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NEW MATRIX INTERPOLATING INEQUALITIES

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ABSTRACT. The main goal of this article is to present new generalizations and new forms of some well known matrix inequalities. These inequalities can be thought of as interpolating inequalities between the arithmetic-geometric mean inequality and Cauchy-Schwarz inequalities, generalizing some recent results in this direction.

1. INTRODUCTION

In the sequel, the algebra of all $n \times n$ complex matrices will be denoted by $\mathbb{M}_n$, the cone of positive semi-definite (or simply positive) matrices will be denoted by $\mathbb{M}_n^+$ while $\mathbb{M}_n^{++}$ will stand for the cone of strictly positive matrices in $\mathbb{M}_n$. The notations $X \geq 0$ or $X > 0$ will be used to mean that $X \in \mathbb{M}_n^+$ or $\mathbb{M}_n^{++}$, respectively.

For $X \in \mathbb{M}_n$, the singular values $\{s_j(X)\}$ are the eigenvalues of $|X| = (X^*X)^{1/2}$ arranged in a decreasing order. The following inequality for singular values is crucial [3]

$$s_j(A^tB^{1-t}) \leq s_j(tA + (1-t)B) \;\; j = 1, 2, ..., n, \tag{1.1}$$

valid for the positive matrices A, B and for $0 \leq t \leq 1$.

Clearly, the inequality (1.1) implies

$$\|A^tB^{1-t}\| \leq \|tA + (1-t)B\|, 0 \leq t \leq 1, \tag{1.2}$$

for any unitraily invariant norm $\| \;\; \|$. We recall here that a unitarily invariant norm $\| \cdot \|$ on $\mathbb{M}_n$ is a matrix norm that satisfies $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$.

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The fact that (1.1) implies (1.2) follows immediately from the Fan dominance theorem, which can be found in [7, Theorem IV.2.2].

On the other hand, the following Hölder inequality was proved in [10] for the positive matrices A, B and any matrix $X \in \mathbb{M}_n$,

$$\|A^t X B^{1-t}\| \leq \|AX\|^t \|XB\|^{1-t}, 0 \leq t \leq 1. \tag{1.3}$$

The inequality (1.3) can be manipulated to show that the function $t \mapsto \|A^t X B^{1-t}\|$ is log-convex on $[0, 1]$, see [5, 12]. But then, the function $t \mapsto \|A^t X B^{1-t}\| \, \|A^{1-t} X B^t\|$ can be shown to be log-convex too, [5, 12]. We should remark here that in [9], it is proved that $t \mapsto \|A^t X B^{1-t}\|$ is convex. However, the proof given there in fact implies log-convexity.

Letting $g(t) = \|(A^*A)^t X (B^*B)^{1-t}\| \, \|(A^*A)^{1-t} X (B^*B)^t\|$, it is proved in [5] that this log-convex function satisfies, for $A, B, X \in \mathbb{M}_n$ and $t \in [0, 1]$,

$$\begin{aligned} ||AXB^*||^2 &= g(1/2) \\ &\leq g(t) \\ &\leq ||tA^*AX + (1-t)XB^*B|| \; ||(1-t)A^*AX + tXB^*B||. \end{aligned} \tag{1.4}$$

So, in particular we have

$$||AXB^*||^2 \leq \|(A^*A)^t X (B^*B)^{1-t}\| \; \|(A^*A)^{1-t} X (B^*B)^t\|. \tag{1.5}$$

This inequality was shown in [2] using a singular value argument. The inequalities (1.4) were aimed to extend some inequalities from [2, 4, 20]. In [4], the inequality

$$\|AB^*\|^2 \leq \|t\, A^*A + (1-t)B^*B\| \; \|(1-t)A^*A + t\, B^*B\|, 0 \leq t \leq 1, \tag{1.6}$$

was proved for $A, B \in \mathbb{M}_n$ and any unitarily invariant norm $\| \quad \|$. The significance of this inequality is the way it interpolates between the Cauchy-Schwarz inequality, when $t = 0$, and the arithmetic-geometric mean inequality when $t = \frac{1}{2}$.
A generalization of (1.6) was given in [20], where the inequality

$$\|AXB^*\|^2 \leq \|t\, A^*AX + (1-t)XB^*B\| \; \|(1-t)A^*AX + t\, XB^*B\|, 0 \leq t \leq 1, \tag{1.7}$$

was proved for $A, B, X \in \mathbb{M}_n$. Therefore, (1.4) provides a refinement of (1.7); by inserting the function g.

In the case of the Hilbert-Schmidt norm, the inequality (1.7) was refined in [2] as follows.

$$\|AXB^*\|_2^2 \leq \left(\|t\, A^*AX + (1-t)XB^*B\|_2^2 - r_0^2\|AX - XB\|_2^2\right)^{\frac{1}{2}}$$
$$\times \left(\|(1-t)\, A^*AX + t\, XB^*B\|_2^2 - r_0^2\|AX - XB\|_2^2\right)^{\frac{1}{2}}, 0 \leq t \leq 1, \tag{1.8}$$

for $A, B, X \in \mathbb{M}_n$ and $r_0 = \min\{t, 1-t\}$.

The idea of log-convexity will be used to present a reverse of (1.8). See section 2.2 below.

The following lemma will be needed in our work [19].

Lemma 1.1. *Let $A, B, X \in \mathbb{M}_n$ such that $A, B > 0$. For $r \geq 0$, define the function*

$$f(t) = ||\ |A^tXB^{1-t}|^r\ ||\ ||\ |A^{1-t}XB^t|^r\ ||.$$

Then f is log-convex on the interval $[0,1]$. Moreover, since f is symmetric about $t = \frac{1}{2}$, it is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and attains its minimum at $1/2$.

Another inequality that we will need in our proofs is the following inequality from [6]

$$\left\|X^{1/2}(X+Y)Y^{1/2}\right\| \leq \frac{1}{2}\left\|(X+Y)^2\right\|, X, Y \in \mathbb{M}_n^+. \tag{1.9}$$

In this article, we present new forms that generalize some of the above inequalities. For example, we prove that

$$||\ |A^{1/2}B^{1/2}|^r\ ||^2 \leq ||\ |A^tB^{1-t}|^r\ ||\ ||\ |A^{1-t}B^t|^r\ ||$$
$$\leq ||(tA + (1-t)B)^r||\ ||((1-t)A + tB)^r||,$$

for positive A, B, $0 \leq t \leq 1$ and $r > 0$. This provides a generalization of (1.6) for positive matrices. Moreover, this will be a generalization of the Cauchy-Schwarz inequality and a well known arithmetic-geometric mean inequality. See Theorem 2.1

below and the comments following it.

Moreover, we prove the other generalization

$$||AB||^2 \leq g(t)$$
$$\leq \frac{1}{4t(1-t)} || (tA + (1-t)B)^2 || \, || ((1-t)A + tB)^2 ||,$$

valid for the positive matrices A, B and $0 \leq t \leq 1$, and for some log-convex function g, refining the corresponding result from [20].

Further, the Hilbert-Schmidt norm inequality (1.8) will be manipulated to get a refinement of the inequality [6]

$$4\|AB\|_2 \leq \|(A+B)^2\|_2.$$

In the end, we present a refinement and a reverse of (1.8).

We refer the reader to [1, 2, 4, 5, 8, 11, 12, 13, 14, 20, 15, 16, 17, 18] as a list of references that treat matrix inequalities, where refinements, reverses and interpolation are emphasized.

2. Main Results

We present our results in two parts, where general unitarily invariant norms are considered first. Then the Hilbert-Schmidt norm will be discussed, where some new results will be found.

2.1. General Unitarily Invariant Norms.

In this part we present some new results that can be thought of as interpolating inequalities between various inequalities. We begin with a simple lemma.

Lemma 2.1. *Let $A, B \geq 0$ and $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function. Then*

$$\|\varphi(|A^t B^{1-t}|)\| \leq \|\varphi\,(t\, A + (1-t)B)\,\|,$$

for any unitarily invariant norm $\| \ \|$.

In particular, for any $r \geq 0$,

$$|| \, |A^t B^{1-t}|^r \, || \leq ||(tA + (1-t)B)^r||. \tag{2.1}$$

Proof. Since φ is increasing, it follows for each $1 \leq j \leq n$,

$$\begin{aligned} s_j(\varphi\left(|A^tB^{1-t}|\right)) &= \varphi\left(s_j(|A^tB^{1-t}|)\right) \\ &= \varphi\left(s_j(A^tB^{1-t})\right) \\ &\leq \varphi\left(s_j(tA+(1-t)B)\right) \\ &= s_j(\varphi(tA+(1-t)B)). \end{aligned}$$

This implies

$$\|\varphi(|A^tB^{1-t}|)\| \leq \|\varphi\left(t\ A+(1-t)B\right)\|.$$

This completes the proof of the first inequality. For the second one, let $\varphi(x)=x^r, r>0$, then (2.1) follows.

□

Combining Lemmas 2.1 and 1.1 (with $X=I$) implies the following.

Theorem 2.1. *Let $A, B \in \mathbb{M}_n^+$ and $r \geq 0$. Then*

$$\begin{aligned} \|\, |A^{1/2}B^{1/2}|^r\, \|^2 &\leq \|\, |A^tB^{1-t}|^r\, \|\ \|\, |A^{1-t}B^t|^r\, \| \\ &\leq \|(tA+(1-t)B)^r\|\ \|((1-t)A+tB)^r\|. \end{aligned}$$

Proof. Let

$$g_r(t) = \|\, |A^tB^{1-t}|^r\, \|\ \|\, |A^{1-t}B^t|^r\, \|.$$

Then g is log-convex on the interval $[0,1]$, decreasing on $[0,1/2]$, increasing on $[1/2,1]$ and attains its minimum at $1/2$. This together with Lemma 2.1 imply the result. □

In Theorem 2.1, if $t=0$ we have Cauchy-Schwarz inequality for positive matrices

$$\|\, |A^{1/2}B^{1/2}|^r\, \|^2 \leq \|A^r\|\ \|B^r\|.$$

On the other hand, if $t=1/2$ we have a generalization of the arithmetic-geometric mean inequality for positive matrices,

$$\|\, |A^{1/2}B^{1/2}|^r\, \| \leq \frac{1}{2^r}\|(A+B)^r\|.$$

Notice that the result is still valid for any increasing function φ to have

$$\|\varphi\left(|A^{1/2}B^{1/2}|\right)\| \leq \left\|\varphi\left(\frac{A+B}{2}\right)\right\|.$$

Theorem 2.2. *Let $A, B \in \mathbb{M}_n^+$. Then a log-convex function g exists, such that, for $0 \leq t \leq 1$,*

$$\begin{aligned}||AB||^2 &= g\left(\frac{1}{2}\right)\\ &\leq g(t)\\ &\leq \frac{1}{4t(1-t)}||\left(tA+(1-t)B\right)^2||\;||\left((1-t)A+tB\right)^2||\\ &\leq \frac{1}{4t(1-t)}||tA^2+(1-t)B^2||\;||(1-t)A^2+tB^2||.\end{aligned}$$

Proof. Let g be as in [5], which satisfies (1.4), with $X = A^{1/2}B^{1/2}$. Then

$$\begin{aligned}||AB||^2 &= g(1/2)\\ &\leq g(t)\\ &\leq ||tA^{3/2}B^{1/2}+(1-t)A^{1/2}B^{3/2}||\;||(1-t)A^{3/2}B^{1/2}+tA^{1/2}B^{3/2}||\end{aligned}$$

Now, noting (1.9),

$$\begin{aligned}||tA^{3/2}B^{1/2}+(1-t)A^{1/2}B^{3/2}|| &= ||A^{1/2}(tA+(1-t)B)B^{1/2}||\\ &= \frac{1}{\sqrt{t(1-t)}}||(tA)^{1/2}\left(tA+(1-t)B\right)\left((1-t)B\right)^{1/2}||\\ &\leq \frac{1}{2\sqrt{t(1-t)}}||\left(tA+(1-t)B\right)^2||.\end{aligned}$$

Similarly,

$$||(1-t)A^{3/2}B^{1/2}+tA^{1/2}B^{3/2}|| \leq \frac{1}{2\sqrt{t(1-t)}}||\left((1-t)A+tB\right)^2||.$$

Hence,

$$\begin{aligned}||AB||^2 &= g(1/2)\\ &\leq g(t)\\ &\leq \frac{1}{4t(1-t)}||\left(tA+(1-t)B\right)^2||\;||\left((1-t)A+tB\right)^2||.\\ &\leq \frac{1}{4t(1-t)}||tA^2+(1-t)B^2||\;||(1-t)A^2+tB^2||,\end{aligned}$$

where we have used the fact that the function $f(x) = x^2$ is operator convex to obtain the last inequality. This completes the proof. $\square$

In particular, for $t = 1/2$ we have

$$||AB|| \leq \frac{1}{4}||\,(A+B)^2\,||, \tag{2.2}$$

which is a celebrated result of Bhatia and Kittaneh [6].

2.2. **The Hilbert-Schmidt Norm.**

Theorem 2.3. *Let* $A, B \geq 0$*,* $t \in [0,1]$*,* $r_0 = \min\{t, 1-t\}$ *and* $C = ||A^{3/2}B^{1/2} - A^{1/2}B^{3/2}||_2$*. Then*

$$\begin{aligned} ||AB||_2^2 &\leq ||A^{t+1/2}B^{3/2-t}||_2\;||A^{3/2-t}XB^{t+1/2}||_2 \text{ (it is a log-convex function)} \\ &\leq \frac{1}{4t(1-t)}\left(||(tA+(1-t)B)^2||_2^2 - r_0^2 t(1-t)C^2\right)^{1/2} \times \\ &\times \left(||((1-t)A+tB)^2||_2^2 - r_0^2 t(1-t)C^2\right)^{1/2} \end{aligned}$$

Proof. Let $A, B \geq 0$, $t \in [0,1]$, $r_0 = \min\{t, 1-t\}$ and $C = ||A^{3/2}B^{1/2} - A^{1/2}B^{3/2}||_2$.

$$h(t) = ||A^tXB^{1-t}||_2\;||A^{1-t}XB^t||_2, \text{ for } t \in [0,1].$$

It was proved in [5] that

$$\begin{aligned} ||A^{1/2}XB^{1/2}||_2^2 &= h(1/2) \\ &\leq h(t) \\ &\leq \left(||tAX+(1-t)XB||_2^2 - r_0^2||AX-XB||_2^2\right)^{1/2} \\ &\quad \times \left(||(1-t)AX+tXB||_2^2 - r_0^2||AX-XB||_2^2\right)^{1/2}, \end{aligned}$$

where $r_0 = \min\{t, 1-t\}$.

Let $X = A^{1/2}B^{1/2}$ and $C = ||A^{3/2}B^{1/2} - A^{1/2}B^{3/2}||_2$. Using (1.9) we have

$$\begin{aligned}||AB||_2^2 &\le ||A^{t+1/2}B^{3/2-t}||_2\, ||A^{3/2-t}XB^{t+1/2}||_2\\ &\le \left(||tA^{3/2}B^{1/2}+(1-t)A^{1/2}B^{3/2}||_2^2-r_0^2||A^{3/2}B^{1/2}-A^{1/2}B^{3/2}||_2^2\right)^{1/2}\\ &\quad\times\left(||(1-t)A^{3/2}B^{1/2}+tA^{1/2}B^{3/2}||_2^2-r_0^2||A^{3/2}B^{1/2}-A^{1/2}B^{3/2}||_2^2\right)^{1/2}\\ &\le \frac{1}{4t(1-t)}\left(||(tA+(1-t)B)^2||_2^2-4r_0^2t(1-t)C^2\right)^{1/2}\times\\ &\qquad\times\left(||((1-t)A+tB)^2||_2^2-4r_0^2t(1-t)C^2\right)^{1/2}\end{aligned}$$

This completes the proof. □

In particular, if $t=\frac{1}{2}$ in the above inequality we get the following refinement of the inequality $4\|AB\|_2 \le \|(A+B)^2\|_2$ proved in [6]. However, this inequality was shown for any unitarily invariant norm in this reference. Here, we present a refinement for the Hilbert-Schmidt norm.

Corollary 2.1. *Let $A, B \in \mathbb{M}_n$ be positive. Then*

$$16\,||AB||_2^2+||A^{3/2}B^{1/2}-A^{1/2}B^{3/2}||_2^2 \le ||(A+B)^2||_2^2. \tag{2.3}$$

It should be remarked that in [6], the refinement

$$16\,||AB||_2^2+||A^2+B^2-2AB||_2^2 \le ||(A+B)^2||_2^2 \tag{2.4}$$

was proved. Numerical examples show that neither (2.3) nor (2.4) is uniformly better than the other.

Our next result is the following refinement of (1.8). In this next theorem, we use the notations $r_0=\min\{t,1-t\}, r_1=\min\{t,1-2t\}, r_2=\min\{2t-1,2-2t\}$. Moreover, it has been shown in [13] that for $0\le t\le \frac{1}{2}$ and $a,b>0$ one has the inequality

$$(a^{1-t}b^t)^2+r_0^2(a-b)^2+r_1(a-\sqrt{ab})^2 \le ((1-t)a+tb)^2. \tag{2.5}$$

On the other hand, if $\frac{1}{2}\le t\le 1$,

$$(a^{1-t}b^t)^2+r_0^2(a-b)^2+r_2(b-\sqrt{ab})^2 \le ((1-t)a+tb)^2. \tag{2.6}$$

Theorem 2.4. *Let $A, B, X \in \mathbb{M}_n$. Let $\|A^*AX - XB^*B\|_2^2 = T_1$, $\|(A^*A)^{1/2}X(B^*B)^{1/2} - XB^*B\|_2^2 = T_2$ and $\|(A^*A)^{1/2}X(B^*B)^{1/2} - A^*AX\|_2^2 = T_3$. Then for $0 \le t \le \frac{1}{2}$,*

$$\begin{aligned}&\|AXB^*\|_2^2\\ &\le \left(\|t\ A^*AX + (1-t)XB^*B\|_2^2 - r_0^2 T_1 - r_1 T_2\right)^{\frac{1}{2}}\\ &\times \left(\|(1-t)\ A^*AX + t\ XB^*B\|_2^2 - r_0^2 T_1 - r_1 T_3\right)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, if $\frac{1}{2} \le t \le 1$,

$$\begin{aligned}&\|AXB^*\|_2^2\\ &\le \left(\|t\ A^*AX + (1-t)XB^*B\|_2^2 - r_0^2 T_1 - r_2 T_2\right)^{\frac{1}{2}}\\ &\times \left(\|(1-t)\ A^*AX + t\ XB^*B\|_2^2 - r_0^2 T_1 - r_2 T_3\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. For $0 \le t \le \frac{1}{2}$, we have (2.5). It is a standard argument to obtain a corresponding Hilbert-Schmidt norm inequality. For the positive matrices A, B and any X, one can easily use (2.5) to prove, for positive A, B,

(2.7)
$$\|A^t XB^{1-t}\|_2^2 + r_0^2\|AX - XB\|_2^2 + r_1\|A^{1/2}XB^{1/2} - XB\|_2^2 \le \|t\ AX + (1-t)XB\|_2^2$$

and

(2.8)
$$\|A^{1-t}XB^t\|_2^2 + r_0^2\|AX - XB\|_2^2 + r_1\|AX - A^{1/2}XB^{1/2}\|_2^2 \le \|t\ AX + (1-t)XB\|_2^2.$$

Replacing (A, B) with (A^*A, B^*B) in (2.7), (2.8), then using (1.5) we have the desired inequality for $0 \le t \le \frac{1}{2}$. Similar computations implies the desired inequality for $\frac{1}{2} \le t \le 1$. □

Interestingly, we can present a reverse of (1.8), as follows. A well known reverse of the Young inequality has the form [11, 14]

$$\|A^t XB^{1-t}\|_2^2 + R^2\|AX - XB\|_2^2 \ge \|t\ AX + (1-t)XB\|_2^2, 0 \le t \le 1, \tag{2.9}$$

where $R_0 = \max\{t, 1-t\}$. Before proceeding to the result, notice that a log convex function f satisfies the following for $0 \le t \le \frac{1}{2}$,

(2.10)

$$f(t) = f\left(2t \cdot \frac{1}{2} + (1-2t)\cdot 0\right) \le f\left(\frac{1}{2}\right)^{2t} f(0)^{1-2t} \Rightarrow f\left(\frac{1}{2}\right)^{2t} \ge f(0)^{2t-1} f(t).$$

Therefore, if f is, in addition, symmetric about $\frac{1}{2}$, we have for $\frac{1}{2} \le t \le 1$,

$$f(t) = f(1-t) \Rightarrow f\left(\frac{1}{2}\right)^{2(1-t)} \ge f(0)^{1-2t} f(t). \tag{2.11}$$

We remark that the proof of (1.5) given in [2] does not permit us to get reversed inequalities. This is the advantage of the new proof. The proof of the following theorem is a standard computational application of (2.9), (2.10) and (2.11), hence we omit it.

Theorem 2.5. *Let* $A, B, X \in \mathbb{M}_n$*. Then, for* $0 \le t \le 1$,

$$\begin{aligned}
&\|AXB^*\|_2^{4r_0}\\
&\ge (\|A^*AX\|_2 \|XB^*B\|_2)^{-|2t-1|}\\
&\times \left(\|t\ A^*AX + (1-t)XB^*B\|_2^2 - R_0^2\|A^*AX - XB^*B\|_2^2\right)^{\frac{1}{2}}\\
&\times \left(\|(1-t)\ A^*AX + t\ XB^*B\|_2^2 - R_0^2\|A^*AX - XB^*B\|_2^2\right)^{\frac{1}{2}},
\end{aligned}$$

where $r_0 = \min\{t, 1-t\}$ *and* $R_0 = 1 - r_0$.

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