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DIFFERENCE CESÁRO SEQUENCE SPACE DEFINED BY MUSIELAK-ORLICZ FUNCTIONS

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Abstract. The main goal of this paper is to study the topological and algebraic properties of the new constructed Cesàro sequence space of difference operator by means of Musielak-Orlicz functions. We also make an effort to study the properties of composite of Museilak-Orlicz function.

1. Introduction

Let ω denotes the space of all complex valued sequences. For $1 < t < \infty$, the Cesàro sequence space Ces_t of real sequences (χ_j) is defined by,

$$Ces_t = \Big\{\chi = (\chi_j) : \sum_{i=1}^{\infty}(\frac{1}{i}\sum_{j=1}^{i}|\chi_j|)^t < \infty\Big\}.$$

It is a Banach space under the norm

$$\|\chi_j\| = \bigg(\sum_{i=1}^{\infty}(\frac{1}{i}\sum_{j=1}^{i}|\chi_j|)^t\bigg)^{\frac{1}{t}}.$$

This space is useful in theory of matrix operators and was first introduced by Shiue [17]. Many authors studied some geometric properties of this space (see, e.g., [7, 8, 16]) and references therin.

An Orlicz function $\mathcal{M} : [0,\infty) \to [0,\infty)$ is continous, non-decreasing and convex such that $\mathcal{M}(0) = 0, \mathcal{M}(\chi) > 0$ for $\chi > 0$ and $\mathcal{M}(\chi) \to \infty$ as $\chi \to \infty$. Based on Orlicz function, Lindenstrauss and Tzafriri [9] defined the space of all scalar sequences,

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denoted by $l_{\mathcal{M}}$, such that $\sum_{j=1}^{\infty} \mathcal{M}(\frac{\chi_j}{\rho}) < \infty$.
This space is called Orlicz sequence space and it is a Banach space equipped with the norm

$$\|\chi_j\| = \inf\{\rho > 0 : \sum_{j=1}^{\infty} \mathcal{M}(\frac{\chi_j}{\rho}) \leq 1\}.$$

A sequence, $\mathcal{M} = (\Im_j)$ of Orlicz function is called Musielak-Orlicz function (see [10, 15]). For more details about the sequence spaces defined by Musielak-Orlicz function, we may refer to ([11, 12, 13, 14, 18, 19, 20]) and references therein.
Kızmaz [6] introduced the notion of difference space. Further, it was generalised by Et. and Çolak [4], the difference squence space as below:

$$Z(\Delta) = \{\chi = (\chi_j) \in \omega : (\Delta^\nu \chi_j) \in \digamma\}$$

for $\digamma = l_\infty,\ c$ and c_0, where ν is non-negative integer and

$$\Delta^\nu \chi_j \Delta^{\nu-1}\chi_j - \Delta^{\nu-1}\chi_{j-1}, \Delta^0 \chi_j = \chi_j \ for\ all\ j \in \mathbb{N},$$

or equivalently,

$$\Delta^\nu \chi_j = \sum_{w=0}^{j} (-1)^w \binom{u}{w} \chi_{j+w}.$$

Et. and Başasir [5] generalised these spaces by taking $\digamma = l_\infty(p),\ c(p)$ and $c_0(p)$.
Dutta [3] introduced the following difference sequence spaces (a new difference operator):

$$Z(\Delta_\eta) = \{\chi = (\chi_j) \in \omega : \Delta_n(\chi) \in \digamma\}$$

for $\digamma = l_\infty,\ c$ and c_0, where $\Delta_\eta \chi = (\Delta_\eta \chi_j) = \{\chi_j - \chi_{j-n}\}$ for all $k, n \in \mathbb{N}$.
Başar and Atlay A1 introduced the generalized difference matrix $B = (b_{\eta j})$ for all

$j, \eta \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator, by

$$b_{\eta j} = \begin{cases} \alpha & \text{when } \ j = \eta \\ \beta & \text{when } \ j = \eta - 1 \\ 0 & \text{when } \ j > \eta \, or \ (0 \leq j < \eta - 1) \end{cases}$$

Başarir and Kayikçi [2] defined the matrix $B^{\nu}(b^{\nu}_{\eta k})$ which reduced the difference matrix Δ^{ν}_{1} in case $\alpha = 1, \ \beta = 1$. The generalized B^{μ}-difference operator is equivalent to the following binomial representation:

$$B^{\nu}\chi = B^{\nu}(\chi_j) = \sum_{w}^{\nu} \binom{\nu}{0} r^{\nu - w} s^{w} \chi_{j-w}.$$

Let $\mathcal{M} = (\Im_i)$ be Musielak-Orlicz function. For a bounded sequence $t = (t_i)$ of positive real numbers, we define the difference Cesàro space defined by Museilak-Orlicz function as follows:

$$Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t) = \{\chi \in \omega : \sum_{i=1}^{\infty} \left[\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B^{\nu}_{\Lambda} \chi_j|}{\phi_1} \right) \right]^{t_i} < \infty\}.$$

Let $\mathcal{M} = (\Im_i)$ be Musielak-Orlicz function. Then, the multipliers of the sequence space $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ denoted by $S(Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t))$, is given by

$$S(Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)) = \{a \in \omega : a\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t) \text{ for all } \chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)\}.$$

Let $\mathcal{M} = (\Im_i)$ be Musielak-Orlicz function and $\vee$ be a fixed natural number. Then we define

$$Ces(\mathcal{M}^{\vee}, B^{\nu}_{\Lambda}, t) = \{\chi \in \omega : \sum_{i=1}^{\infty} \left[\Im_i^{\vee} \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B^{\nu}_{\Lambda} \chi_j)|}{\phi} \right) \right]^{t_i} < \infty \ \text{ for some } \ \phi > 0\}.$$

The following inequality will be used throughout the paper. If $0 \leq t_i \leq \sup t_i = C, \ D = \max(1, 2^{C-1})$, then

$$|a_i + b_i|^{t_i} \leq C(|a_i|^{t_i} + |b_i|^{t_i}), \tag{1.1}$$

for all i and $a_i, \ b_i \in \mathbb{C}$. Also $|a|^{t_i} \leq max(1, a|^{C})$ for all $a \in \mathbb{C}$.

2. Main Results

Theorem 2.1. *For any Musielak-Orlicz function $\mathcal{M} = (\Im_i)$, the space $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ is linear over $\mathbb{C}$.*

Proof. Let $\chi,\ \xi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist $\phi_1 > 0$ and $\phi_2 > 0$ such that

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi_1}\right)\right]^{t_i} < \infty$$

and

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\xi_j|}{\phi_2}\right)\right]^{t_i} < \infty.$$

Define $\phi_3 = \max(2|\alpha|\phi_1, 2|\beta|\phi_2)$. Since $\Im_i$ is non-decreasing and convex, $\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{n}\sum_{j=1}^{i}|\alpha B^{\nu}_{\Lambda}\chi_j+\beta B^{\nu}_{\Lambda}\xi_j|}{\phi_3}\right)\right]^{t_i}$

$$\begin{aligned}
&\le \sum_{i=1}^{\infty}\frac{1}{2^{t_i}}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi_1}\right)+\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\xi_j|}{\phi_2}\right)\right]^{t_i}\\
&\le \max(1, 2^{C-1})\left(\sum_{i=1}^{\infty}\left[\Im_i(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi_1})\right]^{t_i}+\sum_{i=1}^{\infty}\left[\Im_i(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\xi_j)|}{\phi_2})\right]^{t_i}\right)\\
&< \infty.
\end{aligned}$$

Hence, the required result. □

Theorem 2.2. *For any Musielak-Orlicz function $\mathcal{M} = (\Im_i)$, the space $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ is paranormed, with a paranorm defined as*

$$\gamma(\chi) = \left(\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi_1}\right)\right]^{t_i}\right)^{\frac{1}{K}}, \tag{2.1}$$

where $C = sup\ t_i$ and $K = max(1, C)$.

Proof. For this, we only need to show that γ is subadditive and multiplication is continous. For this, let $\chi,\ \xi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ and by using the Minkowski's inequality, we have

$$\left(\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B_\Lambda^\nu(\chi_j+\xi_j)|}{\phi}\right)\right]^{t_i}\right)^{\frac{1}{K}}$$

$$\leq \left(\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{1}{i}\sum_{j=i}^{i}\left(\frac{|B_\Lambda^\nu\chi_j|}{\phi}+\frac{|B_\Lambda^\nu\xi_j|}{\phi}\right)\right)\right]^{t_i}\right)^{\frac{1}{K}}$$

$$\leq \left(\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B_\Lambda^\nu\chi_j|}{\phi}\right)\right]^{t_i}\right)^{\frac{1}{K}}+\left(\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B_\Lambda^\nu\xi_j|}{\phi}\right)\right]^{t_i}\right)^{\frac{1}{K}}.$$

Thus, $\gamma(x)$ is subadditive. To complete the proof, let $\delta \in \mathbb{C}$. By definition, we have

$$\gamma(\delta\chi)=\left(\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B_\Lambda^\nu\delta\chi_j|}{\phi}\right)\right]^{t_i}\right)^{\frac{1}{K}} \leq T_\delta^{\frac{C}{K}}\gamma(x),$$

where $T_\delta \in \mathbb{N}_0$ such that $|\delta| \leq T_\delta$. Let $\delta \to 0$ and for fixed χ, $\gamma(\chi)=0$. By definition for $|\delta|<1$, we have

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|\delta B_\Lambda^\nu\chi_j|}{\phi}\right)\right]^{t_i} < \epsilon,\ \ i > i_0(\epsilon) \tag{2.2}$$

Also, for $1 \leq i \leq i_0$, for sufficiently small δ. Since $\mathcal{M}=(\Im_i)$ is continous, we have

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|\delta B_\Lambda^\nu\chi_j|}{\phi}\right)\right]^{t_i} < \epsilon. \tag{2.3}$$

By the above equations, it imply that $\gamma(\delta x) \to 0$ as $\delta \to 0$. This completes the proof that the space $Ces(\mathcal{M}, B_\Lambda^\nu, t)$ is a paranormed space.

□

Theorem 2.3. *For any Musielak-Orlicz function $\mathcal{M}=(\Im_i)$, the space $Ces(\mathcal{M}, B_\Lambda^\nu, t)$ is a complete paranormed space with paranorm defined in equation (2.1)*

Proof. For this, it is sufficient to prove the completeness property $Ces(\mathcal{M}, B_\Lambda^\nu, t)$. Let $(\chi^{(m)})$ be a Cauchy sequence in $Ces(\mathcal{M}, B_\Lambda^\nu, t)$. Let $r\chi_0$ be fixed. Then for each $\frac{\epsilon}{r\chi_0}>0$, $\exists N \in \mathbb{N}_0$ such that

$$\gamma(B_\Lambda^\mu\chi^{(m)}) - B_\Lambda^\nu\chi^{(n)}) < \frac{\epsilon}{r\chi_0},$$

for all $m,\ n \geq N$,

Using (2.1), we have

$$\left(\sum_{i=1}^{\infty} \left[\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)} - B_\Lambda^\nu \chi_j^{(n)}|}{\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi^{(n)})} \right) \right]^{t_i} \right)^{\frac{1}{K}} \leq 1.$$

Thus

$$\sum_{i=1}^{\infty} \left[\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)} - B_\Lambda^\nu \chi_j^{(n)}|}{\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi^{(n)})} \right) \right]^{t_i} \leq 1.$$

Since $1 \leq t_i < \infty$, it follows that $\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)} - B_\Lambda^\nu \chi_j^{(n)}|}{\gamma(B_\Lambda^\nu x^{(m)} - B_\Lambda^\nu x^{(n)})} \right) \leq 1$, for each $i \geq 1$.

We choose $r > 0$ such that $(\frac{\chi_0}{2}) rt(\frac{\chi_0}{2}) \geq 1$, where t is the kernel associated with $\Im_i$.

Hence,

$$\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)} - B_\Lambda^\nu \chi_j^{(n)}|}{\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu x^{(n)})} \right) \leq (\frac{\chi_0}{2}) rt(\frac{\chi_0}{2})$$

for each $i \in \mathbb{N}$. Using the integral representation of Orlicz function, we get $\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)} - B_\Lambda^\nu \chi_j^{(n)}| \leq \frac{r\chi_0}{2} \gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi^{(n)}) < \frac{\epsilon}{2}$, for all $m, n \geq N$.

Hence for each fixed j, $(B_\Lambda^\nu \chi_j^{(m)})$ is Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $(B_\Lambda^\nu \chi_j^{(m)}) \to (B_\Lambda^\nu \chi_j)$ as $m \to \infty$. For given $\epsilon > 0$, choose an integer $i_0 > 1$ such that $\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi^{(n)}) < \epsilon$ for all $m, n \geq i_0$, such that $\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi^{(n)}) < \rho < \epsilon$.

Since

$$\left(\sum_{i=1}^{\infty} \left[\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)} - B_\Lambda^\nu \chi_j^{(n)}|}{\phi} \right) \right]^{t_i} \right)^{\frac{1}{K}} \leq 1$$

for all $m, n \geq i_0$.

Now, using continuity of $\Im_i$ and taking $n \to \infty$ in above equality, we have

$$\left(\sum_{i=1}^{\infty} \left[\Im_i \left(\frac{\frac{1}{i} \sum_{j=1}^{i} |B_\Lambda^\nu \chi_j^{(m)})) - B_\Lambda^\nu \chi_j|}{\phi} \right) \right]^{t_i} \right)^{\frac{1}{K}} \leq 1$$

for all $m \geq i_0$.

Letting $m \to \infty$, we get $\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi) < \epsilon$ for all $m, n \geq i_0$, such that $\gamma(B_\Lambda^\nu \chi^{(m)} - B_\Lambda^\nu \chi^{(n)}) < \rho < \epsilon$ for all $m \geq i_0$. Thus $(B_\Lambda^\nu \chi^{(m)})$ converges to $(B_\Lambda^\nu \chi)$ with respect to the pranorm of $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$. Since $(B^{\nu}_{\Lambda}\chi^{(m)}) \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ and $\Im_i$ is continous, it follows that $(B^{\nu}_{\Lambda}\chi) \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$. □

Theorem 2.4. *Let $\mathcal{M} = (\Im_i)$ be Musielak-Orlicz function and $t = (t_i), s = (s_i)$ are bounded sequence of positive real numbers with $0 < t_i \leq s_i < \infty$ for each i. Then $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t) \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, s)$.*

Proof. Let $\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, p)$. Then, $\exists\ \phi > 0$ such that

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{p_i} < \infty.$$

This implies that $\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right) \leq 1$ for sufficiently large value of i, say $i \geq i_0$ for some fixed $i_0 \in \mathbb{N}$. Since $\Im_i$ is non-decreasing and $t_i \leq s_i$, we have

$$\sum_{i\geq i_0}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{s_i} \leq \sum_{i\geq i_0}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i} < \infty.$$

Therefore, $\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, q)$ and hence, the result. □

Theorem 2.5. *Let $\mathcal{M} = (\Im_i)$ be Museilak-Orlicz function and $r = r_i, t = (t_i)$ be bounded sequences of positive real numbers with $0 < r_i, t_i < \infty$ and if $p_i = \min(r_i, t_i), q_i = \max(r_i, t_i)$, then $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, p) = Ces(\mathcal{M}, B^{\nu}_{\Lambda}, r) \cap Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ and $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, q) = D$, where D is subspace of ω generated by $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, r) \cup Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$.*

Proof. By above result, $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, p) \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, r) \cap Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$ and $D \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, q)$.

For any $\delta \in \mathbb{C}$, $|\delta|^{p_i} \leq \max(|\delta|^{r_i}, |\delta|^{t_i})$ and thus $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, r) \cap Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t) \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, p)$.

Let $E = \{i : r_i \geq t_i\}$ and $F = \{i : r_i < t_i\}$. If $\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, q)$, we write

$$\xi_i = \chi_i (i \in E) \text{ and } \xi_i = 0 (i \in F);$$

$$\varsigma_i = 0 (i \in E) \text{ and } \varsigma_i = \chi_i (i \in F).$$

Since $\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, q),\ \exists\ \phi > 0$ such that

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{q_i} < \infty.$$

Now

$$\begin{aligned}\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\xi_j|}{\phi}\right)\right]^{r_i} &= \sum_{n\in E}+\sum_{n\in F}\\ &= \sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\xi_j|}{\phi}\right)\right]^{q_i} < \infty.\end{aligned}$$

Therefore, $\xi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, r) \subset D$. Likewise, $B^{\mu}_{\Lambda}\varsigma \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t) \subset D$. Thus, $\chi = \xi + \varsigma \in D$ and hence, $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, q) \subset D$, which is the desired result. □

Theorem 2.6. *Let $\mathcal{M} = (\Im_i)$ be Musielak-Orlicz function which satisfies Δ_2 condition. Then $l_\infty \subset S(Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t))$.*

Proof. Let $\varsigma = \varsigma_i \in l_\infty,\ K = \sup_i|\varsigma_i|$ and $\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$. Then

$$\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i} < \infty\ for some\ \phi > 0.$$

Since $\Im_i$ satisfies Δ_2-condition,$\exists\ N > 0$ such that

$$\begin{aligned}\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|\varsigma_i B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i} &\leq \sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}\varsigma_i||B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i}\\ &\leq \sum_{i=1}^{\infty}\left[\Im_i\left(1+[K]\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i}\\ &\leq (N(1+[T]))^{C}\sum_{i=1}^{\infty}\left[\Im_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i}\\ &< \infty,\end{aligned}$$

where $[T]$ is the integer part of N. Hence, the result. □

Theorem 2.7. *Let $\mathcal{M}$ be Museilak-Orlicz function and $v \in \mathbb{N}$.*

(i) If there exists a constant $\beta \geq 1$ such that $\Im_i(y) \geq \beta y$ for all $\xi \geq 0$, then $Ces(\mathcal{M}^{\vee}, B^{\nu}_{\Lambda}, t) \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$

(ii) Suppose there exists a constant α, $0 < \alpha \leq 1$such that $\Im_i(y) \leq \alpha y$ for all $y \geq 0$ and let r, $\vee \in \mathbb{N}$ be such that $r < \vee$. Then $Ces(B^{\nu}_{\Lambda}, t) \subset Ces(\mathcal{M}^r, B^{\nu}_{\Lambda}, t) \subset Ces(\mathcal{M}^{\vee}, B^{\nu}_{\Lambda}, t)$.

Proof. (i) Since $\Im_i(y) \geq \beta y$ for all $y \geq 0$ and $\Im_i$ is non-decreasing and convex, we have $\Im^v_i(y) \geq \beta^v y$ for each $v \in \mathbb{N}$. Let $\chi \in Ces(\mathcal{M}^{\vee}, B^{\nu}_{\Lambda}, t)$. Using above inequality, we have

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|\right)^{t_i} \leq \max(1,\phi^C)\max(1,\beta^{-vC})\sum_{i=1}^{\infty}\left[\Im^v_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i}.$$

Hence,$\chi \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$.

(ii) Since $\Im_i(y) \leq \alpha y$ for all $y \geq 0$ and $\Im_i$ is non-decreasing and convex, we have $\Im^r_i \leq \alpha^r y$ for each $m \in \mathbb{N}$. We can easily prove the first inclusion. To prove the next one, suppose that $\vee - r = s$ and let $\chi \in Ces(\mathcal{M}^r, B^{\nu}_{\Lambda}, t)$. Again, using above inequality, we have

$$\sum_{i=1}^{\infty}\left[\Im^{\vee}_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i} \leq \max(1,\alpha^{sH})\sum_{i=1}^{\infty}\left[\Im^r_i\left(\frac{\frac{1}{i}\sum_{j=1}^{i}|B^{\nu}_{\Lambda}\chi_j|}{\phi}\right)\right]^{t_i}$$

and hence $\chi \in Ces(\mathcal{M}^{\vee}, B^{\nu}_{\Lambda}, t)$.

□

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