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HOMOTOPY ANALYSIS METHOD FOR SOLVING THE BACKWARD PROBLEM FOR THE TIME-FRACTIONAL DIFFUSION EQUATION

MOHAMMAD F. AL-JAMAL[1,2]

Abstract. This paper deals with the backward problem of a nonhomogeneous time-fractional diffusion equation, that is, the problem of determining the past distribution of the substance from present measurements. By the separation of variables method, exact solutions of the forward and backward problems are obtained in terms of eigenfunctions and Mittag-Leffler functions. Contrary to the forward problem, i.e., determining the present solution from given initial data, the backward problem, i.e., the problem of recovering the initial condition from noisy measurements of the final data, is proved to be ill-posed and highly unstable with respect to perturbations in the final data, and thus, some regularization technique is required. The novelty of the current work stems from utilizing the homotopy analysis method as a tool to obtain a regularization scheme to tackle the instability of the backward problem. Stability and convergence results of the proposed method are proved, and optimal convergence rates of the regularized solution are given under both *a priori* and *a posteriori* parameter choice rules. The resulted algorithm is very efficient and computationally inexpensive. Numerical examples are presented to illustrate the validity and accuracy of the proposed homotopy method.

1. Introduction

Fractional calculus is a branch of mathematics that attempts to generalize the classical calculus to include derivatives and integrals of arbitrary orders. It can be traced back to 1695 in a letter from L'Hospital to Leibniz asking of the meaning of

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the derivative of order 1/2. Since then several definitions have been given trying to accommodate the meaning of non-integer order derivatives and integrals, most noticeably, the Riemann-Liouville and Caputo definitions [1,2]. Recently, new definitions of fractional derivatives have emerged, most popularly, the Caputo-Fabrizio fractional derivative [3, 4], Atangana-Baleanu definition [5], and the memory-dependent derivative [6].

Nowadays, fractional differential equations have gained extra popularity due to its potential applications in physics, engineering, biology, and other branches, and due to its ability to model complex real-life systems, such as, dissipation mechanisms with memory effects, hearing loss due to Mumps virus, optical solitons, predator-prey dynamical systems, anomalous diffusion, viscoelasticity, and economic growth with memory effect; see [2, 4, 6–11] and the references therein.

In this paper, we consider the backward problem of determining the initial condition g in the nonhomogeneous time-fractional diffusion problem:

$$\begin{cases} D_t^\alpha u(x,t) = u_{xx}(x,t) + f(x,t), & 0 < x < l, \ 0 < t < T, \\ u(0,t) = u(l,t) = 0, & 0 < t < T, \\ u(x,0) = g(x), & 0 < x < l, \end{cases} \tag{1.1}$$

where the time-fractional derivative $D_t^\alpha u$ is taken in the Caputo sense [1] defined by

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}\, d\tau, \quad 0 < \alpha < 1.$$

Time-fractional differential equations are fairly examined and their analytical aspects and numerical treatment are well developed. They are found to be adequate models to describe anomalous diffusion, such as the sub-diffusion, which have been observed, for example, in transport processes in porous media, protein diffusion within cells, and movement of a material along fractals. We refer the reader to [1, 9–18] and the references therein. For instance, equations (1.1) models a slow diffusion process of a contaminant in a subdiffusive medium (fractal or porous media) that takes place in a straight pipe of finite length l over the time period $0 \leq t \leq T$.

Solving equations (1.1) for $u(x,t)$ from given initial condition $g(x)$ and source term $f(x,t)$ is usually termed as the forward or direct problem. In this paper, we are interested in the backward problem of determining the initial condition $g(x)$, and hence the solution $u(x,t)$, from a noisy measurement $q^\delta(x)$ of the final data $q(x) = u(x,T)$ satisfying the *a priori* error bound

$$\|q^\delta - q\|_0 \leq \delta, \tag{1.2}$$

where $\delta > 0$ is the noise level. Thus, the **backward problem** under consideration can be stated as follows:

Backward problem: *given a noisy measurement $q^\delta(x)$ of $q(x) = u(x,T)$, estimate the initial state $g(x) = u(x,0)$, $x \in [0,l]$.*

Such inverse problem can be used to recover the initial concentration of a contaminant (or the initial temperature profile in the case of a heat conduction problem) in a sub-diffusive media which is important for example in environmental engineering, hydrology, and physics, and it can be applied to other disciplines such as image deblurring.

Despite its importance in a diverse set of applications in science and engineering, only a humble number of results dealing with inverse problems in time-fractional differential equations have been established. In [19], Wang et al. considered the backward problem in time-fractional diffusion equations with variable coefficients in general domain. They used Tikhonov regularization to solve the corresponding Fredholm integral equation. Wang and Liu [20] used the total variation regularization to solve the backward problem from given internal measurements. In [21], Deng and Yang proposed a numerical method based on the idea of reproducing kernel approximation to reconstruct the unknown initial heat distribution from a scattered measurements of transient temperature. Kokila and Nair [22] used the Fourier truncation method for solving the nonhomogeneous time-fractional backward heat conduction problem. In [23], Yang et al. used the truncation regularization technique to solve the backward problem for nonhomogeneous time-fractional diffusion-wave equations. Tuan et al. [24] used filter regularization method to determine the initial data from final value with deterministic and random noise. In [25], Zhang and Xu considered the problem of identifying the time-independent source term from boundary data. In [26], Li and Guo considered the identification of the diffusion coefficient and the order of the fractional derivative from the boundary data. Uniqueness and stability results concerning the reconstruction of the initial condition from interior measurements are obtained by Al-Jamal [6]. See also [27–32] for a recent account of the theory of inverse problems in fractional differential equations.

In the recent years, the homotopy analysis method and its modifications have been used to solve fractional differential equations. We refer the reader to [33–35] and the references therein. In this paper, we utilize the homotopy analysis method to obtain approximate expansion of the initial condition. The novelty of this approach is twofold. First, the use of the convergence parameter of the homotopy analysis method as a filter factor for the regularized solution, which in turn can be chosen to exploit different regularization schemes. Second, the proposed approach can be used to solve the backward in the case of nonhomogeneous source term f, which is a limitation for many of the existing methods. Moreover, we give convergence rates under *a priori* and *a posteriori* parameter choice rules of the regularization parameter. The results of the numerical experiments are in excellent agreement with our theoretical analysis.

The rest of this paper is organized as follows. In Section 2, we present some definitions and essential functions and function spaces, then we derive the solution of the forward problem and investigate the instability of the backward problem. In Section 3, we transform the nonhomogeneous problem (1.1) into a homogeneous problem, then an approximate analytic solution is obtained via HAM, from which our regularization technique is developed, and after that, the main results are stated and proved. Section 4 is devoted for the practical implementation of the proposed method, and some numerical experiments to validate our theoretical results.

2. Preliminaries

2.1. **Relevant functions and function spaces.** Let $L^2(0,l)$ be the space of square-integrable functions with inner product and norm given by

$$(v,w)_0 = \int_0^l v(x)w(x)dx, \quad \|v\|_0 = \left(\int_0^l |v(x)|^2\, dx\right)^{\frac{1}{2}}.$$

Let $\{(\lambda_n, X_n) : n = 1, 2, \ldots\}$ be the orthonormal eigensystem for the eigenvalue problem

$$-X''(x) = \lambda X(x), \quad X(0) = X(l) = 0.$$

The eigenvalues and the normalized (with respect to the L^2-norm) eigenfunctions are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \; X_n(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$$

In the analysis below, we find it useful to define the function space $\mathbf{H}^p(0,l)$ by

$$\mathbf{H}^p(0,l) = \left\{ v \in L^2(0,l) : \sum_{n=1}^{\infty} \left|(v, X_n)_0\right|^2 \lambda_n^{2p} < \infty \right\},$$

which is a Hilbert space with the norm

$$\|v\|_p = \left(\sum_{n=1}^{\infty} \left|(v, X_n)_0\right|^2 \lambda_n^{2p} \right)^{\frac{1}{2}}.$$

One of the key ingredients when dealing with fractional differential equations is the Mittag-Leffler [1,12]. The Mittag-Leffler function of index (α, β) is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are constants. For brevity, we will use the notation $E_\alpha(z)$ to denote $E_{\alpha,1}(z)$. We cite some relevant properties of the Mittag-Leffler function [1].

Lemma 2.1. *Let $\lambda > 0$.*

(1) *For $0 < \alpha < 1$, $t > 0$, we have*

$$D_t^\alpha E_\alpha(-\lambda t^\alpha) = -\lambda E_\alpha(-\lambda t^\alpha).$$

(2) *For $\alpha > 0$, $t > 0$, we have*

$$\frac{d}{dt}\left[E_\alpha(-\lambda t^\alpha)\right] = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha).$$

(3) *For $0 < \alpha < 1$, $t \geq 0$, we have*

$$E_{\alpha,\alpha}(-t) \geq 0, \; 0 < E_\alpha(-t) \leq 1.$$

From Lemma 2.1, we can easily deduce the following corollary.

Corollary 2.1. *For $\alpha, \lambda, t > 0$, we have*

$$\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda\tau^\alpha)d\tau = \frac{1}{\lambda}\left(1 - E_\alpha(-\lambda t^\alpha)\right).$$

We also need the following asymptotic result which can be found in [31].

Lemma 2.2. *Assume that $0 < \alpha < 1$. Then there exist constants $C_-, C_+ > 0$ depending only on α such that*

$$\frac{C_-}{1+t} \leq E_\alpha(-t) \leq \frac{C_+}{1+t}, \quad \textit{for all } t \geq 0.$$

2.2. **The eigenfunction expansion of the forward solution.** Following the separation of variables method, we formally define the solution of problem (1.1) by

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x).$$

Then, it follows that $T_n(t)$ solves the fractional order initial-value problem

$$D^\alpha T_n(t) + \lambda_n T_n(t) = f_n(t), \quad T_n(0) = g_n, \tag{2.1}$$

where

$$f_n(t) = \int_0^l f(x,t) X_n(x)dx, \;\; g_n = \int_0^l g(x) X_n(x) dx, \quad n = 1, 2, \ldots$$

From [12], the solution of the initial-value problem (2.1) is given by

$$T_n(t) = g_n E_\alpha(-\lambda_n t^\alpha) + F_n(t),$$

where

$$F_n(t) = \int_0^t f_n(t-\tau)\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau.$$

Hence, the formal solution to (1.1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ g_n E_\alpha(-\lambda_n t^\alpha) + F_n(t) \right\} X_n(x).$$

2.3. **Stability of the backward problem.** From the final data $u(x,T) = q(x)$, we observe that

$$q_n = g_n E_\alpha(-\lambda_n T^\alpha) + F_n(T), \quad n = 1, 2, \dots \tag{2.2}$$

where $q_n = (q, X_n)_0$. Thus, the solution to the backward problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{q_n - F_n(T)}{E_\alpha(-\lambda_n T^\alpha)} E_\alpha(-\lambda_n t^\alpha) + F_n(t) \right\} X_n(x). \tag{2.3}$$

Now, we discuss the stability of the backward problem with respect to perturbations in the final data q. To this end, let q^δ be some noisy data satisfying the bound (1.2), and denote by $u^\delta(x,t)$ the solution of the backward problem with q^δ in place of q, that is,

$$u^\delta(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{q_n^\delta - F_n(T)}{E_\alpha(-\lambda_n T^\alpha)} E_\alpha(-\lambda_n t^\alpha) + F_n(t) \right\} X_n(x), \tag{2.4}$$

where $q_n^\delta = \left(q^\delta, X_n\right)_0$. Then, from equations (2.3) and (2.4), for any $t \in (0,T]$, we have

$$u^\delta(x,t) - u(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)} \left(q_n^\delta - q_n\right) \right\} X_n(x).$$

From Lemma 2.2, we have the inequality

$$\frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)} \le \left(\frac{C_+}{C_-}\right) \left(\frac{1 + \lambda_n T^\alpha}{1 + \lambda_n t^\alpha}\right) \le \left(\frac{C_+}{C_-}\right) \left(\frac{T^\alpha}{t^\alpha}\right),$$

and so, by using the Parseval's identity, we obtain

$$\left\|u^\delta(\cdot,t) - u(\cdot,t)\right\|_0^2 = \sum_{n=1}^{\infty} \left| \frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)} \right|^2 \left|q_n - q_n^\delta\right|^2 \le \left(\frac{C_+}{C_-}\right)^2 \left(\frac{T^\alpha}{t^\alpha}\right)^2 \delta^2.$$

This shows that the backward problem is in fact stable for $t \in (0,T]$. However, the problem becomes unstable for $t = 0$. To see this, take, for instance, perturbations of the final data as

$$q^{\delta_k}(x) = q(x) + \delta_k X_k(x),$$

with $\delta_k = E_\alpha(-\lambda_k T^\alpha)^{\frac{1}{2}}$, to compute the backward solution $g^{\delta_k}(x) = u^{\delta_k}(x,0)$ as given by the expansion (2.4). Then,

$$g^{\delta_k}(x) = \sum_{n=1}^{\infty} \left\{ \frac{(q + \delta_k X_k, X_n)_0}{E_\alpha(-\lambda_n T^\alpha)} \right\} X_n(x) = g(x) + \delta_k^{-\frac{1}{2}} X_k(x).$$

On the one hand, we have

$$\left\|q^{\delta_k} - q\right\|_0 = \delta_k \to 0,$$

as $k \to \infty$, but on the other hand, we get

$$\left\|g^{\delta_k} - g\right\|_0 = \delta_k^{-\frac{1}{2}} \to \infty,$$

as $k \to \infty$. Thus, the problem of determining the initial data from the final data q is unstable, and hence, it is ill-posed. To deal with the instability issue, we shall discuss regularization schemes in the following section.

3. Regularization scheme via the homotopy analysis method

3.1. **Transformation into homogeneous problem.** We can transform the non-homogeneous problem (1.1) into a homogeneous problem by introducing the function v given by

$$v(x,t) = u(x,t) - w(x,t),$$

where $w(x,t) = \sum_{n=1}^{\infty} F_n(t) X_n(x)$ is the solution to the time-fractional diffusion equations

$$\begin{cases} D_t^\alpha w(x,t) = w_{xx}(x,t) + f(x,t), & 0 < x < l, \ \ 0 < t < T, \\ w(0,t) = w(l,t) = 0, & 0 < t < T, \\ w(x,0) = 0, & 0 < x < l. \end{cases} \tag{3.1}$$

Then, v satisfies the homogeneous time-fractional diffusion problem

$$\begin{cases} D_t^\alpha v(x,t) = v_{xx}(x,t), & 0 < x < l, \ \ 0 < t < T, \\ v(0,t) = v(l,t) = 0, & 0 < t < T, \\ v(x,0) = u(x,0) = g(x), & 0 < x < l. \end{cases} \tag{3.2}$$

Therefore, if we set

$$\overline{q}(x) = q(x) - w(x,T), \quad \overline{q}^\delta(x) = q^\delta(x) - w(x,T),$$

then the original backward problem defined by equations (1.1) is equivalent to the backward problem of recovering $g(x)$ in equations (3.2) from the noisy measurement

$\overline{q}^\delta(x)$ of the final data $\overline{q}(x) = v(x,T)$. Moreover, from (1.2), we have the *a priori* error bound

$$\|\overline{q}^\delta - \overline{q}\|_0 \leq \delta. \tag{3.3}$$

From the last section, we deduce the following expansions of the solution v:

$$v(x,t) = \sum_{n=1}^{\infty} \{g_n E_\alpha(-\lambda_n t^\alpha)\} X_n(x) = \sum_{n=1}^{\infty} \left\{ \frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)} \overline{q}_n \right\} X_n(x), \tag{3.4}$$

where $\overline{q}_n = (\overline{q}, X_n)_0 = q_n - F_n(T)$. In particular, by setting $t = 0$, we get

$$g(x) = \sum_{n=1}^{\infty} \{g_n\} X_n(x) = \sum_{n=1}^{\infty} \left\{ \frac{\overline{q}_n}{E_\alpha(-\lambda_n T^\alpha)} \right\} X_n(x). \tag{3.5}$$

In the next subsection, we provide an alternative method to obtain approximate and exact solutions to problem (3.2) by utilizing the homotopy analysis method (HAM). The resulted solution will provide an insight for the suggested regularization method.

3.2. **The backward solution via HAM.** To present our homotopy based approach, let us define the following sequence of auxiliary problems:

$$D_t^\alpha V_n = (V_n)_{xx}, \;\; 0 < x < l, \;\; 0 < t < T, \tag{3.6}$$

subject to the conditions

$$\begin{cases} V_n(0,t) = V_n(l,t) = 0, & 0 < t < T, \\ V_n(x,T) = \overline{q}_n X_n(x), & 0 < x < l. \end{cases} \tag{3.7}$$

Then, to solve (3.6)-(3.7) by means of the homotopy analysis method, we introduce the so-called *zero-order deformation equation* given by

$$(1-p)\mathcal{L}\left[\phi_n(x,t;p) - v_0(x,t)\right] = \hbar_n p \mathcal{L}\left[\phi_n(x,t;p)\right],$$

where $\mathcal{L}$ is an auxiliary linear operator, $\hbar_n$ is a nonzero auxiliary parameter, $p \in [0,1]$ is the embedding parameter, and v_0 is some initial guess of the solution V_n of (3.6). If we define the auxiliary linear operator $\mathcal{L}$ by

$$\mathcal{L}[v] = D_t^\alpha v - v_{xx},$$

then, as p varies from 0 to 1, the solution $\phi_n(x,t;p)$ deforms from the initial guess $\phi_n(x,t;0) = v_0(x,t)$ to the solution $\phi_n(x,t;1) = V_n(x,t)$ of (3.6). Now, a formal power series expansion of the function $\phi_n(x,t;p)$ about $p=0$ is

$$\phi_n(x,t;p) = v_0(x,t) + \sum_{k=1}^{\infty} v_k(x,t)p^k,$$

and therefore, the solution to equation (3.6) has a formal expansion of the form

$$V_n(x,t) = \phi_n(x,t;1) = v_0(x,t) + \sum_{k=1}^{\infty} v_k(x,t).$$

To enforce conditions (3.7), we may choose $v_0(x,t) = \overline{q}_n X_n(x)$, and thus, we may assume that

$$v_k(x,T) = 0, \quad k = 1,2,\ldots \tag{3.8}$$

Then, to determine the unknown functions v_k, we substitute the expansion of $\phi(x,t;p)$ in the zero-order deformation equation and equate the coefficients of equal powers of p. This yields

$$\begin{aligned} \mathcal{L}[v_1(x,t)] &= \hbar_n \lambda_n \overline{q}_n X_n(x), \\ \mathcal{L}[v_{k+1}(x,t)] &= (1+\hbar_n)^k \mathcal{L}[v_1(x,t)], \quad k = 1,2,\ldots \end{aligned} \tag{3.9}$$

One can directly verify that the solutions to (3.8)-(3.9) are given by

$$\begin{aligned} v_1(x,t) &= -\hbar_n \left[\frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)} \overline{q}_n - \overline{q}_n \right] X_n(x), \\ v_{k+1}(x,t) &= (1+\hbar_n)^k v_1(x,t), \quad k = 1,2,\ldots \end{aligned}$$

Thus, we can form the so-called mth-order homotopy approximate solution to problem (3.6)-(3.7) given by

$$V_n^m(x,t) = v_0(x,t) + \sum_{k=1}^{m} v_k(x,t) = v_0(x,t) + \left(\sum_{k=1}^{m} (1+\hbar_n)^{k-1} \right) v_1(x,t),$$

which can be simplified to

$$V_n^m(x,t) = \xi_n \left(\frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)} \overline{q}_n \right) X_n(x) + (1-\xi_n)\overline{q}_n X_n(x),$$

where $\xi_n = 1 - (1+\hbar_n)^m$.

If we assume that $-1 < \hbar_n < 0$, then the exact solution to the auxiliary problem (3.6)-(3.7) is

$$V_n(x,t) = \lim_{m\to\infty} V_n^m(x,t) = \left(\frac{E_\alpha(-\lambda_n t^\alpha)}{E_\alpha(-\lambda_n T^\alpha)}\overline{q}_n\right) X_n(x).$$

Moreover, from the expansion (3.4), we see that the solution to problem (3.2) can be formed as

$$v(x,t) = \sum_{n=1}^{\infty} V_n(x,t).$$

By replacing V_n by its mth-order approximation V_n^m, we suggest the approximate HAM solution to the backward problem (3.2) as

$$v_H(x,t) = \sum_{n=1}^{\infty} V_n^m(x,t).$$

In particular, if we set $t = 0$, then the exact initial condition $g(x)$ can be approximated by

$$g_H(x) = \sum_{n=1}^{\infty} \xi_n \left(\frac{\overline{q}_n}{E_\alpha(-\lambda_n T^\alpha)}\right) X_n(x) + \sum_{n=1}^{\infty}(1-\xi_n)\overline{q}_n X_n(x). \tag{3.10}$$

We conclude by the following consistency result which will play a key role in proving the main results of this paper.

Lemma 3.1. *Assume that $g \in \mathbf{H}^p(0,l)$ for some $p > 0$. Then, the following bound holds*

$$\|g_H - g\|_0 \leq (\mu_p) \sup_{n\geq 1} \{\mathcal{B}_n\},$$

where

$$\mu_p = \left(\frac{1+\lambda_1 T^\alpha}{\lambda_1 C_-}\right)^p \|g\|_p, \;\; \mathcal{B}_n = (1-\xi_n)E_\alpha(-\lambda_n T^\alpha)^p.$$

Proof. From equations (3.5)-(3.10) it follows that

$$g_H(x) - g(x) = \sum_{n=1}^{\infty}(1-\xi_n)\left(E_\alpha(-\lambda_n T^\alpha) - 1\right) g_n X_n(x).$$

Then, by the Parseval's identity and the fact that $0 < E_\alpha(-t) < 1$ for $t \geq 0$, we get

$$\|g_H - g\|_0^2 \leq \sum_{n=1}^{\infty}(1-\xi_n)^2 g_n^2 \leq \sup_{n\geq 1}\left\{\mathcal{B}_n^2\right\} \sum_{n=1}^{\infty} \frac{g_n^2}{E_\alpha(-\lambda_n T^\alpha)^{2p}}.$$

Using Lemma 2.2, we can bound the last term as

$$\sum_{n=1}^{\infty} \frac{g_n^2}{E_\alpha(-\lambda_n T^\alpha)^{2p}} \leq \sum_{n=1}^{\infty} \left| \frac{1+\lambda_n T^\alpha}{C_-} \right|^{2p} g_n^2 \leq \left| \frac{1+\lambda_1 T^\alpha}{C_- \lambda_1} \right|^{2p} \sum_{n=1}^{\infty} \lambda_n^{2p} g_n^2$$

$$= \left| \frac{1+\lambda_1 T^\alpha}{C_- \lambda_1} \right|^{2p} \|g\|_p^2, \tag{3.11}$$

which finishes the proof. □

3.3. **Regularization scheme via HAM.** The expansion in (3.10) will be the basis of the regularization scheme presented in this paper. More specifically, we suggest the following regularized solution to our backward problem:

$$g_H^\delta(x) = \sum_{n=1}^{\infty} \xi_n \left(\frac{\overline{q}_n^\delta}{E_\alpha(-\lambda_n T^\alpha)} \right) X_n(x) + \sum_{n=1}^{\infty} (1-\xi_n) \overline{q}_n^\delta X_n(x), \tag{3.12}$$

where $\overline{q}_n^\delta = \left(\overline{q}^\delta, X_n\right)_0 = q_n^\delta - F_n(T)$. In the context of regularization theory, the numbers $\xi_n \in (0,1)$ are treated as the *filter factors* of regularization method.

Now, we prove some results concerning the recovery of the initial condition g by the proposed homotopy method. We shall need the following stability result.

Lemma 3.2. *It holds that*

$$\left\| g_H^\delta - g_H \right\|_0 \leq \left(\sup_{n \geq 1} \{\mathcal{A}_n\} + 1 \right) \delta,$$

where

$$\mathcal{A}_n = \frac{\xi_n}{E_\alpha(-\lambda_n T^\alpha)}.$$

Proof. By equations (3.12)-(3.10), the triangle inequality, the Parseval's identity, and the fact that $\xi_n \in (0,1)$, we have

$$\begin{aligned} \left\| g_H^\delta - g_H \right\|_0 &\leq \left\| \sum_{n=1}^{\infty} \xi_n \left(\frac{\overline{q}_n^\delta - \overline{q}_n}{E_\alpha(-\lambda_n T^\alpha)} \right) X_n \right\|_0 + \left\| \sum_{n=1}^{\infty} (1-\xi_n) \left(\overline{q}_n^\delta - \overline{q}_n \right) X_n \right\|_0 \\ &\leq \sup_{n \geq 1} \left\{ \frac{\xi_n}{E_\alpha(-\lambda_n T^\alpha)} \right\} \delta + \delta, \end{aligned}$$

as required. □

Combining Lemma 3.1 and Lemma 3.2 together with the triangle inequality, we obtain the first main result:

Theorem 3.1. *Assume that* $g \in \mathbf{H}^p(0,l)$ *for some* $p > 0$*. Then*

$$\left\|g_H^\delta - g\right\|_0 \leq (\mu_p) \sup_{n\geq 1}\{\mathcal{B}_n\} + \left(\sup_{n\geq 1}\{\mathcal{A}_n\} + 1\right)\delta.$$

Several regularization methods can be designed by choosing different filter factors. In this paper, we will focus on the filter factors given by

$$\xi_n = \frac{E_\alpha(-\lambda_n T^\alpha)^2}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}, \quad n = 1, 2, \ldots \tag{3.13}$$

where $\beta > 0$ is a fixed constant which plays the rule of the *regularization parameter* of the regularization scheme. We will utilize Theorem 3.1 to obtain some convergence results under this particular choice of the filter factors. To this end, we need the following auxiliary result.

Lemma 3.3. *If the factors* ξ_n *are chosen as in (3.13), then*

$$\mathcal{A}_n = \frac{E_\alpha(-\lambda_n T^\alpha)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}, \;\; \text{and } \mathcal{B}_n = \frac{\beta E_\alpha(-\lambda_n T^\alpha)^p}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta},$$

with

$$\sup_{n\geq 1}\{\mathcal{A}_n\} \leq \frac{1}{2\sqrt{\beta}}, \;\; \text{and } \sup_{n\geq 1}\{\mathcal{B}_n\} \leq \begin{cases} \beta^{\frac{p}{2}}, & p < 2, \\ \beta, & p \geq 2. \end{cases}$$

Proof. Consider the functions

$$f(x) = \frac{x}{x^2+\beta}, \quad g(x) = \frac{\beta x^p}{x^2+\beta}, \quad x > 0,\ p < 2.$$

Then it easy to verify that f attains its maximum value at $x_0 = \sqrt{\beta}$, while g attains its maximum value at $x_1 = \sqrt{\frac{p\beta}{2-p}}$. Hence,

$$\sup_{n\geq 1}\{\mathcal{A}_n\} = \sup_{n\geq 1}\left\{\frac{E_\alpha(-\lambda_n T^\alpha)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right\} \leq f(x_0) = \frac{1}{2\sqrt{\beta}},$$

and, for $p < 2$, we have

$$\sup_{n\geq 1}\{\mathcal{B}_n\} = \sup_{n\geq 1}\left\{\frac{\beta E_\alpha(-\lambda_n T^\alpha)^p}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right\} \leq g(x_1) = \left(\frac{1}{2}(2-p)^{1-\frac{p}{2}} p^{\frac{p}{2}}\right)\beta^{\frac{p}{2}} \leq \beta^{\frac{p}{2}}.$$

For the case $p \geq 2$, we use Lemma 2.1 (3) to see that

$$\sup_{n\geq 1}\{\mathcal{B}_n\} = \sup_{n\geq 1}\left\{\frac{\beta E_\alpha(-\lambda_n T^\alpha)^p}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right\} \leq \sup_{n\geq 1}\left\{E_\alpha(-\lambda_n T^\alpha)^{p-2}\beta\right\} \leq \beta.$$

This completes the proof. $\square$

The following result gives error estimates on the regularized approximate solution using the filter factors defined by (3.13).

Lemma 3.4. *Assume that $g \in \mathbf{H}^p(0,l)$ for some $p > 0$. If ξ_n is chosen as in (3.13), then the following error estimates hold.*

(1) *For $p < 2$, we have*

$$\left\|g_H^\delta - g\right\|_0 \leq \mu_p \beta^{\frac{p}{2}} + \frac{\delta}{2\sqrt{\beta}} + \delta.$$

(2) *For $p \geq 2$, we have*

$$\left\|g_H^\delta - g\right\|_0 \leq \mu_p \beta + \frac{\delta}{2\sqrt{\beta}} + \delta.$$

Proof. The proof follows directly from Theorem 3.1 and Lemma 3.3. □

Finally, we cite the following remark regarding the convergence rates of the regularization scheme under an *a priori* choice rule of the regularization parameter β.

Remark 1. *Under the hypotheses of Lemma 3.4, if we choose $\beta = C\delta^\gamma$ for some $\gamma \in (0,2)$ and constant $C > 0$, then*

$$\left\|g_H^\delta - g\right\|_0 \to 0,$$

as $\delta \to 0$. For a given value of $p > 0$, the convergence rate is optimal when

$$\gamma = \begin{cases} \frac{2}{p+1}, & p < 2, \\ \frac{2}{3}, & p \geq 2, \end{cases}$$

in which case we have

$$\left\|g_H^\delta - g\right\|_0 = \begin{cases} O\left(\delta^{\frac{p}{p+1}}\right), & p < 2, \\ O\left(\delta^{\frac{2}{3}}\right), & p \geq 2. \end{cases}$$

Thus, we obtain the fastest convergence when $p \geq 2$. In this case, we have

$$\left\|g_H^\delta - g\right\|_0 = O\left(\delta^{\frac{2}{3}}\right),$$

provided $\beta = C\delta^{\frac{2}{3}}$.

3.4. **Convergence under Morozov's discrepancy principle.** Next, we prove convergence results under *a posteriori* parameter choice rule for the regularization parameter β. In this paper, we focus on the Morozov's discrepancy principle, which amounts to choosing $\beta > 0$ such that

$$\left\| Lg_H^\delta - \overline{q}^\delta \right\|_0 = \eta\delta, \tag{3.14}$$

where $\eta > 1$ is some given constant, and L denotes the forward map from the initial data to the final data, which from (3.4), is given by

$$Lz = \sum_{n=1}^{\infty} \left\{ (z, X_n)_0 \, E_\alpha(-\lambda_n T^\alpha) \right\} X_n.$$

We have the following auxiliary results:

Lemma 3.5. *It holds that*

$$\|Lg_H - \overline{q}\|_0 \leq (\eta + 2)\delta.$$

Proof. Using the triangle inequality, equation (3.14), and inequality (3.3), we see that

$$\begin{aligned} \|Lg_H - \overline{q}\|_0 &\leq \left\| Lg_H - Lg_H^\delta \right\|_0 + \left\| Lg_H^\delta - \overline{q}^\delta \right\|_0 + \left\| \overline{q}^\delta - \overline{q} \right\|_0 \\ &\leq \left\| Lg_H - Lg_H^\delta \right\|_0 + \eta\delta + \delta. \end{aligned}$$

Since $0 < E_\alpha(-\lambda_n T^\alpha) \leq 1$, and

$$g_H(x) - g_H^\delta(x) = \sum_{n=1}^{\infty} \left\{ \frac{E_\alpha(-\lambda_n T^\alpha) + \beta}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \left(\overline{q}_n - \overline{q}_n^\delta \right) \right\} X_n(x),$$

it follows that

$$\begin{aligned} \left\| Lg_H - Lg_H^\delta \right\|_0^2 &= \sum_{n=1}^{\infty} \left| \frac{E_\alpha(-\lambda_n T^\alpha)^2 + \beta E_\alpha(-\lambda_n T^\alpha)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^2 \left| \overline{q}_n - \overline{q}_n^\delta \right|^2 \\ &\leq \sum_{n=1}^{\infty} \left| \overline{q}_n - \overline{q}_n^\delta \right|^2 \leq \delta^2, \end{aligned}$$

which concludes the proof. ◻

Lemma 3.6. *Assume that $g \in \mathbf{H}^p(0, l)$ for some $p > 0$. Then*

$$\|g_H - g\|_0 \leq \left((\eta + 2)^p \mu_p \right)^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}.$$

Proof. From equations (3.5)-(3.10) we get

$$g_H(x) - g(x) = \sum_{n=1}^{\infty} \left\{ \frac{\beta}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} (\overline{q}_n - g_n) \right\} X_n(x),$$

$$Lg_H(x) - \overline{q}(x) = \sum_{n=1}^{\infty} \left\{ \frac{\beta E_\alpha(-\lambda_n T^\alpha)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} (\overline{q}_n - g_n) \right\} X_n(x).$$

Then, using Holder's inequality, we have

$$\begin{aligned} \|g_H - g\|_0^2 &= \sum_{n=1}^{\infty} \left| \frac{\beta}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^2 |\overline{q}_n - g_n|^2 \\ &= \sum_{n=1}^{\infty} \left| \frac{\beta E_\alpha(-\lambda_n T^\alpha)(\overline{q}_n - g_n)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^{\frac{2p}{p+1}} \left| \frac{\beta E_\alpha(-\lambda_n T^\alpha)^{-p}(\overline{q}_n - g_n)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^{\frac{2}{p+1}} \\ &\leq S_1 S_2, \end{aligned}$$

where

$$S_1 = \left(\sum_{n=1}^{\infty} \left| \frac{\beta E_\alpha(-\lambda_n T^\alpha)(\overline{q}_n - g_n)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^2 \right)^{\frac{p}{p+1}},$$

$$S_2 = \left(\sum_{n=1}^{\infty} \left| \frac{\beta E_\alpha(-\lambda_n T^\alpha)^{-p}(\overline{q}_n - g_n)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^2 \right)^{\frac{1}{p+1}}.$$

For the first term, from Lemma 3.5, we have

$$S_1 = \left(\|Lg_H - \overline{q}\|_0^2 \right)^{\frac{p}{p+1}} \leq ((\eta + 2)\, \delta)^{\frac{2p}{p+1}},$$

and by that fact that $0 < E_\alpha(-\lambda_n T^\alpha) \leq 1$ and inequality (3.11), we have

$$\begin{aligned} S_2 &= \left(\sum_{n=1}^{\infty} \left| \frac{\beta}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta} \right|^2 \frac{|\overline{q}_n - g_n|^2}{E_\alpha(-\lambda_n T^\alpha)^{2p}} \right)^{\frac{1}{p+1}} \\ &\leq \left(\sum_{n=1}^{\infty} \frac{|\overline{q}_n - g_n|^2}{E_\alpha(-\lambda_n T^\alpha)^{2p}} \right)^{\frac{1}{p+1}} \\ &= \left(\sum_{n=1}^{\infty} \frac{[E_\alpha(-\lambda_n T^\alpha) - 1]^2 |g_n|^2}{E_\alpha(-\lambda_n T^\alpha)^{2p}} \right)^{\frac{1}{p+1}} \\ &\leq (\mu_p)^{\frac{2}{p+1}}, \end{aligned}$$

which completes the proof. $\square$

Lemma 3.7. *Assume that $g \in \mathbf{H}^p(0,l)$ for some $p > 0$. Then*

$$\left\|g_H^\delta - g_H\right\|_0 \leq \delta + \frac{1}{2}\begin{cases}(C_p)^{\frac{-1}{p+1}}\delta^{\frac{p}{p+1}}, & p < 1,\\ (C_p)^{\frac{-1}{2}}\delta^{\frac{1}{2}}, & p \geq 1,\end{cases}$$

where $C_p = (\eta - 1)/\mu_p$.

Proof. We first observe that

$$\left(g_H^\delta, X_n\right)_0 = \frac{E_\alpha(-\lambda_n T^\alpha)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\overline{q}_n^\delta + \frac{\beta}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\overline{q}_n^\delta,$$

and so

$$\begin{aligned} Lg_H^\delta - \overline{q}^\delta = &\sum_{n=1}^{\infty}\left\{\frac{\beta(E_\alpha(-\lambda_n T^\alpha) - 1)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\left(\overline{q}_n^\delta - \overline{q}_n\right)\right\}X_n \\ &+ \sum_{n=1}^{\infty}\left\{\frac{\beta(E_\alpha(-\lambda_n T^\alpha) - 1)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\left(\overline{q}_n\right)\right\}X_n. \end{aligned}$$

Because $0 < E_\alpha(-\lambda_n T^\alpha) \leq 1$, we have

$$\begin{aligned} &\left\|\sum_{n=1}^{\infty}\left\{\frac{\beta(E_\alpha(-\lambda_n T^\alpha) - 1)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\left(\overline{q}_n^\delta - \overline{q}_n\right)\right\}X_n\right\|_0^2 \\ &= \sum_{n=1}^{\infty}\left|\frac{\beta(E_\alpha(-\lambda_n T^\alpha) - 1)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right|^2\left|\overline{q}_n^\delta - \overline{q}_n\right|^2 \leq \delta^2, \end{aligned}$$

and by equation (3.5) and inequality (3.11), we have

$$\begin{aligned} &\left\|\sum_{n=1}^{\infty}\left\{\frac{\beta(E_\alpha(-\lambda_n T^\alpha) - 1)}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\left(\overline{q}_n\right)\right\}X_n\right\|_0^2 \\ &= \sum_{n=1}^{\infty}\left|\frac{\beta E_\alpha(-\lambda_n T^\alpha)^{p+1}}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right|^2\frac{(E_\alpha(-\lambda_n T^\alpha) - 1)^2\left|g_n\right|^2}{E_\alpha(-\lambda_n T^\alpha)^{2p}} \\ &\leq \sup_{n\geq 1}\left|\frac{\beta E_\alpha(-\lambda_n T^\alpha)^{p+1}}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right|^2\left(\mu_p^2\right). \end{aligned}$$

By utilizing Lemma 3.3, we see that

$$\sup_{n\geq 1}\left|\frac{\beta E_\alpha(-\lambda_n T^\alpha)^{p+1}}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\right| \leq \begin{cases}\beta^{\frac{p+1}{2}}, & p < 1,\\ \beta, & p \geq 1.\end{cases}$$

Therefore, in view of condition (3.14), and the last two inequalities, we see that

$$\eta\delta = \left\|Lg_H^\delta - \overline{q}^\delta\right\|_0 \leq \delta + \mu_p \begin{cases} \beta^{\frac{p+1}{2}}, & p < 1, \\ \beta, & p \geq 1, \end{cases}$$

and consequently, this yields that

$$\beta \geq \begin{cases} (C_p)^{\frac{2}{p+1}} \delta^{\frac{2}{p+1}}, & p < 1, \\ C_p\delta, & p \geq 1. \end{cases}$$

Hence, from the last inequality, Lemma 3.2, and Lemma 3.3, we get

$$\left\|g_H^\delta - g_H\right\|_0 \leq \delta + \frac{\delta}{2\sqrt{\beta}} \leq \delta + \frac{1}{2} \begin{cases} (C_p)^{\frac{-1}{p+1}} \delta^{\frac{p}{p+1}}, & p < 1, \\ (C_p)^{\frac{-1}{2}} \delta^{\frac{1}{2}}, & p \geq 1, \end{cases}$$

which proves the result. □

Combining Lemma 3.6 and Lemma 3.7, we obtain the main convergence result of the proposed scheme under the Morozov's discrepancy principle given by condition (3.14).

Theorem 3.2. *Assume that $g \in \mathbf{H}^p(0,l)$ for some $p > 0$. Then*

$$\left\|g_H^\delta - g\right\|_0 \leq ((\eta+2)^p \mu_p)^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}} + \delta + \frac{1}{2} \begin{cases} (C_p)^{\frac{-1}{p+1}} \delta^{\frac{p}{p+1}}, & p < 1, \\ (C_p)^{\frac{-1}{2}} \delta^{\frac{1}{2}}, & p \geq 1. \end{cases}$$

Remark 2. *In view of Theorem 3.2, we see that under the Morozov's discrepancy principle (3.14), the proposed method is of order $O(\delta^{\frac{p}{p+1}})$ if $p < 1$, with optimal convergence rate $O(\delta^{\frac{1}{2}})$ when $p \geq 1$.*

4. Numerical illustrations

Next, we will show how to implement the proposed scheme for a practical problem. Since it is often the case that the final data is just a discrete noisy readings of the exact final data, we will assume that the final data q^δ is generated using the formula

$$q^\delta(x_i) = q(x_i) + r_i q(x_i), \quad i = 0, 1, \ldots, N,$$

where x_i are uniform grid points of $[0, l]$, and r_i are uniform random real numbers in $[-1, 1]$. The noise level δ is computed using the root-mean-square norm:

$$\delta = \sqrt{\frac{1}{N+1}\sum_{i=0}^{N}\left(q^\delta(x_i) - q(x_i)\right)^2}.$$

The computations of the Fourier coefficients q_n^δ and $f_n(t)$ can be done extremely quickly using the fast Fourier transform. To evaluate $F_n(T)$, we use the midpoint quadrature rule over the grid $t_j = j(T/M)$, $j = 0, 1, \ldots, M$, and we utilize Corollary 2.1, this yields the estimates

$$F_n(T) \doteq \sum_{i=0}^{M/2-1}\left(\frac{E_\alpha(-\lambda_n t_{2i}^\alpha) - E_\alpha(-\lambda_n t_{2i+2}^\alpha)}{\lambda_n}\right) f_n(t_{M-2i-1}),\ \ n = 1, 2, \ldots, N-1.$$

Then, the discretized solution by the homotopy regularization scheme (3.12)-(3.13) is given by

$$g_H^{\delta,\beta}(x_i) \doteq \sum_{n=1}^{N-1}\left\{\frac{E_\alpha(-\lambda_n T^\alpha) + \beta}{E_\alpha(-\lambda_n T^\alpha)^2 + \beta}\left(q_n^\delta - F_n(T)\right)\right\} X_n(x_i),\ \ i = 1, 2, \ldots, N-1.$$

To assess the error in the computed solution, we use the relative root-mean-square error denoted and given by

$$\mathcal{E}(\delta,\beta) = \frac{\sqrt{\sum_{i=0}^{N}\left(g_H^{\delta,\beta}(x_i) - g(x_i)\right)^2}}{\sqrt{\sum_{i=0}^{N}\left(g(x_i)\right)^2}}.$$

In the experiments below, we fix $T = 1.0$, $N = 1000$, $M = 500$, and for the *a posteriori* rule we take $\eta = 1.05$. We utilize MATHEMATICA for the computations of the Mittag-Leffler function and the FFT as well.

4.1. **Example 1.** In the first example, we consider the fractional diffusion equation

$$\begin{cases} D_t^\alpha u(x,t) = u_{xx}(x,t) + f(x,t), & 0 < x < \pi,\ 0 < t < 1, \\ u(0,t) = u(\pi,t) = 0, & 0 < t < 1, \\ u(x,0) = \sin(2x) + \sin(3x), & 0 < x < \pi, \end{cases}$$

with

$$f(x,t) = \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 4t + 4\right)\sin(2x) + 9\sin(3x).$$

In this case, the forward solution to this problem is given by

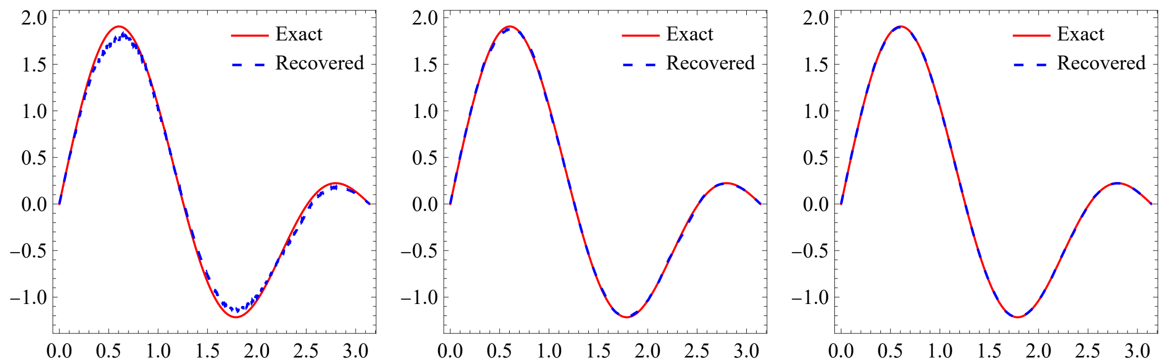
$$u(x,t) = (t+1)\sin(2x) + \sin(3x).$$

Error results for several noise levels δ and orders α using *a priori* (β_{pri}) and *a posteriori* (β_{pos}) choice rules of β are summarized in Table 1. Plots for the exact and recovered initial condition for different noise levels when $\alpha = 0.5$ are shown in Figure 1. Plots for the solution error versus data error using *a priori* and *a posteriori* choice rules are depicted in Figure 2.

From Figure 1, we see that the regularized solution via the proposed approach converges to the exact initial condition g as the noise level $\delta \to 0$. It is evident from Table 1 that the proposed method converges to the exact solution for diverse set of values of the fractional order α. Moreover, since $g \in \mathbf{H}^p(0,l)$ for $p \geq 2$, it follows from Remark 1 that the theoretical optimal convergence rate with the *a priori* rule $\beta = \beta_{pri} = C\delta^{2/3}$ is $O(\delta^{2/3})$, while from Remark 2, the theoretical convergence rate under the *a posteriori* rule (3.14) is $O(\delta^{1/2})$. From Figure 2, we see that the observed order of convergence is consistent with our theory and very close to the theoretical *a priori* and *a posteriori* orders $O(\delta^{2/3})$ and $O(\delta^{1/2})$, respectively. We point out that the value of the constant C in the *a priori* rule $\beta = \beta_{pri} = C\delta^{2/3}$ is taken in such a way the equation $\mathcal{E}(0.01, \beta_{pri}) = \mathcal{E}(0.01, \beta_{pos})$ is satisfied.

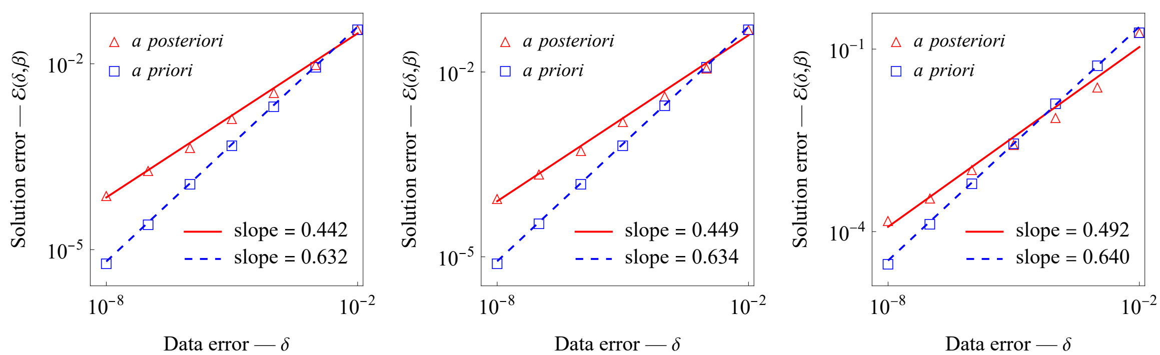
Table 1. Relative errors $\mathcal{E}(\delta, \beta)$ in the computed initial condition for Example 1 using *a priori* and *a posteriori* choice rules of the regularization parameter β for several noise levels δ and fractional orders α.

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
δ	$\mathcal{E}(\delta, \beta_{pri})$	$\mathcal{E}(\delta, \beta_{pos})$	$\mathcal{E}(\delta, \beta_{pri})$	$\mathcal{E}(\delta, \beta_{pos})$	$\mathcal{E}(\delta, \beta_{pri})$	$\mathcal{E}(\delta, \beta_{pos})$
0.01	0.0382	0.0382	0.0518	0.0518	0.1986	0.1986
0.001	0.0094	0.0102	0.0128	0.0123	0.0571	0.0254
0.0001	0.0021	0.0036	0.0030	0.0042	0.0134	0.0079



(A) $\delta = 0.01$ (B) $\delta = 0.001$ (C) $\delta = 0.0001$

FIGURE 1. Exact and recovered initial condition for Example 1 with fractional order $\alpha = 0.5$ for several noise levels δ using *a posteriori* choice rule.



(A) $\alpha = 0.1$ (B) $\alpha = 0.5$ (C) $\alpha = 0.9$

FIGURE 2. Solution error versus data error in Example 1 using *a priori* and *a posteriori* choice rules along with slope of linear fit. Plots are in log-log scale.

4.2. **Example 2.** In this example, we consider the fractional diffusion equation

$$\begin{cases} D_t^{\alpha} u(x,t) = u_{xx}(x,t), & 0 < x < 2 \ \ 0 < t < 1, \\ u(0,t) = u(2,t) = 0, & 0 < t < 1, \\ u(x,0) = g(x), & 0 < x < 2. \end{cases}$$

with

$$g(x) = \begin{cases} 5x, & 0 \le x \le 1, \\ 10 - 5x, & 1 < x \le 2. \end{cases}$$

The forward solution to this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ g_n E_\alpha(-(n\pi/2)^2 t^\alpha) \right\} \sin(n\pi x/2), \quad g_n = \frac{40\sin(n\pi/2)}{\pi^2 n^2}.$$

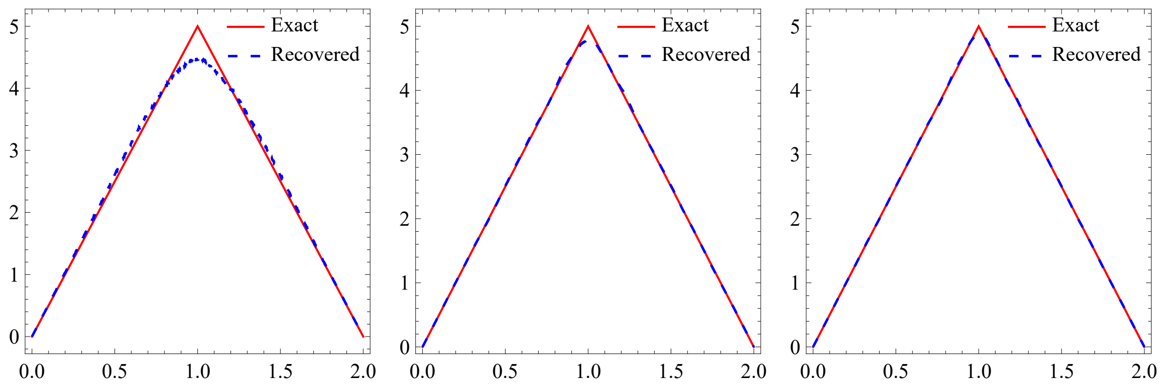
Error results for several noise levels δ and orders α using *a priori* (β_{pri}) and *a posteriori* (β_{pos}) choice rules of β are summarized in Table 2. Plots for the exact and recovered initial condition for different noise levels when $\alpha = 0.5$ are shown in Figure 3. Plots for the solution error versus data error using *a priori* and *a posteriori* choice rules are depicted in Figure 4.

From Figure 3, we see that the regularized solution via the proposed approach converges to the exact initial condition g as the noise level $\delta \to 0$. It is evident from Table 2 that the proposed method converges to the exact solution for diverse set of values of the fractional order α. Moreover, since $g \in \mathbf{H}^p(0,l)$ for all $p < 3/4$, it follows from Remark 1 that the theoretical optimal convergence rate with the *a priori* rule $\beta = \beta_{pri} = C\delta^{8/7}$ is at best $O(\delta^{3/7})$, while from Remark 2, the theoretical convergence rate under the *a posteriori* rule (3.14) is also at best $O(\delta^{3/7})$. From Figure 4, we see that the observed order of convergence is consistent with our theory and very close to the theoretical *a priori* and *a posteriori* order $O(\delta^{3/7})$. We point out that the value of the constant C in the *a priori* rule $\beta = \beta_{pri} = C\delta^{8/7}$ is taken in such a way the equation $\mathcal{E}(0.01, \beta_{pri}) = \mathcal{E}(0.01, \beta_{pos})$ is satisfied.

TABLE 2. Relative errors $\mathcal{E}(\delta,\beta)$ in the computed initial condition for Example 2 using *a priori* and *a posteriori* choice rules of the regularization parameter β for several noise levels δ and fractional orders α.

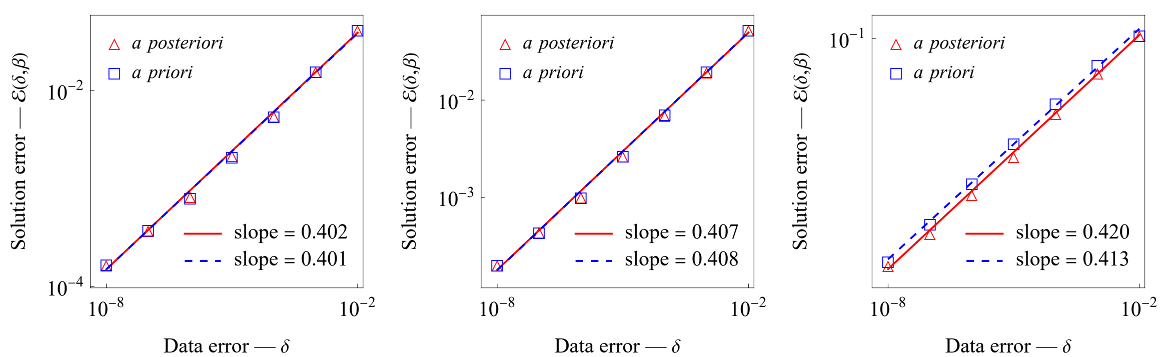
	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
δ	$\mathcal{E}(\delta,\beta_{pri})$	$\mathcal{E}(\delta,\beta_{pos})$	$\mathcal{E}(\delta,\beta_{pri})$	$\mathcal{E}(\delta,\beta_{pos})$	$\mathcal{E}(\delta,\beta_{pri})$	$\mathcal{E}(\delta,\beta_{pos})$
0.01	0.0424	0.0424	0.0540	0.0540	0.1104	0.1104
0.001	0.0159	0.0161	0.0202	0.0201	0.0532	0.0436
0.0001	0.0055	0.0056	0.0072	0.0071	0.0205	0.0159

The results of the last two examples show that the method is stable relative to noise δ and order α. Moreover, the observed convergence rates coincides with our



(A) $\delta = 0.01$ (B) $\delta = 0.001$ (C) $\delta = 0.0001$

FIGURE 3. Exact and recovered initial condition for Example 2 with fractional order $\alpha = 0.5$ for several noise levels δ using *a posteriori* choice rule.



(A) $\alpha = 0.1$ (B) $\alpha = 0.5$ (C) $\alpha = 0.9$

FIGURE 4. Solution error versus data error in Example 2 using *a priori* and *a posteriori* choice rules along with slope of linear fit. Plots are in log-log scale.

theoretical analysis. The convergence rate in the second example deteriorates since the initial condition for the first example is smoother than the initial condition in the second example; this has been predicted by our theoretical results.

5. CONCLUSION

We considered a new regularization scheme based on the homotopy analysis method to solve the backward problem of identifying the initial data for a one-dimensional nonhomogeneous time-fractional diffusion equation from noisy final data. The proposed method allows to tackle not only homogeneous problems, but also nonhomogeneous problems, which is a shortcoming of most existing methods. Moreover, we proved the consistency and stability of the proposed method, and most importantly, we gave optimal convergence rates under both *a priori* and *a posteriori* parameter choice rules. Numerical realization is also given to elucidate and validate the proposed method. Numerical experiments showed noteworthy results.

The results of the numerical examples are in excellent agreement with theoretical ones. Moreover, the examples show that the method is robust with respect to the order of fractional derivative and noise level. The provided analysis can be carried out to problems in higher-spatial domains, which will be our emphasis in a future work.

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(1) Department of Mathematics and Natural Sciences, College of Sciences and Human Studies, Prince Mohammad Bin Fahd University, Al Khobar 31952, Saudi Arabia.

Email address: maljamal@pmu.edu.sa

(2) Department of Mathematics, Faculty of Science, Yarmouk University, Irbid 21163, Jordan.

Email address: mfaljamal@yu.edu.jo