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ON HEREDITARY, SEMIHEREDITARY AND QUASI-HEREDITARY TERNARY RINGS

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Abstract. In this paper, we define right hereditary, semihereditary and quasi-hereditary ternary rings, as previously introduced in binary rings. We show that, if a ternary ring T is completely reducible, then every right ideal I of T is of the form $I = e.1.T$, where e is an idempotent. Consequently, T is a right hereditary ternary ring. We also prove that, if a reduced ternary ring T satisfies the minimal condition on right annihilators of idempotents, and if $I \neq 0$ is a right ideal satisfying the condition: (for every right ideal $K \subset I$, there exists a minimal right ideal H of T such that $H \subset K$), then I is projective as a right T–module. Finally, we show that if T is a semiprimary ternary ring in which the Jacobson radical $= \{0\}$, then T is a right hereditary and quasi-hereditary ternary ring.

1. Introduction

The concept of the ternary rings has been produced by W. G. Lister in 1971, where some special elements, ideals and regularity of these rings were also presented [8]. Some earlier work on ternary rings may be found in [1], [2], [3], [4] and [6]. In this paper we introduce the notion of right hereditary, right semi-hereditary and

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quasi-hereditary ternary rings. Also, we obtain some properties of right hereditary, right semi-hereditary and quasi-hereditary ternary rings. The most important results are Proposition 3.4 and Proposition 3.7. We studied right hereditary, semi-hereditary and quasi-hereditary ternary rings taking advantage of what was previously studied in binary rings. For further studies about right hereditary, semihereditary and quasi-hereditary rings, we may refer to the references [5] and [7]. All ternary rings will be associative with identity. A ternary ring T is called right hereditary ternary ring if every right ideal is projective as a right T -module. Any completely reducible ternary ring is an example of right hereditary ternary ring [2]. In this paper we proved that, if T is a reduced and a right artinian ternary ring, then T is a right hereditary ternary ring. We also proved that, If T is a strongly clean ternary ring, then T is a right semihereditary ternary ring. Then we proved that, if T is a semiprimary and a semiprime ternary ring, then T is a right hereditary and a quasi-hereditary ternary ring. Finally, we show that, if T is a quasi-hereditary ternary ring, then $T/RadT$ is a right hereditary and quasi-hereditary ternary ring.

2. Basic Properties

Definition 2.1. [1] A nonempty set T is called a ternary ring if T is an additive commutative group satisfying the following properties:

$(i)(a.b.c).d.e = a.(b.c.d).e = a.b.(c.d.e)$

$(ii)(a + b).c.d = a.c.d + b.c.d$

(iii) a.(b+c).d=a.b.d+a.c.d

(iv)a.b.(c+d)=a.b.c+a.b.d

for all $a, b, c, d, e \in T$

Definition 2.2. [1] An additive subgroup A of a ternary ring T is called a ternary subring of T if $a.b.c \in A$ for all $a, b, c \in A$.

Definition 2.3. [1]A ternary ring T is called commutative if:
$a.b.c = a.c.b = b.a.c = b.c.a = c.a.b = c.b.a$ for all $a, b, c \in T$

Definition 2.4. [1] In a ternary ring T. If there exists an element $e \in T$ such that: $e.e.x = e.x.e = x.e.e = x$ for all $x \in T$, then e is called an identity element of T.

Definition 2.5. [1] Let T be a ternary ring with identity 1. An element x of a ternary ring T is said to be a right (left, lateral) invertible element in T if there exist y, z in T such that $x.y.z = 1$ (respectively $y.z.x = 1, y.x.z = 1$). If x is a left, a right and a lateral invertible element of T, then x is called an invertible element of T. The set of all invertible elements in T will be denoted by U_T.

Definition 2.6. [1] An element x of a ternary ring T is called idempotent, if $x^3 = x$. The set of all idempotent elements in T will be denoted by Id_T

Definition 2.7. [1] An element x of a ternary ring T is called nilpotent, if $x^n = 0$ for some odd positive integer n.
The set of all nilpotent elements in T will be denoted by N_T.

Definition 2.8. [1] An additive subgroup A of a ternary ring T is called a left (right, lateral) ideal of T if $b.c.a \in A, \forall a \in A$ and $b, c \in T$(respectively $a.b.c \in A, b.a.c \in A, \forall a \in A$ and $b, c \in T$). If A is a left, a right and a lateral ideal of T, then A is called an ideal of T.
If A is a left and right ideal of T then A is called a two sided ideal of T.

Definition 2.9. [6] Let T be a ternary ring. We say that T is left (right, lateral) artinian (Noetherian), if T satisfies the descending chain condition (DCC) (ascending chain condition (ACC)) on left (respectively, right lateral) ideals. That is, for any chain of left (respectively, right lateral) ideals $I_1 \supseteq I_2 \supseteq ..., (I_1 \subseteq I_2 \subseteq ...$ there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = ...$

A ternary ring T is an artinian (Noetherian) ternary ring, if T is a left, right and lateral artinian (Noetherian).
A ternary ring T is a two-sided artinian (Noetherian), if T is both a left and right artinian (Noetherian).

Definition 2.10. [3] A module P is called projective if given any epimorphism $f : M \to N$ and any homomorphism $g : P \to N$ there exists a homomorphism $h : P \to M$ with $f \circ h = g$.

Definition 2.11. [3] A ternary ring T is called a right hereditary ternary ring if every right ideal of T is projective as a right T-module.

Definition 2.12. [3] A ternary ring T is called a right semihereditary ternary ring if every right finitely generated ideal of T is projective as a right T-module.

Definition 2.13. [6] A ternary ring T is called a completely reducible ternary ring if T can be written as a sum of all right minimal ideals of T.

Lemma 2.1. *[6] A ternary ring T is completely reducible if and only if there exists $n \in \mathbb{N}$ such that $T = \oplus_{i=1}^{n} m_i$, where $m_i (i = 1, 2, ...n)$ is a minimal right ideal of T.*

Definition 2.14. A ternary ring T is called a semiprimary ternary ring if the Jacobson radical (denoted by $RadT$) is a nilpotent ideal and $T/RadT$ is completely reducible.

Definition 2.15. An ideal I of ternary ring T is called hereditary ideal if I is idempotent (i.e. $I^3 = I$), projective as a T-module and satisfies $I.RadT.I = \{0\}$.

Definition 2.16. A semiprimary ternary ring T is called quasi-hereditary ternary ring if there exists a chain of ideals of T : $\{0\} = I_0 \subset I_1 \subset I_2 \subset ... \subset I_n = T$ such that I_m/I_{m-1} is a hereditary ideal of $T/I_{m-1}.\forall m = 1, 2, ..., n$ (such a chain is called a hereditary chain.)

Lemma 2.2. *[3] Every free right T-module is a projective T-module.*

Lemma 2.3. *[3] A right T-module M is projective if, and only if M is a direct summand of free T-module.*

Definition 2.17. A ternary ring T is called a reduced ternary ring if T has no nilpotent elements different from 0 (i.e. $N_T = \{0\}$).

Lemma 2.4. *[2] If T is a ternary ring and e is an idempotent of T , then $e' = 1 - (e.e.1)$ is an idempotent, and $e.e'.1 = e'.e.1 = 0$. Therefore $T = e.T.T \oplus e'.T.T$.*

Lemma 2.5. *[6] Let T be a ternary ring and K be a minimal right ideal of T . Then either $K^3 = \{0\}$ or $K = e.K.K = e.T.T$, where e is an idempotent of K.*

Definition 2.18. [2] A ternary ring T is called a regular ternary ring if for every element x of T there exists y of T such that $x = x.y.x$.

Lemma 2.6. *[2] A ternary ring T is regular if and only if every principal right ideal of T is generated by an idempotent element.*

Lemma 2.7. *[2] If T is a regular ternary ring, then every right ideal of T is an idempotent ideal.*

Lemma 2.8. *[4] If T is a regular ternary ring, then T is a right semihereditary ternary ring*

Definition 2.19. [1,6] A ternary ring T is called a semiprime ternary ring if $\{0\}$ is a right semiprime ideal of T (i.e. if J is a right ideal of T such that $J^3 \subseteq \{0\}$ then $J \subseteq \{0\}$).

Lemma 2.9. *[6] If T is a right semiprime ternary ring, then T doesn't have any nilpotent right ideal different from {0}.*

Lemma 2.10. *[6] If T is a right artinian ternary ring and I is a proper ideal of T, then T/I is a right artinian ternary ring.*

Lemma 2.11. *[3,6] If T is a ternary ring, then the following statements are equivalent: 1) T is a completely reducible ternary ring.*
2) The lattice of right ideals of T is complement.
3) T is a right artinian and a regular ternary ring.
4) T is a right artinian and a J-semisimple ternary ring (i.e. Jacobson radical=$\{0\}$).
5) T is a right artinian and a semiprime ternary ring.
6) T is a right Noetherian and a regular ternary ring
7) Every right ideal of T is injective.
8) Every right module over T is injective.
9) Every exact sequence of the form $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ of right T-modules splits.
10) Every right module over T is projective.
11) Every right module over T is completely reducible.
If a ternary ring T satisfies any condition of Lemma 2.11, then T is a right hereditary ternary ring.

Definition 2.20. [6] A ternary ring T is called a PrP - ternary ring if every principal right ideal of T is projective as a T-module.

Definition 2.21. [6] A ternary ring T is called a zc-ternary ring if T satisfies: $1.x.y = 0 \Longrightarrow 1.y.x = 0, \forall x, y \in T$

Definition 2.22. [6] A ternary ring T is called a zi-ternary ring if T satisfies: $1.x.y = 0 \Longrightarrow x.z.y = 0, \forall x, y, z \in T$

Definition 2.23. [6] An element x of a ternary ring T is called a semicentral element if: $x.x.y = y.x.x, \forall y \in T$

Lemma 2.12. *[6] If T is a right Noetherian and a PrP-ternary ring, then the following conditions are equivalent: 1) T is a zc-ternary ring.*
2) T is a zi-ternary ring.
3) the idempotents of T are semicentral.

Definition 2.24. A ternary ring T is called a strongly clean ternary ring if for every element x of T there exists an idempotent e and an invertible element u of T such that: $x = u.e.e = e.u.e = e.e.u$.

3. Main Result

In the following Proposition we will give the necessary conditions for the ternary ring to be a Noetherian ternary ring or a Prp-ternary ring.

Proposition 3.1. *Let T be a ternary ring.*
1) If for every element x of T, there exists $n \in \mathbb{N}$ such that $x = n.1.e$ where e is an idempotent in T, then T is a PrP-ternary ring.
2) If for every sequence $a_1, a_2, ..., a_n$ of elements of T there exists $s \in \mathbb{N}$, where $1 \leq s < n$ and $t_1, t_2, ..., t_s$ of T such that $a_{s+1} = a_1.t_1.1 + a_2.t_2.1 + ... + a_s.t_s.1$ then T is a right Noetherian ternary ring.

Proof. 1) Let x be an element of T. Then by hypothesis, x is of the form $x = n.1.e$, where $n \in \mathbb{N}$ and $e^3 = e$. Hence, $T \subseteq n.1.T$. But $n.1.T \subseteq T$, so $T = n.1.T$. Consequently, the right principal ideal $x.T.T$ is a direct summand of T because we have $T = e.T.T \oplus e^{'}.T.T$ by Lemma 2.4. So

$$T = n.1.T = n.1.e.T.T \oplus n.1.e^{'}.T.T = x.T.T \oplus n.1.e^{'}.T.T$$

Thus, $x.T.T$ is projective by Lemma 2.3. So T is a PrP-ternary ring.
2) Suppose that T is not right Noetherian then there exists an ascending chain of right ideals $I_1 \subseteq I_2 \subseteq ...$ such that $I_n \subsetneqq I_{n+1}$ for all $n \in \mathbb{N}$. Let us take the

sequence $a_1, a_2, ...a_n$ of elements of T such that $a_i \in I_i$, $\forall i = 1, 2, ...n$ and $a_i \notin I_{i-1}$, $\forall i = 2, .., n$. Then by hypothesis, there exist $1 \leq s < n$ and $t_1, t_2, ...t_s$ of T such that $a_{s+1} = a_1.t_1.1 + a_2.t_2.1 + ... + a_s.t_s.1$ Therefore $a_{s+1} \in I_s$. This is a contradiction, and hence T is a right Noetherian ternary ring. □

Corollary 3.1. *If T is a ternary ring satisfying the two conditions of Proposition 3.1. Then the following statements are equivalent:*
1) T is a zc-ternary ring
2) T is a zi-ternary ring
3) The idempotents of T are semicentral

Proof. It follows by Lemma 2.12 □

Corollary 3.2. *Let T be a regular ternary ring and suppose that for every sequence $a_1, a_2, ...a_n$ of elements of T, there exists $s \in \mathbb{N}$, where $1 \leq s < n$ and $t_1, t_2, ..., t_s$ of T such that $a_{s+1} = a_1.t_1.1 + a_2.t_2.1 + ... + a_s.t_s.1$ Then the following statements are equivalent:*
1) T is a zc-ternary ring
2) T is a zi-ternary ring
3) The idempotents of T are semicentral

Proof. From (2) of Proposition 3.1, T is a right Noetherian ternary ring. Since T is a regular ternary ring, T is a right Noetherian and a PrP-ternary ring by Lemma 2.11. Now the result follows from Lemma 2.12. □

In Propositions 3.2 and 3.3 we will give the necessary conditions for the ternary ring to be a completely reducible ternary ring.

Proposition 3.2. *For any ternary ring T, the following two statements are equivalent:*

1) T is a completely reducible ternary ring.

2) Every right ideal I of T is of the form $I = i.1.T$, where i is an idempotent.

Proof. ($1 \longrightarrow 2$): Assume that T is completely reducible. Let I be a right ideal of T.Then I is a direct summand of T by Lemma 2.11. So there exists a right ideal J such that $T = I \oplus J$. Consequently, $1 = i+j$ for some $i \in I$ and $j \in J$. Therefore, $i = 1.1.i = (i+j).1.i = i.1.i + j.1.i$, where $i.1.i \in I$ and $j.1.i \in J$ .So $j.1.i \in I \cap J = \{0\}$. Thus, $i = i.1.i = 1.i.i = (i+j).i.i = i^3 + j.i.i$ so $j.i.i \in I \cap J = \{0\}$. Finally,$i = i^3$. Also $i.1.T \subseteq I$ and for any $a \in I$, $a = 1.1.a = (i+j).1.a = i.1.a + j.1.a$, where $j.1.a \in I \cap J = \{0\}$. Therefore, $a = i.1.a \in i.1.T$. Consequently, $I = i.1.T; i^3 = i$
($2 \longrightarrow 1$): Let I be a right ideal of T. Then $I = e.1.T$ where e is an idempotent. So, $e^{'} = 1 - (e.e.1)$ is an idempotent by Lemma 2.4. Also $J = e^{'}.1.T$ is a right ideal of T and $T = I \oplus J$ Hence, the lattice of right ideals of T is complement. So T is a completely reducible ternary ring by Lemma 2.11. □

Corollary 3.3. *If a ternary ring T satisfies any condition of Proposition 3.2 or Lemma 2.11, then T is a right hereditary ternary ring in which every right ideal is generated by an idempotent.*

Proof. It follows by Lemma 2.11. □

Proposition 3.3. *If T is a completely reducible ternary ring, then there exist non trivial idempotents $e_1, e_2, ..., e_n \in T$ ($e_t \neq 0, \forall t = 1, 2, ...n$) such that:*

i) $1 = e_1 + e_2 + ... + e_n$

ii) $e_1, e_2, ..., e_n$ are orthogonal idempotents (i.e. if $\forall i, j \in 1, 2, ...n$ and $i \neq j$, then $e_i.1.e_j = e_j.1.e_i = 0$.

iii) $e_t (\forall t = 1, 2, ...n)$ can't be written as a sum of two idempotents e_r and e_s, where $r, s \in 1, 2, ..., n$ and $e_r \neq 0$ and $e_s \neq 0$

Proof. Since the completely reducible ternary ring T can be written as a finite direct sum of minimal right ideals, i.e. $T = \oplus_{i=1}^{n} L_i$, where $L_i (i = 1, \dots, n)$ is a minimal right ideal of T by Lemma 2.1, then, there exists $e_i \in L_i (i = 1, ..., n)$ such that $1 = e_1+e_2+...+e_n$. Thus, $e_i = 1.e_i.1 = (e_1+e_2+...+e_n).e_i.1 = e_1.e_i.1+e_2.e_i.1+...+e_n.e_i.1$ but

$$e_i-(e_i.e_i.1) = e_1.e_i.1 + ... + e_{i-1}.e_i.1 + e_{i+1}.e_i.1 + ... + e_n.e_i.1 \in L_i \cap \oplus_{j=1,j\neq i}^{n} L_j = \{0\}$$

So, $e_i = e_i.e_i.1 = 1.e_i.e_i = (e_1 + e_2 + ... + e_n).e_i.e_i = e_i.e_i.e_i = e_i^3$. Hence, e_i is an idempotent. Also, for every $e_i \in L_i$ and $e_j \in L_j (i, j = 1, 2, ..., n)$ and $i \neq j$ We have

$$e_i.e_j.1 = (0, ..., 0, e_i, 0, ..., 0).(0, ..., 0, e_j, 0, ..., 0).(1, ..., 1) = (0, ..., 0)$$

Assume that one of the idempotents, say, $e_t (t \in \{1, 2, ...n\})$ can be written as a sum of two idempotents e_r and e_s , where $r, s \in \{1, 2, ..., n\}$. Then for every element x of L_t we have $x = 1.1.x = (e_1 + e_2 + ... + e_n).1.x = e_1.1.x + ... + e_t.1.x + ... + e_n.1.x$ So,

$$x - e_t.1.x \in L_t \cap (L_1 \oplus ... \oplus L_{t-1} \oplus L_{t+1} \oplus ... \oplus L_n) = L_t \cap \oplus_{j=1,j\neq t}^{n} L_j = \{0\}$$

therefore $x = e_t.1.x \in e_t.1.T$. It follows that $L_t \subseteq e_t.1.T$. So,$L_t = e_t.1.T$. For every element x of $(e_r + e_s).1.T$ we have:

$$x = (e_r + e_s).1.t = e_r.1.t + e_s.1.t \in e_r.1.T + e_s.1.T$$

Also, for every element x of $e_r.1.T + e_s.1.T$, there exist $t_r, t_s \in T$ such that
$x = e_r.1.t_r + e_s.1.t_s = (e_r)^3.1.t_r + (e_s.1.e_r).e_r.t_r + (e_s)^3.1.t_s + (e_r.1.e_s).e_s.t_s$
$= (e_r + e_s).1.(e_r.e_r.t_r) + (e_r + e_s).1.(e_s.e_s.t_s)$
$= (e_r + e_s).1.[e_r.e_r.t_r + e_s.e_s.t_s] \in (e_r + e_s).1.T.$
So, $L_t = e_t.1.T = (e_r + e_s).1.T = e_r.1.T + e_s.1.T$. But $e_r.1.T$ and $e_s.1.T$ are right ideals of T, so L_t is not minimal. This is a contradiction. □

In the following Proposition we will give the necessary conditions for a right ideal of a ternary ring T to be projective as a right T-module.

Proposition 3.4. *Let T be a reduced ternary ring which satisfies the minimal condition on right annihilators of idempotents and let $I \neq \{0\}$ be a right ideal satisfying the condition (for every right ideal $L \subset I$, there exists a minimal right ideal K of T such that $K \subset L$) Then I is projective as a right T–module.*

Proof. If I is a minimal right ideal of T , then $I^3 \neq \{0\}$ because T is a reduced ternary ring. Thus , there exists an idempotent e^* of I such that $I = e^*.T.T$ by Lemma 2.5. Thus I is a free right T-module. So, I is projective by Lemma 2.2.

If I is not a minimal right ideal, then by hypothesis, there exists a minimal right ideal K such that $K^3 \neq \{0\}$ and $K \subset I$. Accordingly, there exists an idempotent $e^{'}$ such that $K = e^{'}.T.T \subset I$ by Lemma 2.5. Also, it can be easily proved that $A = r.ann_I(e^{'}) = \{t \in I; e^{'}.1.t = 0\}$ is a right ideal of T. Let H=$\{r.ann_{_I}(e); e^3 = e \neq 0$ and $e.T.T \subset I\}$. Then, $A \in H$ and since $A = r.ann_I(e^{'}), (e^{'})^3 = (e^{'})$ and $(e^{'}).T.T \subset I$,we have $H \neq \phi$. So, by hypothesis, H has a minimal element E. So, $E \subset I$ and there exists an idempotent e of T such that $E = r.ann_I(e); e^3 = e \neq 0$ and $e.T.T \subset I$ Suppose $E \neq \{0\}$. Then by hypothesis , there exists a minimal right ideal F of T such that $f^3 = f \neq 0; \{0\} \neq F = f.T.T \subset I$ by Lemma 2.5. Thus $0 \neq f \in f.T.T \subseteq r.ann_I(e) = E$, so $e.1.f = 0$. Therefore $f.(e.1.f).e = 0$, hence $e.(f.e.1.f.e).1 = 0$. Thus, $f.(e.f.e.1.f.e.1).1 = 0$. So, $f.e.1.f.e.1.f.e.1 = (f.e.1)^3 = 0$. Hence, $f.e.1 = 0$. since T is a reduced ternary ring. Let $g = e + f$. Then

$g^3 = (e + f).(e + f).(e + f) = e^3 + e.e.1.f.1 + e.1.f.1.1.e + e.1.f.1.f + f.e.1.1.1.e$
$+f.e.1.f.1+f.f.e.1.1+f^3 = e+0+0+0+0+0+0+f = g$ i.e. g is an idempotant. Also, $e.e.g = e.e.(e + f) = e.e.e + e.e.f = e + e.1.e.1.f = e + e.1.0 = e \neq 0$. So, $e.e.f = 0$ and $g.T.T = (e + f).T.T = e.T.T + f.T.T \subset I$. It follows that ,$r.ann_I(g) \in H$. Also, for every element x of $r.ann_I(g)$ we have $e.e.g.1.x = 0$. Thus, $e.1.x = 0$. Hence $x \in r.ann_I(e)$. Therefore $r.ann_I(g) \subseteq r.ann_I(e) = E$. Since E is minimal element of $H, r.ann_I(g) = E$.

Since, $e.1.f = 0$. So $f \in r.ann_I(e)$ but $g.1.f = e.1.f + f.1.f = f.1.f \neq 0$ because if $f.1.f = 0$ then $f.1.f.1.f = 0$. Therefore $f^3 = f = 0$, this is a contradiction. So $f \notin r.ann_I(g)$. Thus, $r.ann_I(g) \neq r.ann_I(e)$. This is a contradiction. So $E = \{0\}$. Also, for every element x of I we have $(e.e.e).1.x = e.1.x$,so $e.1.e.e.x = e.1.x = 0$. Hence $e.1.(e.e.x - x) = 0$. Thus $e.e.x - x \in E = \{0\}$, therefore $x = e.e.x \in e.T.T$ so $I \subseteq e.T.T$. But $e.T.T \subseteq I$, so $I = e.T.T$, i.e. I is generated by the idempotant e. Thus I is a free T-module. So,I is a projective right module by Lemma 2.2. Finally, I is a projective right ideal. □

Corollary 3.4. *If T is a reduced and a right artinian ternary ring, then T is a right hereditary ternary ring.*

Proposition 3.5. *If T is a strongly clean ternary ring, then T is a right semihereditary ternary ring.*

Proof. Let x be an element of T. Then, x is of the form $x = u.e.e = e.e.u = e.u.e$, where e is an idempotent and u is an invertible element of T (i.e. there exist $u_1, u_2 \in T$, where $u.u_1.u_2 = 1$). Therefore, $I = x.T.T$ is a principal right ideal which satisfies: $I = x.T.T = e.e.u.T.T \subseteq e.T.T = e.1.T$, so $I \subseteq e.1.T$. Also, for every element f of $e.T.T = e.1.T$ there exists $t \in T$ such that $f = e.1.t$, so $f = e.1.t = e.e.e.1.t = e.e.1.e.t = e.e.u.u_1.u_2.e.t = x.u_1.u_2.e.t \in x.T.T = I$ So, e.1.$T \subseteq I.Consequently$,I=e.1.T=e.T.T. Accordingly, every principal right ideal of T is generated by an idempotent. Therefore, T is a regular ternary ring by Lemma 2.6. Finally, T is a semihereditary ternary ring by Lemma 2.8. □

Proposition 3.6. *If T is a ternary ring such that $T = U_T \cup Id_T$, where U_T is the set of invertible elements of T and Id_T is the set of idempotents elements of T, then T is a semihereditary ternary ring.*

Proof. Since trivially every invertible element or an idempotent element is a regular element,$T = U_T \cup Id_T$ is a regular ternary ring. Thus T is a semihereditary ternary ring by Lemma 2.8. □

If T is a ternary ring such that $T = U_T \cup Id_T$ and Id_T is finite, then T is a hereditary ternary ring.

Proposition 3.7. *: Let T be a semiprimary ternary ring in which $RadT = \{0\}$. Then T is a right hereditary and quasi-hereditary ternary ring.*

Proof. Since $RadT = \{0\}$, $T/RadT = T$ is completely reducible (Definition 2.14), T is a right hereditary by Lemma 2.11. Let $I \neq \{0\}$ be a proper ideal of T will prove that $0 \subset I \subset T$ is a heredity chain Since T is a right hereditary ternary ring, I is a projective right T-module. Also, T is regular. So, I is an idempotent ideal by Lemma 2.7, and: $I/\{0\}.RadT/\{0\}.I/\{0\} = I.RadT.I = I.\{0\}.I = \{0\}$. Thus $I/\{0\} = I$ is hereditary ideal of $T/\{0\} = T$ (Definition 2.15). Since T/I is a unitary ternary ring, T/I is a free T/I – module, hence T/I is a projective $T/I-$ module by Lemma 2.2. Also T/I is an idempotent ideal of T/I and since T is a completely reducible. So T is a right artinian and regular by Lemma 2.11. Therefore, T/I is a right artinian ternary ring by Lemma 2.10, and a regular ternary ring. Thus, T/I is a J-semisimple ternary ring by Lemma 2.11. So $Rad(T/I) = \{0\}$. Consequently, $T/I.Rad(T/I).T/I = \{0\}$. Therefore T/I is a hereditary ideal of T/I . Finally, $T$is a quasi-hereditary ternary ring. □

Corollary 3.5. *If T is a semiprimary and semiprime, then T is a right hereditary and a quasi-hereditary ternary ring.*

Proof. Since T is semiprime, $T\backslash\{0\}$ does not have a nilpotent proper ideal by Lemma 2.9. Also, T is semiprimary, so $RadT$ is a nilpotent ideal by Definition 2.14. Therefore,

$RadT = \{0\}$. Hence, T is a right hereditary and a quasi-hereditary ternary ring by Proposition 3.7. □

Corollary 3.6. *If T is a regular and semiprimary ternary ring, then T is a right hereditary and a quasi-hereditary ternary ring.*

Proof. Since T is regular, $RadT$ is an idempotent ideal by Lemma 2.7. Also, since T is a semiprimary ternary ring, $RadT$ is a nilpotent ideal of T by Definition 2.14. Consequently, $RadT = Rad^kT = \{0\}$. So, T is a right hereditary and a quasi-hereditary ternary ring by Proposition 3.7. □

Proposition 3.8. *If T is a completely reducible ternary ring, then T is a right hereditary and a quasi-hereditary ternary ring.*

Proof. Since T is a completely reducible ternary ring, then T is a J-semisimple ternary ring by Lemma 2.11. Thus, T is a semiprimary by Definition 2.14. So, T is a right hereditary and quasi-hereditary ternary ring by Proposition 3.7. □

Corollary 3.7. *If a ternary ring T satisfies any condition of Proposition 3.2 or Lemma 2.11, then T is a right hereditary and a quasi-hereditary ternary ring.*

Proposition 3.9. *If T is a quasi-hereditary ternary ring, then $T/RadT$ is a right hereditary and a quasi-hereditary ternary ring.*

Proof. since T is a quasi-hereditary ternary ring, then T is a semiprimary ternary ring (Definition 2.16). So $T/RadT$ is a completely reducible ternary ring (Definition 2.14). Then, $T/RadT$ is a right hereditary and quasi-hereditary ternary ring by Proposition 3.8. □

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