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ON A CLASS OF DEGENERATE FRACTIONAL $p(.)$-LAPLACIAN PROBLEMS WITH VARIABLE ORDER AND VARIABLE EXPONENT

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Abstract. The aim of this paper is to study a class of a degenerate elliptic problem driven by the fractional $p(.)$-Laplacian operator with variable order and variable exponent, the main tool used here is the variational method combined with the theory of variable-order fractional Sobolev spaces with variable exponent.

1. Introduction

This paper is devoted to study the existence and uniqueness question of weak solutions for the fractional $p(x)$-Laplacian problem

$$(1.1)\qquad \begin{cases} u + \left(-\Delta_{p(x)}^{s(x)}\right)(u - \Theta(u)) + \alpha(u) = f(x,u) \quad in\ \Omega, \\ u = 0 \quad on\ \ \partial\Omega, \end{cases}$$

where $(-\Delta)_{p(x)}^{s(x)}$ is the fractional $p(x)$-Laplacian operator with variable order which can be defined as

$$\left(-\Delta_{p(x)}^{s(x)}\right) u(x) = P \cdot V \cdot \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+s(x,y)p(x,y)}} dy, for\ all\ x \in \Omega,$$

and $P.V.$ is a commonly used abbreviation in the principal value sense. Ω is a bounded open domain of $\mathbb{R}^N (N \geq 3)$. $p(.)$ and $s(.)$ are two continuous variable exponents with $s(x,y)p(x,y) < N$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. α is a non decreasing continuous real function defined on $\mathbb{R}$ and Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}$, the datum f is a Carathéodory function satisfying some growth conditions.

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The terminology variable-order fractional Laplace operator indicates that $s(.)$ and $p(.)$ are functions and not real numbers. This operator is then a generalization of the fractional Laplacian $(-\Delta)^s$, which corresponds to $p(.) \equiv 2$ and $s(.) \equiv s \in (0,1)$ constant, and of the p -Laplacian $-\Delta_p$, which corresponds to $p(.) \equiv p \in (1,+\infty)$ constant and $s(.) \equiv 1$.

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. Recently, great attention has been focused on these problems, both for pure mathematical research and in view of concrete real-world applications. Indeed, this type of operator arises in a quite natural way in different contexts, such as the description of several physical phenomena, optimization, population dynamics and mathematical finance. The fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, also provides a simple model to describe some jump Lévy processes in probability theory (see for example [2], [9], [10], [12], [21] and the references therein).

In last years, a large number of papers are written on fractional Sobolev spaces and nonlocal problems driven by this operator (see for example [3], [8], [9], [10], [11], [25], [26] and [27] for further details). Specifically, we refer to Di Nezza, Palatucci and Valdinoci [11], for a full introduction to study the fractional Sobolev spaces and the fractional p-Laplacian operators. On the other hand, attention has been paid to the study of partial differential equations involving the $p(x)$-Laplacian operators (see [14], [15], [16], [17], [19], [22] and the references therein). So the natural question that arises is to see which result can be obtained, if we replace the $p(x)$-Laplacian operator by its fractional version (the fractional $p(x)$-Laplacian operator). Currently, as far as we know, the only results for fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacian operator are obtained by [4], [5], [13], [18] and [31]. In particular, the authors generalized the last operator to fractional case. Then, they introduced an appropriate functional space to study problems in which a fractional variable exponent operator is present. These works are generalized by Reshmi Biswas and Sweta Tiwari in the case of variable order, see[7], they proved interesting properties concerning the spaces of Sobolev with variable order, another work in this direction can be found in [32].

Recently, an always increasing interest has been shown towards non-local problems involving the fractional $p(.)$-Laplacian operator with variable order. In [7], R. Biswas-et-al obtained the multiplicity of weak solutions for a $p(.)$fractional problem with variable order by using the variational method. In [32], Zuo-et-al proved the existence of solutions for the Kirchoff type problems involving the fractional $p(.)$-Laplacian with variable order, in which the critical mountain pass level, combined with a Brzis and Lieb-type lemma for fractional Sobolev spaces with variable order and variable exponent are applied to study this problem. In the case where $p(.) = 2$, Xiang-et-al studied in [28] the existence of multiplicity of solutions for variable-order fractional Kirchhoff equations with nonstandard growth by applying the Nehari manifold approach.

When $p(.) = p$ and $s(.) = s$, Sabri-et-al treated in [23] the problem

$$\begin{cases} (-\Delta)^s_p(u - \Theta(u)) + \alpha(u) = f \quad in\ \Omega, \\ u = 0 \quad on \quad \partial\Omega, \end{cases} \tag{1.2}$$

under the following assumptions,

$\overline{(H_1')}$: α is a continuous function defined on $\mathbb{R}$ such that $\alpha(x).x \geq 0$ and there exists a positive constant λ_1 such that $|\alpha(x)| \leq \lambda_1 |x|^{p-1}$ for all $x \in \mathbb{R}$.

$\overline{(H_2')}$: Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that for all real numbers x, y, we have $|\Theta(x) - \Theta(y)| \leq \lambda_2 |x - y|$, where λ_2 is a real constant such that $0 < \lambda_2 < \frac{1}{2}$.

$\overline{(H_3')}$: $f \in L^\infty(\Omega)$,

They have proved the existence of a weak solution for problem (1.2) by using a special type of operators called the operator of type (M). This problem has been generalized in the case where the exponent p is variable (see[24]) .

Motivated by the above works, we study the problem (1.1), we will show that this problem has a unique weak solution by proving that the operator

$$T\{u\} = \Big\{u \ + \ (-\Delta)^{s(.)}_{p(.)}(u - \Theta(u)) + \alpha(u) - f(x, u)\Big\}$$

satisfies the assertions of Theorem 2.2.

In this paper, we suppose that

(H_1) : α is a non decreasing continuous real function defined on $\mathbb{R}$, and there exists a positive constant λ_1 such that $|\alpha(z)| \leq \lambda_1 |z|^{\overline{p}(x)-1}$ for all $z \in \mathbb{R}$ and $x \in \overline{\Omega}$.

(H_2) : Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that for all real numbers x, y, we have $|\Theta(x) - \Theta(y)| \leq \lambda_2 |x - y|$, where λ_2 is a real constant such that $0 < \lambda_2 < \frac{1}{2}$.

(H_3) : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is nonincreasing with respect to the second variable, i.e.,

$$f(x, t_1) \leq f(x, t_2) \text{ for a.e. } x \in \Omega \text{ and } t_1, t_2 \in \mathbb{R} \text{ with } t_1 \geq t_2,$$

and there exist functions $a \in L^{(p^*_{s(.)})'}(\Omega)$ and $b \in L^{\infty}(\Omega) \cap L^{\gamma(.)}(\Omega)$ such that

$$|f(x,t)| \leq a(x) + b(x)|t|^{\alpha(x)}, \tag{1.3}$$

where $\alpha(x) \in C_+(\bar{\Omega})$ such that $\alpha^- \leq \alpha^+ \leq p^- - 1$ and

$$\gamma(x) = \frac{p^*_{s(.)}(x)}{p^*_{s(.)}(x) - (\alpha(x) + 1)} \text{ for all } x \in \overline{\Omega}.$$

We first give the definition of weak solutions for problem (1.1).

Definition 1.1. A function $u \in X_0$ is called a weak solution to the problem (1.1) if and only if

$$\int_\Omega uv dx + \int_\Omega \int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}(v(x)-v(y))\, dx\, dy + \int_\Omega \alpha(u) v\, dx = \int_\Omega f(x,u) v dx, \tag{1.4}$$

for all $v \in X_0$, where $\psi^u_\Theta(x,y) = u(x) - u(y) - \Theta(u(x)) + \Theta(u(y))$ and X_0 will be introduced in Section 2

Now we are in a position to state the main result as follows:

Theorem 1.1. *Let $p(.)$ and $s(.)$ be two continuous variable exponents satisfying* (2.1), (2.2), (2.3), *and* (2.4) *with $s(x,y)p(x,y) < N$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. If hypotheses $(H_1), (H_2)$ and (H_3) hold, then, the problem* (1.1) *has a unique weak solution.*

2. Preliminaries and notations

In this section, we recall some notations and definitions and we will state some results which will be used in this work.

Let Ω be a smooth bounded open set in $\mathbb{R}^N$, we consider the set

$$C_+(\bar{\Omega}) = \{q \in C(\bar{\Omega}) : 1 < q^- < q(x) < q^+ < \infty \text{ for all } x \in \bar{\Omega}\},$$

where

$$q^- = \inf_{x\in\overline{\Omega}} q(x) \text{ and } q^+ = \sup_{x\in\overline{\Omega}} q(x).$$

For any $q \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(\cdot)}(\Omega) = \left\{ u : \text{ function } u : \Omega \to \mathbb{R} \text{ is measurable with } \int_\Omega |u(x)|^{q(x)} dx < \infty \right\},$$

which is endowed with the so-called Luxemburg norm

$$\|u\|_{q(\cdot)} = \inf \left\{ \gamma > 0 : \int_\Omega \left| \frac{u(x)}{\gamma} \right|^{q(x)} dx \leq 1 \right\}.$$

$\left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)$ is a separable reflexive Banach space see, for example [19].

Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$ and $s : \overline{\Omega} \times \overline{\Omega} \longrightarrow (0, 1)$ be two continuous functions such that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) \leq p(x, y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) < +\infty \tag{2.1}$$

$$0 < s^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} s(x, y) \leq s(x, y) < s^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} s(x, y) < 1 \tag{2.2}$$

$$0 < s^- < s^+ < 1 < p^- \leq p^+ \tag{2.3}$$

We set

$$\overline{p}(x) = p(x, x) \text{ and } \overline{s}(x) = s(x, x) \text{ for all } x \in \overline{\Omega}.$$

We assume that

(2.4)

p and s are symmetric, that is, $p(x, y) = p(y, x)$ and $s(x, y) = s(y, x)$ for all $(x, y) \in \overline{\Omega}\times\overline{\Omega}$.

The variable-order fractional Sobolev spaces with variable exponent via the Gagliardo approach is defined by

$$X = W^{s(\cdot),p(\cdot)}(\Omega)$$

$$= \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}} dxdy < \infty \text{ for some } \gamma > 0 \right\}$$

with the norm $\|u\|_X = \|u\|_{\bar{p}(x)} + [u]_{s(\cdot),p(\cdot)}$, where

$$[u]_{s(\cdot),p(\cdot)} = \inf \left\{ \gamma > 0 : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}} dxdy < 1 \right\}$$

is a Gagliardo seminorm with variable-order and variable exponent. The space X is a separable reflexive Banach space, see [7]. Next we define the subspace X_0 of X as

$$X_0 = X_0^{s(.),p(.)}(\Omega) := \{u \in X : u = 0 \text{ a.e.in } \Omega^c\}$$

endowed by the norm

$$\|u\|_{X_0} := [u]_{s(\cdot),p(\cdot)}.$$

The space X_0 is a separable reflexive Banach space, see [7]. We define the convex modular function $\varrho_{p(\cdot)}^{s(\cdot)} : X_0 \to \mathbb{R}$ by

$$\varrho_{p(\cdot)}^{s(\cdot)}(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy$$

whose associated norm define by

$$\|u\| = \|u\|_{\rho_{p(\cdot)}^{s(\cdot)}} = \inf \left\{ \gamma > 0 : \varrho_{p(\cdot)}^{s(\cdot)} \left\{ \frac{u}{\gamma} \right\} \leq 1 \right\},$$

which is equivalent to the norm $\|\cdot\|_{X_0}$.

Proposition 2.1. [7] Let $u \in X_0$ and $\{u_n\} \subset X_0$, then

(1) $\|u\|_{X_0} < 1(\text{ resp. } = 1, > 1) \iff \rho_{p(\cdot)}^{s(\cdot)}(u) < 1(\text{ resp. } = 1, > 1)$,
(2) $\|u\|_{X_0} < 1 \Rightarrow \|u\|_{X_0}^{p^+} \leq \rho_{p(\cdot)}^{s(\cdot)}(u) \leq \|u\|_{X_0}^{p^-}$,
(3) $\|u\|_{X_0} > 1 \Rightarrow \|u\|_{X_0}^{p^-} \leq \rho_{p(\cdot)}^{s(\cdot)}(u) \leq \|u\|_{X_0}^{p^+}$
(4) $\lim_{n\to\infty} \|u_n\|_{X_0} = 0(\infty) \iff \lim_{n\to\infty} \rho_{p(\cdot)}^{s(\cdot)}(u_n) = 0(\infty)$,
(5) $\lim_{n\to\infty} \|u_n - u\|_{X_0} = 0 \iff \lim_{n\to\infty} \rho_{p(\cdot)}^{s(\cdot)}(u_n - u) = 0$.

Theorem 2.1. [7] Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and let $p(.)$ and $s(.)$ be two continuous variable exponents satisfying (2.1), (2.2), (2.3), and (2.4) with $s(.)p(.) < N$. Assume that $r : \overline{\Omega} \longrightarrow (1, +\infty)$ is a continuous variable exponent such that

$$p_{s(.)}^*(x) = \frac{N\overline{p}(x)}{N - \overline{s}(x)\overline{p}(x)} > r(x) \geq r^- = \min_{x\in\overline{\Omega}} r(x) > 1 \quad \text{for all } x \in \overline{\Omega}.$$

Then, there exists a positive constant $C = C(N, s, p, r, \Omega)$ such that, for any $u \in X_0$

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{X_0}.$$

Thus, the space X_0 is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1, p^*_{s(.)})$. Moreover, this embedding is compact.

Let $q' \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ for all $x \in \overline{\Omega}$, then we have the following Hölder-type inequality :

Lemma 2.1. [17]**(Hölder-type inequality)**. If $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, then
$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{q^-} + \frac{1}{q'^-}\right) \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{q'(.)}(\Omega)} \leq 2\|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{q'(x)}(\Omega)}.$

Definition 2.1. [20] Let Y be a reflexive Banach space and let P be an operator from Y to its dual Y'. We say that P is monotone if and only if

$$\langle Pu - Pv, u - v\rangle \geq 0, \quad \forall\, u, v \in Y.$$

Theorem 2.2. [20] Let Y be a reflexive real Banach space and $P : Y \longrightarrow Y'$ be a bounded operator, hemi-continuous, coercive and monotone on space Y. Then, the equation $Pu = h$ has at least one solution $u \in Y$ for each $h \in Y'$.

We now recall the basic properties of Nemytsky operators in Lebesgue spaces.

Theorem 2.3. [30] Let Ω be a not necessarily bounded domain of $\mathbb{R}^N$, $p_1, p_2 \in [1, +\infty)$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies the growth condition

$$|f(x, s)| \leq a(x) + b(x)|s|^{\frac{p_1}{p_2}}, \quad x \in \Omega,\ s \in \mathbb{R},$$

where $a \in L^{p_2}(\Omega)$ and b is a non-negative function in $L^\infty(\Omega)$. Then the operator N_f from $L^{p_1}(\Omega)$ into $L^{p_2}(\Omega)$ defined by $(N_f u)(x) = f(x, u(x))$ is a bounded and continuous operator.

Lemma 2.2. [1] For $\xi,\ \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.3. *For $a \geq 0$, $b \geq 0$ and $1 \leq p < +\infty$, we have*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

3. Proof of the main result

In this section, we prove the existence and uniqueness of weak solutions to problem (1.1). Our method is based on the variational method and the properties of Nemytsky operators.

Existence part. Let the operator $T : X_0 \longrightarrow (X_0'$ (where $(X_0'$ is the dual space of X_0 and let

$$T = A + L,$$

where for all $u, v \in X_0$

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega\int_\Omega \frac{|\psi_\Theta^u(x,y)|^{p(x,y)-2}\psi_\Theta^u(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}(v(x)-v(y))dxdy + \int_\Omega \alpha(u)v\,dx \\ &:= \langle A_1u, v\rangle \ + \langle A_2u, v\rangle\end{aligned}$$

and

$$\langle Lu, v\rangle = -\int_\Omega f(x,u)v\,dx + \int_\Omega uv\,dx := \langle L_1u, v\rangle \ + \langle L_2u, v\rangle.$$

The proof of existence part of Theorem 1.1 is divided into several Lemmas.

Lemma 3.1. *The operator T is bounded.*

Proof. On the one hand, we use Hölder-type inequality, hypothesis (H_2) and Lemma 2.3, we have for any $u, v \in X_0$,

$$\begin{aligned}|\langle A_1u, v\rangle| &\le \int_\Omega\int_\Omega \frac{|\psi_\Theta^u(x,y)|^{p(x,y)-1}}{|x-y|^{N+s(x,y)p(x,y)}}|v(x)-v(y)|\,dx\,dy \\ &\le 2^{p^+-2}\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|^{p(x,y)-1}}{|x-y|^{N+s(x,y)p(x,y)}} + \frac{|\Theta(u(x))-\Theta(u(y))|^{p(x,y)-1}}{|x-y|^{N+s(x,y)p(x,y)}}\right)\times \\ &\quad |v(x)-v(y)|\,dx\,dy \\ &\le 2^{p^+-2}(\lambda_2^{p^+-1}+1)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-1}}{|x-y|^{N+s(x,y)p(x,y)}}|v(x)-v(y)|\,dx\,dy \\ &\le C_0\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dx\,dy\right)^{\frac{p(x,y)-1}{p(x,y)}} \times \\ &\quad \left(\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dx\,dy\right)^{\frac{1}{p(x,y)}} \\ &\le C_0\|u\|_{X_0}^{p(x,y)-1}\|v\|_{X_0} \\ &\le C_0\max\left(\|u\|_{X_0}^{p^+-1}, \|u\|_{X_0}^{p^--1}\right)\|v\|_{X_0}.\end{aligned}$$

Since $0 < \lambda_2 < \frac{1}{2}$, we have $C_0 = 2^{p^+-1}(\lambda_2^{p^+-1} + 1)$. This implies that A_1 is bounded. On the other hand, using again Hölder-type inequality, hypothesis (H_1) and Theorem 2.1, we get

$$\begin{aligned} |\langle A_2 u, v\rangle| &\leq \lambda_1 \int_\Omega |u|^{\overline{p}(x)-1}|v|\,dx \\ &\leq 2\lambda_1 \|u\|_{\overline{p}(x)}^{\overline{p}(x)-1}\|v\|_{\overline{p}(x)} \\ &\leq 2\lambda_1 C_1 C_2 \|u\|_{X_0}^{\overline{p}(x)-1}\|v\|_{X_0} \\ &\leq 2\lambda_1 C_1 C_2 \max\left(\|u\|_{X_0}^{p^+-1}, \|u\|_{X_0}^{p^--1}\right)\|v\|_{X_0}, \end{aligned}$$

where C_1, C_2 are two constants of the compact embedding given by Theorem 2.1. Then A_2 is bounded. This allows us to deduce that A is bounded. It remains to show that L is bounded, indeed, we have

$$\begin{aligned} \|L_1(u)\|_{X_0'} &= \sup_{\|v\|=1} |\langle L_1(u), v\rangle| = \sup_{\|v\|=1}\left|\int_\Omega f(x,u)v dx\right| \\ &\leq \sup_{\|v\|=1}\int_\Omega |f(x,u)||v|dx \leq \sup_{\|v\|=1}\int_\Omega \left(a_1(x) + b_1(x)|u|^{\alpha(x)}\right)|v|dx \\ &\leq \sup_{\|v\|=1}\left[\|a\|_{(p^*_{s(.)}(x))'}\|v\|_{p^*_{s(.)}(x)} + \||u|^{\alpha(x)}\|_{\frac{p^*_{s(.)}(x)}{\alpha(x)}}\|b\|_{\frac{p^*_{s(.)}(x)}{p^*_{s(.)}(x)-\alpha(x)-1}}\|v\|_{p^*_{s(.)}(x)}\right] \\ &\leq C_3\|a_1\|_{(p^*_{s(.)}(x))'} + C_4 M^{\alpha^+}\|b\|_{\gamma(x)}, \end{aligned}$$

where C_3, C_4 are two constants of the compact embedding given by Theorem 2.1 and $M > 0$ such that $\|u\|_{X_0} \leq M$. Thus, L_1 is bounded. Finally, by Hölder-type inequality, we get immediately the boundedness of L_2. This allows us to say that L is bounded. Hence, T is bounded. □

Lemma 3.2. *The operator T is hemi-continuous.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ and $u \in X_0$ such that u_n converges strongly to u in X_0. Firstly, we will prove that A_1 is continuous on X_0, indeed

$$
\begin{aligned}
&\langle A_1 u_n - A_1 u, v\rangle \\
= \;& \int_\Omega\int_\Omega \left(\frac{|\psi_\Theta^{u_n}(x,y)|^{p(x,y)-2}\psi_\Theta^{u_n}(x,y) - |\psi_\Theta^{u}(x,y)|^{p(x,y)-2}\psi_\Theta^{u}(x,y)}{|x-y|^{N+s(x,y)p(x,y)}} \right) \times \\
&(v(x)-v(y))\,dx\,dy \\
= \;& \int_\Omega\int_\Omega \left(\frac{|\psi_\Theta^{u_n}(x,y)|^{p(x,y)-2}\psi_\Theta^{u_n}(x,y)}{|x-y|^{(N+s(x,y)p(x,y))\frac{p(x,y)-1}{p(x,y)}}} - \frac{|\psi_\Theta^{u}(x,y)|^{p(x,y)-2}\psi_\Theta^{u}(x,y)}{|x-y|^{(N+s(x,y)p(x,y))\frac{p(x,y)-1}{p(x,y)}}} \right) \times \\
&\frac{(v(x)-v(y))}{|x-y|^{\frac{N+s(x,y)p(x,y)}{p(x,y)}}}\,dx\,dy.
\end{aligned}
$$

Let us set

$$
F_{\theta,n}(x,y) = \frac{|\psi_\Theta^{u_n}(x,y)|^{p(x,y)-2}\psi_\Theta^{u_n}(x,y)}{|x-y|^{(N+s(x,y)p(x,y))\frac{p(x,y)-1}{p(x,y)}}} \;\in\; L^{p'(x,y)}(\Omega\times\Omega),
$$

$$
F_{\theta}(x,y) = \frac{|\psi_\Theta^{u}(x,y)|^{p(x,y)-2}\psi_\Theta^{u}(x,y)}{|x-y|^{(N+s(x,y)p(x,y))\frac{p(x,y)-1}{p(x,y)}}} \;\in\; L^{p'(x,y)}(\Omega\times\Omega),
$$

$$
\varphi(x,y) = \frac{(v(x)-v(y))}{|x-y|^{\frac{N+s(x,y)p(x,y)}{p(x,y)}}} \;\in\; L^{p(x,y)}(\Omega\times\Omega),
$$

where $\frac{1}{p(x,y)} + \frac{1}{p'(x,y)} = 1,$ for all $x,y \in \overline{\Omega}\times\overline{\Omega}$.
Then, we have by Hölder-type inequality

$$
\langle A_1 u_n - A_1 u, v\rangle \le 2\|F_{\theta,n} - F_\theta\|_{L^{p'(x,y)}(\Omega\times\Omega)}\|v\|_{L^{p(x,y)}(\Omega\times\Omega)}.
$$

This implies that

$$
\|A_1 u_n - A_1 u\|_{X_0'} = \sup_{\|v\|_{L^{p(x,y)}(\Omega\times\Omega)}\le 1} |\langle A_1 u_n - A_1 u, v\rangle| \le 2\|F_{\theta,n} - F_\theta\|_{L^{p'(x,y)}(\Omega\times\Omega)}.
$$

Now, we denote

$$
Z_{\theta,n}(x,y) = \frac{\psi_\Theta^{u_n}(x,y)}{|x-y|^{\frac{N+s(x,y)p(x,y)}{p(x,y)}}} \in L^{p(x,y)}(\Omega\times\Omega)
$$

$$
Z_{\theta}(x,y) = \frac{\psi_\Theta^{u}(x,y)}{|x-y|^{\frac{N+s(x,y)p(x,y)}{p(x,y)}}} \in L^{p(x,y)}(\Omega\times\Omega).
$$

Since u_n converges to u strongly in X_0, then

$$
Z_{\theta,n}(x,y) \longrightarrow Z_\theta(x,y) \;\text{ in }\; L^{p(x,y)}(\Omega\times\Omega).
$$

Hence, for a subsequence of $Z_{\theta,n}(x,y)$, we get $Z_{\theta,n}(x,y) \longrightarrow Z_{\theta}(x,y)$ in $\Omega\times\Omega$ and there exists an $h \in L^{p(x,y)}(\Omega\times\Omega)$ such that $|Z_{\theta,n}(x,y)| \leq h(x,y)$.
So, we have

$$F_{\theta,n}(x,y) \longrightarrow F_{\theta}(x,y) \text{ a.e in } \Omega\times\Omega$$

and

$$|F_{\theta,n}(x,y)| = |Z_{\theta,n}(x,y)|^{p(x,y)-1} \leq |h(x,y)|^{p(x,y)-1}.$$

Then, by Dominated Convergence Theorem, we deduce that

$$F_{\theta,n}(x,y) \longrightarrow F_{\theta}(x,y) \;\text{ in }\; L^{p'(x,y)}(\Omega\times\Omega).$$

Consequently

$$A_1 u_n \longrightarrow A_1 u \;\text{ in }\; X_0^{'}.$$

This implies that the operator A_1 is continuous on X_0. Secondly, by application of hypothesis (H_1), we get immediately the continuity of operator A_2. Now, we will prove that L is strongly continuous. For that, we show that L_1 is strongly continuous. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n \rightharpoonup u$ in X_0, so $(u_n)_{n\in\mathbb{N}}$ is bounded in X_0. Define, for $k>0$, the set

$$B_k = \{x \in \Omega : |x| < k\}$$

and $\Omega_k = \Omega\backslash B_k$. From (1.3), Hölder inequality and Theorem 2.1, we have

$$\begin{aligned}
&\left|\int_{\Omega_k} \big((f(x,u_n) - f(x,u)\big)v dx\right| \\
\leq\;& \int_{\Omega_k} |(f(x,u_n)||v|dx \;+\; \int_{\Omega_k} |(f(x,u)||v|dx \\
\leq\;& \int_{\Omega_k} \Big(a(x) + b(x)\,|u_n|^{\alpha(x)}\Big)\,|v|dx \;+\; \int_{\Omega_k} \Big(a(x) + b(x)\,|u|^{\alpha(x)}\Big)\,|v|dx \\
\leq\;& 2\|a\|_{(p^*_{s(.)}(x))'}\|v\|_{p^*_{s(.)}(x)} + \||u_n|^{\alpha(x)}\|_{\frac{p^*_{s(.)}(x)}{\alpha(x)}}\|b\|_{\gamma(x)}\|v\|_{p^*_{s(.)}(x)} \\
+\;\;& \||u|^{\alpha(x)}\|_{\frac{p^*_{s(.)}(x)}{\alpha(x)}}\|b)\|_{\gamma(x)}\|v\|_{p^*_{s(.)}(x)} \\
\leq\;& 2C_5\|a\|_{(p^*_{s(.)}(x))'}\|v\|_{X_0} + C_5\Big(\||u_n|^{\alpha(x)}\|_{\frac{p^*_{s(.)}(x)}{\alpha(x)}} + \||u|^{\alpha(x)}\|_{\frac{p^*_{s(.)}(x)}{\alpha(x)}}\Big)\|b_1\|_{\gamma(x)}\|v\|_{X_0},
\end{aligned}$$

where C_5 is the constant of compact embedding given by Theorem 2.1. Then, for k sufficiently large, we get

$$\left|\int_{\Omega_k} \left(f\left(x,u_n\right) - f(x,u)\right) v dx\right| \to 0 \text{ as } n\to\infty. \tag{3.1}$$

By using Theorem 2.1, we obtain the compact embedding

$$X_0(B_k) \hookrightarrow L^{\alpha^-(p^*_{s(.)}(x))'}(B_k)$$

$\left(\text{because } p^*_{s(.)}(x) - \alpha^-\left(p^*_{s(.)}(x)\right)' = p^*_{s(.)}(x)\left(\frac{p^*_{s(.)}(x)-\alpha^- -1}{p^*_{s(.)}(x)-1}\right) \text{ and } p^*_{s(.)}(x) - (\alpha^- + 1) \geq p^*_{s(.)}(x) - p^- > 0\right)$, and then $u_n \to u$ in $L^{\alpha^-(p^*_{s(.)}(x))'}(B_k)$. Hölder-type inequality and Theorem 2.3 allows us to deduce that

$$f(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot)) \text{ in } L^{(p^*_{s(.)}(x))'}(B_k).$$

So

$$\begin{aligned} \left|\int_{B_k}((f(x,u_n) - f(x,u))v\,dx\right| &\leq \|(f(x,u_n) - f(x,u)\|_{L^{(p^*_{s(.)}(x))'}(B_k)}\|v\|_{L^{p^*_{s(.)}(x)}(B_k)} \\ &\leq C_6\|(f(x,u_n) - f(x,u)\|_{L^{(p^*_{s(.)}(x))'}(B_k)}\|v\|_{X_0(B_k)}. \end{aligned}$$

As a result

$$\int_{B_k}(f(x,u_n) - f(x,u))\,v\,dx \to 0, \text{ as } n \to \infty. \tag{3.2}$$

Therefore, from (3.1) and (3.2), we obtain

$$\int_\Omega (f(x,u_n) - f(x,u))\,v\,dx \to 0, \text{ as } n \to \infty.$$

Consequently, L_1 is strongly continuous. Hence T is hemi-continuous on X_0. ☐

Lemma 3.3. *The operator T is coercive.*

Proof. For any $u \in X_0$, we have

$$\begin{aligned} &\langle Tu, u\rangle \\ = &\int_\Omega u^2\,dx + \int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}(u(x)-u(y))\,dx\,dy \\ + &\int_\Omega \alpha(u)u\,dx - \int_\Omega f(x,u)u\,dx \\ \geq &\int_\Omega\int_\Omega \frac{|\psi^u_\Theta(x,y)|^{p(x,y)-2}\psi^u_\Theta(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}(u(x)-u(y))\,dx\,dy \\ + &\int_\Omega \alpha(u)u\,dx - \int_\Omega f(x,u)u\,dx. \end{aligned}$$

Firstly we deal with A, on the one hand, by application of hypothesis (H_1), we have

$$\int_\Omega \alpha(u)u\,dx \geq 0.$$

On the other hand, using Lemma 2.2, we obtain

$$\begin{aligned}
&\langle A_1 u, u\rangle \\
= & \int_\Omega\int_\Omega \frac{|\psi_\Theta^u(x,y)|^{p-2}\psi_\Theta^u(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}(u(x)-u(y))\,dx\,dy \\
\geq & \int_\Omega\int_\Omega \frac{|u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))|^{p(x,y)}-|\Theta(u(x))-\Theta(u(y))|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}\,dx\,dy.
\end{aligned}$$

And Lemma 2.3 allows us to deduce that

$$\begin{aligned}
&\frac{1}{2^{p^+-1}}|u(x)-u(y)|^{p(x,y)} \\
= & \frac{1}{2^{p^+-1}}|u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))+(\Theta(u(x))-\Theta(y))|^{p(x,y)} \\
\leq & |u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))|^{p(x,y)}+|\Theta(u(x))-\Theta(u(y))|^{p(x,y)}.
\end{aligned}$$

Then

$$\frac{1}{2^{p^+-1}}|u(x)-u(y)|^{p(x,y)}-|\Theta(u(x))-\Theta(u(y))|^{p(x,y)} \leq |u(x)-u(y)-(\Theta(u(x))-\Theta(u(y)))|^{p(x,y)}.$$

Consequently

$$\begin{aligned}
&\langle A_1 u, u\rangle \\
\geq & \int_\Omega\int_\Omega \frac{1}{p(x,y)}\left[\frac{1}{2^{p^+-1}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}-\frac{2|\Theta(u(x))-\Theta(u(y))|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\right]dx\,dy \\
\geq & \int_\Omega\int_\Omega \frac{1}{p(x,y)}\frac{1}{2^{p^+-1}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}-\frac{2\lambda_2^{p(x,y)}}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dx\,dy \\
\geq & \frac{1}{p^+}\left(\frac{1}{2^{p^+-1}}-2\lambda_2^{p^+}\right)\|u\|_{X_0}^{p(x,y)} \\
\geq & \frac{1}{p^+}\left(\frac{1}{2^{p^+-1}}-2\lambda_2^{p^+}\right)\min\left(\|u\|_{X_0}^{p^+},\|u\|_{X_0}^{p^-}\right).
\end{aligned}$$

So, the choice of constant λ_2 in (H_2) gives the existence of a positive constant C_6 such that

$$\langle A_1 u, u\rangle \geq C_6\|u\|_{X_0}^{\delta}, \tag{3.3}$$

where

$$\delta = \begin{cases} p^- & \text{if } \|u\|_{X_0} > 1 \\ p^+ & \text{if } \|u\|_{X_0} < 1. \end{cases}$$

Secondly, we have

$$\langle L_1 u, u\rangle = \int_\Omega f(x,u)u\,dx$$

From (1.3), Hölder- type inequality and Theorem 2.1, we have

$$-\int_\Omega f(x,u)u\,dx \geq -\int_\Omega \big(a(x)|u| + b(x)|u|^{\alpha(x)}|u|\big)dx \tag{3.4}$$

$$\geq -\|a\|_{(p^*_{s(.)}(x))'}\|u\|_{p^*_{s(.)}(x)} - \|u\|^{\alpha(x)+1}_{p^*_{s(.)}(x)}\|b\|_{\gamma(x)} \tag{3.5}$$

$$\geq -C_7\|a\|_{(p^*_{s(.)}(x))'}\|u\|_{X_0} - C_8\|u\|^{\alpha(x)+1}_{X_0}\|b\|_{\gamma(x)} \tag{3.6}$$

$$\geq -C_7\|a\|_{(p^*_{s(.)}(x))'}\|u\|_{X_0} - C_8\|u\|^{\mu+1}_{X_0}\|b\|_{\gamma(x)} \tag{3.7}$$

where

$$\mu = \begin{cases} \alpha^- & \text{if } \|u\|_{X_0} > 1 \\ \alpha^+ & \text{if } \|u\|_{X_0} < 1. \end{cases}$$

Then, from (3.3) and (3.7) we get

$$\langle Tu, u\rangle \geq C_6\|u\|^{\delta}_{X_0} - C_7\|a\|_{(p^*_{s(.)}(x))'}\|u\|_{X_0} - C_8\|u\|^{\mu+1}_{X_0}\|b\|_{\gamma(x)}.$$

Since $\alpha^- \leq \alpha^+ \leq p^- - 1$ then $\mu + 1 \leq \delta$. Consequently

$$\frac{\langle Tu, u\rangle}{\|u\|_{X_0}} \to +\infty \text{ as } \|u\|_{X_0} \to +\infty.$$

Hence, the operator T is coercive. □

Lemma 3.4. *The operator T is monotone.*

Proof. We prove that

$$\langle Tu - Tv, u - v\rangle \geq 0 \quad \text{for all } \ u, v \in X_0.$$

Firstly, we have

$$\langle L_1u - L_1v, u - v\rangle = \int_\Omega \big(f(x,u) - f(x,v)\big)(u - v)dx.$$

As f is a nonincreasing function with respect to the second variable, then

$$\int_\Omega \big(f(x,u) - f(x,v)\big)(u - v)dx \leq 0.$$

Therefore

$$\langle L_1u - L_1v, u - v\rangle \leq 0 \quad \text{for all } \ u, v \in X_0. \tag{3.8}$$

Now, we prove that A is monotone. Firstly ,we have by application of hypothesis (H_1) that

$$\langle A_2u - A_2v, u-v\rangle = \int_\Omega \Big(\alpha(u)-\alpha(v)\Big)(u-v)\,dx \geq 0 \quad \text{for all} \;\; u,v\in X_0.$$

It remains to show that $\langle A_1u - A_1v, u-v\rangle \geq 0$. Indeed, we have

$$\begin{aligned}\langle A_1u - A_1v, u-v\rangle &= \langle A_1u, u\rangle + \langle A_1v, v\rangle - \langle A_1u, v\rangle - \langle A_1v, u\rangle\\ &\geq C_6\mathcal{J}_1(u,v) - C_0\mathcal{J}_2(u,v)\\ &\geq \min(C_0, C_6)(\mathcal{J}_1(u,v) - \mathcal{J}_2(u,v)),\end{aligned}$$

where C_0 and C_6 are the two constants getting in the proof of boundedness and coerciveness of the operator T and

$$\mathcal{J}_1(u,v) = \|u\|_{X_0}^{p(x,y)} + \|v\|_{X_0}^{p(x,y)},$$

$$\mathcal{J}_2(u,v) = \|u\|_{X_0}^{p(x,y)-1}\|v\|_{X_0} + \|v\|_{X_0}^{p(x,y)-1}\|u\|_{X_0}.$$

This implies that

(3.9)

$$\begin{aligned}\langle A_1u - A_1v, u-v\rangle &\geq \min(C_0, C_6)\left[\left(\|u\|_{X_0}^{p(x,y)-1} - \|v\|_{X_0}^{p(x,y)-1}\right)\left(\|u\|_{X_0} - \|v\|_{X_0}\right)\right]\\ &\geq 0.\end{aligned}$$

This implies that A_1 is monotone. Therefore T is monotone. □

Hence, the existence of weak solution for problem (1.1) follows from Theorem 2.2.

Uniqueness part. Let u and w be two weak solutions of problem (1.1). As a test function for the solution u, we take $v = u-w$ in the equality (1.4) and for the solution w we take $v = w-u$ as a test function in (1.4), we have

$$\int_\Omega u(u-w)dx + \int_\Omega\int_\Omega \frac{|\psi_\Theta^u(x,y)|^{p(x,y)-2}\psi_\Theta^u(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}\Big(u(x)-u(y)-\big(w(x)-w(y)\big)\Big)\,dx\,dy$$
$$+\int_\Omega \alpha(u)(u-w)\,dx = \int_\Omega f(x,u)(u-w)dx$$

and

$$\int_\Omega w(w-u)dx + \int_\Omega\int_\Omega \frac{|\psi_\Theta^w(x,y)|^{p(x,y)-2}\psi_\Theta^u(x,y)}{|x-y|^{N+s(x,y)p(x,y)}}\Big(w(x)-w(y)-\big(u(x)-u(y)\big)\Big)\,dx\,dy$$
$$+\int_\Omega \alpha(w)(w-u)\,dx = \int_\Omega f(x,w)(w-u)dx.$$

By summing up the two above equalities, we get

(3.10)
$$\int_\Omega (u-w)^2 dx + \langle A_1u - A_1w, u-w\rangle + \int_\Omega \Big(\alpha(u)-\alpha(w)\Big)(u-w)\,dx = \langle L_1u - L_1w, u-w\rangle.$$

On the one hand, we have by application of hypothesis (H_1) that

$$\int_\Omega \Big(\alpha(u)-\alpha(w)\Big)(u-w)\,dx \geq 0.$$

On the other hand, by using (3.9), we deduce that

$$\langle A_1u - A_1w, u-w\rangle \geq 0.$$

And by (3.8) we get

$$\langle L_1u - L_1w, u-w\rangle \leq 0.$$

Therefore, the equality (3.10) becomes

$$\int_\Omega (u-w)^2 dx \leq 0.$$

This implies that

$$u = w \ \text{ a.e in } \ \Omega.$$

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