n_2\}$, it is found that there is no significant improvement in the estimation process for the different proposed estimators by taking $k > \min\{n_1, \ n_2\}$.

8. Simulation Study and Results

In this section, a Monte-Carlo simulation study is conducted to investigate the properties and performances of the proposed estimators of OVL measures ρ and λ that derived in this paper. The transformation $F_Z(x) = 1 - e^{-x^{\theta}}$ with $\theta = 1, \ 2$ is used for each of the proposed estimator. For sake of comparison, the estimators $\widehat{\rho}_{EM}$ and $\widehat{\lambda}_{EM}$ of ρ and λ that suggested by Eidous and Al-Maqableh (2022) are also considered.

Two random samples are simulated from two Weibull distributions. The first sample $x_1, x_2, \ldots, x_{n_1}$ is simulated from $f_X(x) = We(\alpha_1,\ \beta_1)$, while the second sample $y_1, y_2, \ldots, y_{n_2}$ is generated from $f_Y(y) = We(\alpha_2,\ \beta_2)$, where $f_X(x)$ and $f_Y(y)$ and the corresponding choosing parameters are given in Table 1. Also, the exact values of $\rho\ and\ \lambda$ for each pair of selection are also provided. Despite that the process of selection parameters seems to be arbitrary, we take into account that our selection should vary the exact values of $\rho\ and\ \lambda$ from 0 to 1. That is, the selection gives the values of $\rho\ and\ \lambda$ to be small (near zero) and large (near one). For each pair of densities, the size of simulated data are $(n_1, n_2) = (12, 12)\,,\ (24, 30)\,,\ (504, 504)$. All simulation results are calculated by using Mathematica Version 7.

Table 1. The simulated pair of distributions and the corresponding exact values of $\rho\ and\ \lambda$.

Weibull Distributions	$f_X(x)$	$f_Y(y)$	ρ	λ
Case(1)	We(1,2)	We(1.2,1.8)	0.9793	0.9721
Case(2)	We(1,4)	We(1.8,2.1)	0.6893	0.5672
Case(3)	We(1,2)	We(5,6)	0.055	0.0026

The empirical results were calculated based on a thousand iterations (Rep = 1000). For each of the estimators considered, we computed Relative Bias (RB), Relative Mean Square Error (RMSE) and Efficiency (EFF). If we interested in calculating the RB, RMSE and EFF of an estimator $\widehat{\rho}$, the numerical calculations are performed as follows:

Suppose $\widehat{\rho}$ is an estimator of ρ and let $\widehat{\rho}_{(j)}$ be the value of $\widehat{\rho}$ calculated on the basis of iteration j of a sample, j= 1, 2, …, Rep= 1000 then the estimated expected value of $\widehat{\rho}$ is,

$$\hat{E}\left(\widehat{\rho}\right) = \frac{\sum_{j=1}^{Rep} \widehat{\rho}_{(j)}}{Rep},$$

and the estimated mean square error (MSE) of $\widehat{\rho}$ is,

$$\widehat{MSE}\left(\widehat{\rho}\right) = \frac{\sum_{j=1}^{Rep} \left(\widehat{\rho}_{(j)} - \hat{E}\left(\widehat{\rho}\right)\right)^2}{Rep}.$$

Accordingly, the RB of $\widehat{\rho}$ is given by,

$$RB = \frac{\hat{E}\left(\widehat{\rho}\right) - \rho}{\rho}$$

and the RMSE of $\widehat{\rho}$ is,

$$RMSE = \frac{\sqrt{\widehat{MSE}\left(\widehat{\rho}\right)}}{\rho} .$$

Also, the EFF of $\widehat{\rho}$ with respect to $\widehat{\rho}_{EM}$ is,

$$EFF = \frac{\widehat{MSE}\left(\widehat{\rho}_{EM}\ \right)}{\widehat{MSE}\left(\widehat{\rho}\right)}.$$

All computations and outputs of the simulation study are showed in Table 2 and Table 3, respectively. Based on these outputs we can summarize the results as follows.

(1) **Properties and performances of the various estimators:**

(2) The $|RB|$ and RMSE values of the different estimators of ρ *and* λ decrease with increasing sample sizes. This indicates the various estimators are consistent estimators. For example, from Table 2 and for Case 1 with $\theta = 1$ and $(n_1, n_2) = (12,\ 12)$, the values of $|RB|$ and RMSE of $\widehat{\rho}_{Trap}$ are 0.0528 and 0.0836, respectively, while its $|RB| = 0.001$ and RMSE $= 0.0066$ for large $(n_1, n_2) = (504,\ 504)$.

(3) Most of the RBs values of the proposed estimators were negative (this can be clearly seen by examining the values of RBs of the proposed estimators in Table 1 and 2), indicating that the various proposed estimators are under estimate the exact value of the OVL coefficient. This indicates that some bias correction methods (see Eidous, 2009 and 2011) can be used to improve the performance of the proposed estimators.

(4) It is clear that the RMSE and consequently the values of EFF for the proposed estimators are better than that of Eidous and Al-Maqableh (2022) estimators for all most considered cases. This feature becomes more evident when the exact values of ρ and λ are small. For example and based on Table 2, all values of the EFFs of the proposed estimators $\widehat{\rho}_{Trap}$, $\widehat{\rho}_{Trap}$ and $\widehat{\rho}_{Trap}$ (for Case 3) are all greater than 1, indicating that the RMSEs of the estimator $\widehat{\rho}_{Weib}$ are larger than those corresponding to the proposed estimators. Therefore and at

least for this case, the estimators $\widehat{\rho}_{Trap}$, $\widehat{\rho}_{Trap}$ and $\widehat{\rho}_{Trap}$ are more efficient and then better than the existing estimator $\widehat{\rho}_{Weib}$ of Eidous and Talafha (2020).

(5) **The effect of transformation selection:** To study the effect of the selected transformation on the performance of the proposed estimators of ρ and λ, the two transformations $F_Z(x) = 1 - e^{-x^\theta}$, $\theta = 1,\ 2$ were studied. The two selected transformations (i.e. $\theta = 1,\ 2$) give acceptable results for our proposed estimators. It is clear from the simulation results that these estimators are sensitive to the transformation selection. This can be inferred by examining and comparing the values of RMSEs and then the values of EFFs values associated with the proposed estimators when $\theta = 1$ and when $\theta = 2$ for different sample sizes. However, based on the simulation results, we recommend taking $\theta = 1$ for estimating ρ and $\theta = 2$ for estimating λ. For example, if one examines the values of the EFFs of $\widehat{\rho}_{Trap}$, $\widehat{\rho}_{Simp1}$ and $\widehat{\rho}_{Simp2}$ for Case 1 and with $\theta = 1$, they are found to be larger than their corresponding values for $\theta = 2$. The exact opposite case occurs in the case of examining the EFFs values of the estimators $\widehat{\lambda}_{Trap}$, $\widehat{\lambda}_{Simp1}$ and $\widehat{\lambda}_{Simp2}$ of λ based on Case 1.

(6) **The effect of the selected numerical rule:** The three proposed estimators $\widehat{\rho}_{Trap}$, $\widehat{\rho}_{Simp1}$ and $\widehat{\rho}_{Simp2}$ of ρ are obtained based on trapezoidal, Simpson 1/3 and Simpson 3/8 rules respectively. The same thing can be said for the proposed estimators $\widehat{\lambda}_{Trap}$, $\widehat{\lambda}_{Simp1}$ and $\widehat{\lambda}_{Simp2}$ of λ. By comparing the performances of the proposed estimator of ρ and λ, it is clear that the three rules give similar results for different sample size. In general, their results coincide when the sample sizes get larger. This indicates that using any of these three rules is enough for the process of estimation OVL coefficients ρ and λ. That is, there is no need to use the three methods together. For example, if one considers Case 2 of Table 3 with $(n_1, n_2) = (24,\ 30)$ and $\theta = 1$, then the values of RMSEs for $\widehat{\lambda}_{Trap}$, $\widehat{\lambda}_{Simp1}$ and $\widehat{\lambda}_{Simp2}$ are 0.1723, 0.1720 and 0.1701, respectively. Also, for the same case with $\theta = 2$, their RMSEs values are 0.1702, 0.1701 and 0.1698 respectively. As we can simply see, for each value of θ, the corresponding RMSEs of $\widehat{\lambda}_{Trap}$, $\widehat{\lambda}_{Simp1}$ and $\widehat{\lambda}_{Simp2}$ are very close to each other.

9. Conclusions

This paper focused on developing a new technique to approximate values of the OVL coefficients ρ and λ and then estimating them based on two independent random samples from pair Weibull distributions. The advantage of the proposed technique over others found in the literature is that it can be used without placing any restrictions on the parameters of the Weibull distributions. Based on the results of the simulations conducted in this study, it was evident that the new technique is effectiveness and the performance of the resulting estimators is better than those developed by Eidous and Magableh (2022). Because of the generality of the proposed technique and under the same distributions, it can be applied to estimate some other OVL coefficients mentioned in the literature, such as Weizmann coefficient and also Pianka coefficient (See, Eidous and Al-Talafha, 2020). Also, the same proposed technique can be used when the two random samples are assumed to follow other pair distributions such as the normal distribution, the Lomax distribution, and the inverse Lomax distribution.

Table 2. The RB, RMSE and EFF of the estimators $\widehat{\rho}_{EM}$, $\widehat{\rho}_{Trap}$, $\widehat{\rho}_{Simp1}$ and $\widehat{\rho}_{Simp2}$ when the data are simulated from pair Weibull distributions as given in Table 1.

		$\theta = 1$				$\theta = 2$			
(n_1, n_2)		$\widehat{\rho}_{EM}$	$\widehat{\rho}_{Trap}$	$\widehat{\rho}_{Simp1}$	$\widehat{\rho}_{Simp2}$	$\widehat{\rho}_{EM}$	$\widehat{\rho}_{Trap}$	$\widehat{\rho}_{Simp1}$	$\widehat{\rho}_{Simp2}$
Case 1									
(12,12)	RB	-0.0574	-0.0528	-0.0523	-0.0533	-0.0518	-0.135	-0.109	-0.115
	RMSE	0.0921	0.0836	0.0834	0.0845	0.0837	0.1485	0.1246	0.1308
	EFF	**1**	**1.213**	**1.218**	**1.188**	**1**	**0.318**	**0.452**	**0.41**
(24,30)	RB	-0.0214	-0.0196	-0.0193	-0.0195	-0.0235	-0.0767	-0.0614	-0.0655
	RMSE	0.0425	0.0389	0.0388	0.0388	0.0445	0.0836	0.0697	0.0734
	EFF	**1**	**1.193**	**1.199**	**1.198**	**1**	**0.283**	**0.408**	**0.367**
(504,504)	RB	-0.001	-0.001	-0.001	-0.001	-0.0011	-0.0062	-0.0049	-0.0053
	RMSE	0.0068	0.0066	0.0066	0.0066	0.0067	0.0087	0.0079	0.0081
	EFF	**1**	**1.048**	**1.048**	**1.048**	**1**	**0.599**	**0.736**	**0.695**

		$\theta = 1$				$\theta = 2$			
(n_1, n_2)		$\widehat{\rho}_{EM}$	$\widehat{\rho}_{Trap}$	$\widehat{\rho}_{Simp1}$	$\widehat{\rho}_{Simp2}$	$\widehat{\rho}_{EM}$	$\widehat{\rho}_{Trap}$	$\widehat{\rho}_{Simp1}$	$\widehat{\rho}_{Simp2}$
Case 2									
(12,12)	RB	-0.069	-0.0657	-0.0685	-0.0609	-0.0696	-0.0761	-0.0699	-0.0701
	RMSE	0.2005	0.1793	0.1858	0.172	0.1963	0.1725	0.1731	0.1736
	EFF	**1**	**1.251**	**1.165**	**1.36**	**1**	**1.294**	**1.286**	**1.278**
(24,30)	RB	-0.0318	-0.0303	-0.0303	-0.0305	-0.0259	-0.0255	-0.024	-0.0241
	RMSE	0.1204	0.1048	0.1048	0.1045	0.119	0.0998	0.1004	0.1005
	EFF	**1**	**1.32**	**1.32**	**1.327**	**1**	**1.419**	**1.404**	**1.403**
(504,504)	RB	-0.0008	-0.0015	-0.0015	-0.0015	-0.0023	-0.0024	-0.0024	-0.0024
	RMSE	0.028	0.0225	0.0225	0.0225	0.0271	0.0213	0.0213	0.0213
	EFF	**1**	**1.547**	**1.547**	**1.547**	**1**	**1.617**	**1.617**	**1.617**
Case 3									
(12,12)	RB	-0.331	-0.15	-0.104	-0.113	-0.329	-0.568	-0.535	-0.547
	RMSE	1.013	0.6959	0.7263	0.7193	1.068	0.7058	0.6959	0.6997
	EFF	**1**	**2.12**	**1.947**	**1.985**	**1**	**2.292**	**2.357**	**2.332**
(24,30)	RB	-0.203	-0.0733	-0.0605	-0.0611	-0.157	-0.47	-0.439	-0.451
	RMSE	1.121	0.5086	0.519	0.5176	1.245	0.5502	0.5314	0.5384
	EFF	**1**	**4.86**	**4.67**	**4.69**	**1**	**5.12**	**5.49**	**5.34**
(504,504)	RB	0.0043	0.0001	0.0001	0.0001	0.0066	-0.148	-0.133	-0.138
	RMSE	0.4738	0.1331	0.1331	0.1331	0.4223	0.1808	0.1703	0.174
	EFF	**1**	**12.68**	**12.68**	**12.68**	**1**	**5.46**	**6.15**	**5.89**

TABLE 3. The RB, RMSE and EFF of the estimators $\widehat{\lambda}_{EM}$, $\widehat{\lambda}_{Trap}, \widehat{\lambda}_{Simp1}$ and $\widehat{\lambda}_{Simp2}$ when the data are simulated from pair Weibull distributions as given in Table 1.

		$\theta = 1$				$\theta = 2$			
(n_1, n_2)		$\widehat{\lambda}_{EM}$	$\widehat{\lambda}_{Trap}$	$\widehat{\lambda}_{Simp1}$	$\widehat{\lambda}_{Simp2}$	$\widehat{\lambda}_{EM}$	$\widehat{\lambda}_{Trap}$	$\widehat{\lambda}_{Simp1}$	$\widehat{\lambda}_{Simp2}$
Case 1									
(12,12)	RB	-0.0876	-0.0885	-0.0883	-0.0891	-0.329	-0.0715	-0.0748	-0.0735
	RMSE	0.143	0.1377	0.1377	0.1377	1.068	0.1188	0.1228	0.1213
	EFF	**1**	**1.079**	**1.078**	**1.079**	**1**	**1.375**	**1.287**	**1.318**
(24,30)	RB	-0.0387	-0.0405	-0.0406	-0.0405	-0.157	-0.0311	-0.0326	-0.0321
	RMSE	0.0689	0.0671	0.0671	0.0671	1.245	0.0608	0.0626	0.062
	EFF	**1**	**1.054**	**1.052**	**1.054**	**1**	**1.295**	**1.224**	**1.246**

Case 1									
(504,504)	RB	-0.0021	-0.0019	-0.0019	-0.0019	0.0066	-0.0015	-0.0016	-0.0016
	RMSE	0.0104	0.0094	0.0094	0.0094	0.4223	0.0095	0.0095	0.0095
	EFF	**1**	**1.217**	**1.217**	**1.217**	**1**	**1.175**	**1.166**	**1.169**
Case 2									
(12,12)	RB	-0.0908	-0.106	-0.111	-0.117	-0.0884	-0.0461	-0.0579	-0.0554
	RMSE	0.3127	0.2833	0.2935	0.2852	0.3102	0.2565	0.2566	0.2589
	EFF	**1**	**1.218**	**1.135**	**1.202**	**1**	**1.463**	**1.462**	**1.435**
(24,30)	RB	-0.0511	-0.0612	-0.0607	-0.059	-0.0412	-0.0061	-0.0141	-0.0115
	RMSE	0.1894	0.1723	0.172	0.1701	0.1957	0.1702	0.1701	0.1698
	EFF	**1**	**1.208**	**1.213**	**1.239**	**1**	**1.322**	**1.324**	**1.327**
(504,504)	RB	-0.0021	-0.0035	-0.0035	-0.0035	-0.0019	0.005	0.0036	0.0041
	RMSE	0.0448	0.038	0.038	0.038	0.0463	0.0395	0.0394	0.0395
	EFF	**1**	**1.393**	**1.393**	**1.393**	**1**	**1.374**	**1.38**	**1.378**
Case 3									
(12,12)	RB	0.163	0.856	0.938	0.932	0.547	0.158	0.242	0.206
	RMSE	2.652	3.127	3.325	3.305	4.372	2.577	2.718	2.662
	EFF	**1**	**0.719**	**0.636**	**0.644**	**1**	**2.88**	**2.59**	**2.7**
(24,30)	RB	0.172	0.755	0.749	0.76	0.102	0.0933	0.162	0.135
	RMSE	1.941	2.201	2.202	2.222	1.785	1.255	1.328	1.3
	EFF	**1**	**0.778**	**0.777**	**0.763**	**1**	**2.021**	**1.807**	**1.885**
(504,504)	RB	0.021	0.0091	0.013	0.0101	0.00566	0.424	0.441	0.435
	RMSE	0.3841	0.2556	0.2564	0.2556	0.3801	0.5568	0.5731	0.5678
	EFF	**1**	**2.258**	**2.244**	**2.258**	**1**	**0.466**	**0.44**	**0.448**

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