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ON PSEUDO-BALANCING OF PATH-INDUCED SIGNED GRAPHS

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Abstract. The path decomposition of a graph G is the partition of its edges into distinct paths. Pendant number of a graph G is the least number of terminal vertices involved in the path decomposition of G. If the signed function is induced by the terminal vertices of the path-decomposed graph, it is called path-induced signed graph. In this paper, we identify few classes of balanced signed graphs and also introduce and examine a related concept, namely, pseudo-balancing of path-induced signed graphs.

1. Introduction

A graph $G(V, E)$ is called a *signed graph*, if it is associated with a function σ which takes either of the values $+$ or $-$ to each of its edges. It is denoted by $S(G, \sigma)$ and the function σ is called the *signature* or *sign function* of S. A positive edge of the signed graph $S(G, \sigma)$ is denoted as $\sigma(e) = +1$ and the negative edge as $\sigma(e) = -1$. If the product of signs of edges in each cycle in $S(G, \sigma)$ is positive, then the signed graph is said to be *balanced*. A signed graph is *positively homogeneous* if all of its edges are signed positive and is *negatively homogeneous* if all of its edges are signed negative. We refer to [3] for general terms and definitions of graph theory and to [15, 16] for the terminologies on signed graph.

The partition of edge set of the given graph G into distinct path-graphs is called *path decomposition* of G. By the term *terminus* or a *terminal vertex* of G, we mean a vertex which is an end vertex of a path in the path decomposition of G. The vertices

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other than the terminal vertices are called *internal vertices* of the graph G. The vertices adjacent to a vertex v of the graph G are called the *neighbours* of the vertex v.

Though the concept of an induced signed graph is introduced in [2], it is studied in detail in [13, 12]. The generalised terminology *induced signed graph* is introduced in [1] as follows: An *induced signed graph*, S_G of a graph G is obtained from G by inducing a sign function σ on G using the rule $\sigma(uv) = (-1)^{|\phi(u)-\phi(v)|}$, where $uv \in E(G)$ and ϕ is a specified parameter of $u, v \in V(G)$. In this paper, the parameter ϕ is taken as the pendant number of the graph G. The *pendant number* of a graph G, denoted as $\Pi_p(G)$ is the least number of terminal vertices involved in the path decomposition of G when it is decomposed into distinct path subgraphs. The formal definition of the pendant number of a graph G is given below.

Definition 1.1. [4] Let $V_p(G)$ be the set of all $u \in V(G)$, where $V(G)$ is the vertex set of the graph G, such that u is an end vertex of a path in the path-decomposition of G, then $\Pi_p(G) = \min\{|V_p(G)|\}$.

Definition 1.2. [10]Let $V_p(G)$ be the collection of terminal vertices in the desired path decomposition of the graph G. Let u and v be any two adjacent vertices in G and uv is the edge connecting them. A *path-induced signed graph* of G is the signed graph $S_p(G) = (G, \sigma)$, where the signing function σ is given by

$$\sigma(uv) = \begin{cases} +1, & \text{if } u, v \in V_p(G) \\ -1, & \text{otherwise.} \end{cases}$$

If the context is clear, instead of $S_p(G)$, we may write S_p. Few results on path-induced signed graphs, which are relevant to our discussions, are given below.

Proposition 1.1. [10] The path-induced signed graph of a graph G of order n with pendant number $\Pi_p(G) \geq n - 1$ is balanced.

Proposition 1.2. [10] Let H denote a graph obtained by introducing a new vertex, say w, of degree 2 by subdividing the edge uv of G. Then, w is not a terminal vertex.

Theorem 1.1. [10] Let H denote a graph obtained by subdividing an edge, say uv, of the underlying graph G. Then, the path induced signed graph of H, $S_p(H)$ is balanced if and only if both u and v are terminal vertices of balanced signed graph $S_p(G)$.

This paper refers to [4, 5, 6, 7, 8, 9, 10] for the concept of pendant number of graphs and path-induced signed graphs. We consider simple, connected, finite and cyclic graphs only.

2. Balanced Path-Induced Signed Graphs

Balanced path-induced signed graphs are introduced in [10]. Here, we examine some classes of graphs which are balanced. We begin with complete bipartite graph $K_{m,n}$, where m and n are the number of vertices in each partition. Without lose of generality, we assume $m \leq n$. First, we identify the homogeneous conditions of a path-induced signed graph of a complete bipartite graph and then its balancing.

Proposition 2.1. For every complete bipartite graph $K_{m,n}$ with $m \leq n$, its path-induced signed graph $S_p(K_{m,n})$ is homogeneous.

Proof. Let $K_{m,n}$ be a complete bipartite graph with $m \leq n$ and $S_p(K_{m,n})$ be its path-induced signed graph. Let U and V be the partition of the vertex set of $K_{m,n}$ with $|U| = m$ and $|V| = n$ such that $U = \{u_1, u_2, u_3, \ldots, u_m\}$ and $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex sets of U and V. Consider the following cases:

Case 1: Let both m and n be even. Then, the pendant number of $K_{m,n}$ is 2 (see [4]). Thus, any two vertices among the m vertices of U can be taken as the terminal vertices of every path in $K_{m,n}$. By definition, if there is any edge connecting these two terminal vertices is signed positive. Since every vertex in U is non-adjacent to every other vertex in U, every edge incident to U is signed negative in $S_p(K_{m,n})$. Thus, $S_p(K_{m,n})$ is negatively homogeneous.

Case 2: Let both m and n be odd. Then, the pendant number of $K_{m,n}$ is $m + n$ (see [4]). Since every vertex is a terminal vertex, every edge of $S_p(K_{m,n})$ is signed positive. Thus, $S_p(K_{m,n})$ positively homogeneous.

Case 3: Let m be odd and n be even. Then, the pendant of $K_{m,n}$ is n (see [4]). Here, every vertex in V is a terminal vertex and they are non-adjacent to each other.

Thus, every edge incident to V is negatively signed in $S_p(K_{m,n})$. Hence, $S_p(K_{m,n})$ is negatively homogeneous.

Case 4: Let m be even and n be odd. Then the pendant of $K_{m,n}$ is m (see [4]). Here, every vertex in U is a terminal vertex and is non-adjacent to any other vertex in U. Thus, every edge incident to U is negatively signed in $S_p(K_{m,n})$. Hence, $S_p(K_{m,n})$ is negatively homogeneous. □

Proposition 2.2. For every complete bipartite graph $K_{m,n}$, its path-induced signed graph $S_p(K_{m,n})$ with $m \leq n$, is balanced.

Proof. Let $S_p(K_{m,n})$ be a complete bipartite graph with $m \leq n$. Then, it has mn edges. A complete bipartite path-induced signed graph $S_p(K_{m,n})$ positively homogeneous if both m and n are odd. Otherwise, it is negatively homogeneous. For a bipartite graph, cycles are of even length. Thus, any cycle in $S_p(K_{m,n})$ contains either even number of positively signed edges or even number of negatively signed edges. In both cases, the product of signs of edges in every cycle is positive. Hence, $S_p(K_{m,n})$ is balanced.

□

A wheel graph W_n is the join $C_n + K_1$, where K_1 is called spin of the wheel graph. Next, we examine the balancing of a path-induced signed graph of a wheel graph.

Proposition 2.3. The path-induced signed graph of any wheel graph, $S_p(W_n)$ on $n+1$ vertices is balanced.

Proof. The pendant number of a wheel graph W_n on $n+1$ vertices is (see [8]),

$$\Pi_p(W_n) = \begin{cases} n & \text{if } n \text{ is even;} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Consider the following two cases:

Case 1: Let n be an even number. Then, the only vertex, which is not a terminal vertex in W_n, is the spin and every edge incident to the spin is negatively signed. Thus, any cycle in W_n has either all edges are positively signed or it has exactly a pair of negatively signed edges. Hence, the product of signs of edges in any cycle is positive.

Case 2: Let n be an odd number. Then, the entire vertex set is in the set of terminal vertices. Thus, every edge is positively signed and the corresponding signed graph is positively homogeneous. Hence, the product of signs of edges in any cycle is positive. □

In a similar way, we can prove that the double-wheel graph DW_n on $2n+1$ vertices is balanced. But many other wheel-based graphs such as flower graphs, djembe graphs (which we discuss later) are not balanced. Now, let us move on to the concept of pseudo-balanced graphs.

3. On Pseudo-Balancing of Path-Induced Signed Graphs

The necessary and sufficient condition for a path-induced signed graph $S_p(G)$ to be balanced is that the product of signs of edges of every cycle in it is positive (see [10]). While considering a path-induced signed graph $S_p(G)$ with even number of cycles, there are cases where the product of signs of edges in every cycle is negative. In that case, $S_p(G)$ is not balanced. For example, the friendship graph (also known as the butterfly graph F_2) has pendant number 2 (see [5]) and the path-induced signed graph of it is the join of two negatively signed cycles. Though the product of signs of cycles as well as the product of signs of entire edge set is positive, the graph is not balanced. Moreover, many classes of path-induced signed graphs are outside the realm of balanced graphs. In view of this fact, we introduce a new concept called pseudo-balancing of path-induced signed graphs.

Definition 3.1. A path-induced signed graph $S_p(G)$ of a graph G is said to be *pseudo-balanced* if the product of signs of the entire edge set in $S_p(G)$ is positive.

Proposition 3.1. A balanced path-induced signed graph is always pseudo-balanced.

Proof. Let $S_p(G)$ be a path-induced signed graph of a graph G. Assume that $S_p(G)$ is balanced. Then, the product of signs of edges in every cycle of $S_p(G)$ is positive. This implies if any cycle contains negatively signed edges, then the number of such edges will be an even number. Thus, the product of signs of edges will also be an even number. Hence, it is pseudo-balanced. □

It is clear from the definition that the converse need not be true. That is, a pseudo-balanced path-induced signed graph need not be a balanced path-induced signed graph. The following discussion shows that the class of balanced path-induced signed graphs is a subclass of pseudo-balanced path-induced signed graphs. The pseudo-balancing of the path-induced signed graphs of various graph classes, which are not balanced otherwise can be studied extensively. We attempt a few classes of graphs in the following discussion. Next theorem investigates the pseudo-balancing of the path-induced signed graph of a complete graph K_n. Thus, we have:

Theorem 3.1. A path-induced signed graph $S_p(K_n)$ on a complete graph K_n is pseudo-balanced except for $n \equiv 1 \ (mod\ 4)$.

Proof. Let K_n be a complete graph with n vertices and $m = \frac{n(n-1)}{2}$ edges. Here, we have to consider the following cases:

Case 1: Let $n \equiv 0, 2 \ (mod\ 4)$. That is, n is an even number and $\Pi_p(K_n) = n$ (see [4]). By Proposition 1.1, the path-induced signed graph of a graph G of order n with pendant number $\Pi_p(G) \geq n-1$ is balanced. Hence, $S_p(K_n)$ is pseudo-balanced.

Case 2: Let $n \equiv 3 \ (mod\ 4)$. Then, m is odd and $\Pi_p(K_n) = 2$ (see [4]). Here, exactly one edge is positively signed and all the remaining $m-1$ edges are negatively signed. Since $m-1$ is an even number, the product of signs of these $m-1$ edges is positive. Thus, the product of signs of the entire edge set is positive. Hence, $S_p(K_n)$ is pseudo-balanced.

Case 3: Let $n \equiv 1 \ (mod\ 4)$. Then, m is even and $\Pi_p(K_n) = 2$ (see [4]). Here also, exactly one edge is positively signed and the remaining $m-1$ edges are negatively signed. Since $m-1$ is an odd number, the product of signs of edges is negative. Moreover, the product of signs of the entire edge set is negative. Hence, $S_p(K_n)$ is not pseudo-balanced.

That is, $S_p(K_n)$ is pseudo-balanced except for $n \equiv 1 \ (mod\ 4)$. □

It is proved in [10] that the path- induced signed graphs of odd regular graphs are balanced. Path-induced signed graph of an even regular graph may or may not be balanced. Next, we examine the pseudo-balancing of path-induced signed graph of the even regular graphs.

Proposition 3.2. If G is an even regular graph other than a complete graph, then the path-induced signed graph $S_p(G)$, is pseudo-balanced.

Proof. Let G be an even regular graph with n vertices and m edges and $S_p(G)$ be the path-induced signed graph of G. The pendant number of any even regular graph is identified as 2 (see [4]). Here, we have to consider the following two cases:

Case 1: Let m be an odd number. Since the pendant number is 2, we can take any two adjacent vertices as terminal vertices of every path in the desired path decomposition. Thus, there is exactly one positively signed edge in $S_p(G)$. The remaining $m-1$ edges are negatively signed. Since $m-1$ is an even number, the product of negatively signed edges is positive. Hence, $S_p(G)$ is pseudo-balanced.

Case 2: Let m be an even number. Since G is taken as a regular graph other than a complete graph with pendant number 2, we can take any two non-adjacent vertices as the terminal vertices of the paths in the path decomposition. Then, every edge is negatively signed and the product of even number of negatively signed edges is positive. Thus, $S_p(G)$ is pseudo-balanced. □

The $n-$*fan graph* (F_n) is a graph on $n+2$ vertices in which a single universal vertex is connected to all vertices of a path P_{n+1}. Next, we consider the pseudo-balancing of path-induced signed graph of $n-$fan graphs.

Proposition 3.3. For the path-induced signed graph of n-fan graph $S_p(F_n)$ on $n+2$ vertices;

(i) $S_p(F_n)$ is balanced, if n is even.

(ii) $S_p(F_n)$ is pseudo-balanced, if n is odd.

Proof. Let $S_p(F_n)$ be the path-induced signed graph of n-fan graph on $n+2$ vertices. Consider the two cases:

Case 1: Let n be an even number. Then, the pendant number of F_n is n (see [8]) with exactly two internal vertices, say u and v, of degree two each. Note that both u and v are non-adjacent to each other and the edges incident to them are negatively signed. These are the only negatively signed edges in the given graph. Thus, $S_p(F_n - \{u, v\})$ can be considered as a positively homogeneous path-induced signed graph and is balanced. Now, consider the two neighbours of u and denote them as x and y. Since u is a vertex of degree two, u can be considered as a vertex which subdivide the edge connecting x and y. Remember that both the neighbours of u (x and y) are terminal vertices of some paths. By Proposition 1.2 and Theorem 1.1, the presence of u does not affect the balance of $S_p(F_n - \{u\})$. In a similar way, we can prove that the presence of v does not affect the balance of $S_p(F_n - \{v\})$. Thus, the presence of both the vertices u and v does not affect the balance of $S_p(F_n - \{u, v\})$. Hence, $S_p(F_n)$ is balanced.

Case 2: Let n be an odd number. Here, there are three internal vertices such that the two vertices of degree two, say u and v, are adjacent to the third internal vertex, which is the universal vertex of F_n. Since the product of signs of edges of every cycle passing through the vertices u and v is negative, $S_p(F_n)$ is not balanced. Now, we have to show that it is pseudo-balanced. Among $2n + 1$ edges of $S_p(F_n)$, $n-2$ edges are positively signed and the remaining $n+3$ edges are negatively signed. Since n is odd, the product of signs of $n + 3$ edges is also positive. Hence, $S_p(F_n)$ is pseudo-balanced. □

If each rim vertex of a wheel graph W_n is connected to a corresponding isolated vertex, then the resultant graph G is called *helm graph*. If these pendant vertices of the helm graph are connected to the center of the wheel graph, the new graph is called *flower graph* A_n (see [11]). It has $2n+1$ vertices. If two cycles C_n's are joined to its corresponding vertices and an external vertex is connected to all the $2n$ vertices, such a graph is called *djembe graph* DJ_n (see [14]). It has also $2n + 1$ vertices. Both flower graph and djembe graph are Eulerian graphs with pendant number $\lceil \frac{\Delta}{4} \rceil$, where Δ is the maximum degree of the concerned graph (see [8]). Next, we examine the balancing of path-induced signed graph of flower graphs and djembe graphs.

Proposition 3.4. The path-induced signed graph of the flower graph is pseudo-balanced.

Proof. Let A_n be a flower graph with $2n+1$ vertices and $4n$ edges. A_n has a vertex of degree $2n$ and n vertices of degree 4 and the remaining n vertices are also of degree 2. Since A_n has a lot of chords, its path-induced signed graph $S_p(A_n)$ has a lot of cycles

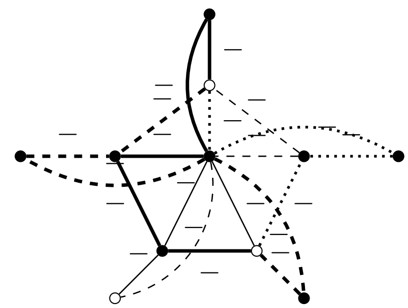

FIGURE 1. A pseudo-balanced path-induced signed graph of a flower graph. The white colored vertices represent the terminal vertices and the different types of lines connecting them represent the distinct paths in the path decomposition.

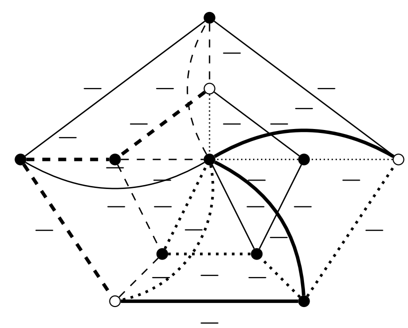

FIGURE 2. A path-induced signed graph of a djembe graph, which is not pseudo-balanced.
The white colored vertices represent the terminal vertices and the different types of lines connecting them represent the distinct paths in the path decomposition.

of length 3 (that is, triangles) with negatively homogeneous cycles. Hence, $S_p(A_n)$ is not balanced. Since A_n is a Eulerian graph with even number of edges and $S_p(A_n)$ is negatively homogeneous, it is pseudo-balanced (see Figure 1). □

Proposition 3.5. The path-induced signed graph of the djembe graph DJ_n is pseudo-balanced, only if n is even.

Proof. Let DJ_n be a djembe graph with $2n+1$ vertices and $5n$ edges. It has a vertex of degree $2n$ and the remaining vertices are of degree 4. Since DJ_n contains cycles with negatively homogeneous triangles, its path-induced signed graph $S_p(DJ_n)$ is not balanced. Though it is Eulerian, when n is odd the number of edges is an odd number.

Thus, the product of signs of edges is negative. Hence, $S_p(DJ_n)$ cannot be pseudo-balanced when n is odd. Whereas, if n is even, the total number of edges is even and the signed graph is negatively homogeneous. Thus, the product of signs of the entire edge set of $S_p(DJ_n)$ is positive. Hence, if n is even, $S_p(DJ_n)$ is pseudo-balanced (see Figure 2). □

The djembe graph is an example for a negatively homogeneous path-induced signed graph of an Eulerian graph which is not pseudo-balanced. Now, we discuss the homogeneous conditions for Eulerian graphs.

Theorem 3.2. Let G be an Eulerian graph with n vertices and m edges. Let $S_p(G)$ be the negatively homogeneous path-induced signed graph of the graph G. $S_p(G)$ is pseudo-balanced if and only if m is even.

Proof. Let G be an Eulerian graph with n vertices and m edges. Let $S_p(G)$ be the path-induced signed graph of the graph G. $S_p(G)$ is negatively homogeneous implies none of the terminal vertices of $S_p(G)$ is adjacent to each other in the desired path decomposition. Assume $S_p(G)$ is pseudo-balanced. Then, the product of signs of the entire edge set is even. It is true only when the number of negatively signed edges is even. Hence, m is even. Conversely, assume the number of edges m is an even number. Since $S_p(G)$ is negatively homogeneous, the product of signs of the entire edge set is an even number. Hence, $S_p(G)$ is pseudo-balanced. □

Proposition 3.6. The path-induced signed graph $S_p(G)$ of an Eulerian graph G can not be positively homogeneous.

Proof. On the contrary, assume that the path-induced signed graph $S_p(G)$ of an Eulerian graph G is positively homogeneous. Then, all the edges of $S_p(G)$ are positively signed. This happens only when all the vertices are terminal vertices of some paths in the desired path decomposition. Thus, pendant number of G is n (see [9]). But, we know that the pendant number of a graph with at least one even degree vertex is less than $n-1$ (see [5]). This leads to a contradiction. Hence, the path-induced signed graph $S_p(G)$ of an Eulerian graph G can not be positively homogeneous. □

Theorem 3.3. Let $S_p(G)$ be the path-induced signed graph of a graph G. If G has exactly two consecutive internal vertices, then $S_p(G)$ is neither balanced nor pseudo-balanced.

Proof. Let $S_p(G)$ be the path-induced signed graph of a graph G. Let exactly two consecutive vertices, say u and v, be internal vertices of G. Then, the degree of each of the vertices u and v is even and the incident edges are negatively signed. Moreover, by definition of path-induced signed graphs, the edge uv is also negatively signed. Any cycle passing through these two vertices always contains three negatively signed edges. Thus, these cycles are negatively homogeneous and consequently $S_p(G)$ can not be balanced.

Now, we examine whether $S_p(G)$ is pseudo-balanced or not. Note that G has exactly two internal vertices and they are consecutive also. It implies that all other vertices are terminal vertices of some paths and the edges connecting them are all positively signed. Thus, in order to identify whether $S_p(G)$ is pseudo-balanced or not, we need to check the product of signs of the edges incident to the vertices u and v only. By the definition of pseudo-balancing of path-induced signed graph, the edge uv and the edges incident to both u and v are all negatively signed. Thus, the product of signs of edges, which are incident to the vertices u and v (the edge $uv+$ the edges incident to $u+$ the edges incident to v), is a negative number. Hence, $S_p(G)$ can not be a pseudo-balanced graph. □

Theorem 3.4. Let $S_p(G)$ be the path-induced signed graph of a graph G. Then, there is no $S_p(G)$ with exactly one negatively signed edge.

Proof. Let $S_p(G)$ be the path-induced signed graph of a graph G. Assume that $S_p(G)$ contains exactly one negatively signed edge, say uv, where u and v are vertices of G. Then, we have to consider the following cases:

Case 1: Let both u and v be terminal vertices of G. Then, the edge uv must be positively signed. This leads to a contradiction.

Case 2: Let both u and v be internal vertices of G. Then, the edge uv and the edges incident to both u and v are also negatively signed. This contradicts our assumption that there is exactly one negatively signed edge in $S_p(G)$.

Case 3: Let either u or v be an internal vertex and the other be a terminal vertex of the graph G. Without loss of generality, assume u is an internal vertex and v is a terminal vertex. Since G is cyclic, every vertex of G is of degree ≥ 2. Then, there exist another vertex w in $S_p(G)$ such that the edge uw is negatively signed. Hence, there is at least one more negatively signed edge in $S_p(G)$ which is again a contradiction to our assumption.

□

4. Conclusion

In this paper, we investigated some balanced path-induced signed graphs. We introduced pseudo-balancing of path-induced signed graph of a graph G and examined various classes of graphs. We identified few homogeneity conditions and other properties of pseudo-balancing of path-induced signed graphs. Examples of path-induced signed graphs which are neither balanced nor pseudo-balanced are also discussed. There is a lot of scope for the future development of the concepts introduced here.

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