Jordan Journal of Mathematics and Statistics (JJMS), 16(3), 2023, pp 463 - 482
DOI: https://doi.org/10.47013/16.3.6

ON HOLOMORPH OF WIP PACC LOOPS

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Abstract. This work investigates the holomorph of a weak inverse property power associative conjugacy closed (WIP PACC) loop. It is shown that the holomorph of a WIP PACC loop is WIP PACC. If Q is a WIP PACC loop and A is the automorphism group of Q, then each $\theta \in A$ is a nuclear automorphism. The $A(Q)$ holomorph of a WIP PACC loop is shown to satisfy the doubly weak inverse property. A necessary and sufficient condition for the holomorph of an arbitrary loop and its automorphism group to produce a WIP PACC loop is established. Finally, if Q is a LWPC (RWPC) loop with $x \in N_\mu(Q)$, then the holomorph of Q is an extra loop.

1. Introduction

George and Jaiyeola in [8] discovered twelve new loop identities of length five which they christened loops of second Bol-Moufang type using a generalized and modified nuclear identification model originally introduced by Drápal and Jedlička [6]. Coincidentally, two of these loops (Q_3 and Q_4) have already appeared in literature in the work of Phillips [15]. Phillips labeled the two identities as LWPC and RWPC and showed that both identities are equivalent to weak inverse property power associative conjugacy closed loops.

Over the years, the theory of holomorph of loops has gained considerable attention. Bruck [2], Robinson [16] and [17], Huthnance [9], Chiboka and Solarin [4], Adeniran et. al. [1], Isere et. al. [11] and Ilojide et.al. [10] have respectively studied the holomorphs of inverse property loop, Bol and extra loops, weak inverse property loop,

2010 *Mathematics Subject Classification.* 20N02, 20N05.
Key words and phrases. left (right) conjugacy closed loop, power associative, weak inverse property, LWPC-loop, RWPC-loop, WIP PACC loop, holomorph.

Received: June 28, 2022 Accepted: Jan. 25, 2023 .

conjugacy closed loops, generalized Bol loops, Osborn loops and on holomorphy of Fenyves BCI-Algebras.
The objectives of this paper is to investigate the holomorph of weak inverse property power associative conjugacy closed loops.

2. Preliminaries

A loop Q is a set with a binary operation $\cdot$ such that for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions x and y in Q and there exists $e \in Q$ such that $ex = xe = x$ for any $x \in Q$. The unique solutions are given by $x = a \setminus b$ and $y = b/a$.
For any $x \in Q$, define the right translation map $R(x)$ and left translation map $L(x)$ of x in $(Q, \cdot)$ by $yR(x) = y \cdot x = yx$ and $yL(x) = x \cdot y = xy$, respectively. It is clear that $(Q, \cdot)$ is a loop if and only if the left and right translation maps are bijections. Since the translation maps are bijections, then the inverse maps $R^{-1}(x)$ and $L^{-1}(x)$ exist and are thus defined by $yR^{-1}(x) = y/x$ and $yL^{-1}(x) = x\backslash y$.

The reader can consult the books [3, 14] for a general background in loop theory.

In a loop Q with identity 1, for all $x \in Q$ there exists a unique right inverse element x^ρ and a unique left inverse element x^λ such that $xx^\rho = x^\lambda x = 1$. Note that x^ρ is not always equal to x^λ in a loop, when they are equal we write $x^\rho = x^\lambda = x^{-1}$ and say the loop has two sided inverse.
A loop Q satisfies the left inverse property (LIP) if $x^\lambda \cdot xy = y$ and the right inverse property (RIP) if $xy \cdot y^\rho = x$. An inverse property loop is a loop that satisfies both the (LIP) and the (RIP). The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop Q are defined by

$$N_\lambda(Q) = \{a \in Q : \ a \cdot xy = ax \cdot y \ \ \forall \ x, \ y \in Q\},$$
$$N_\mu(Q) = \{a \in Q : \ xa \cdot y = x \cdot ay \ \ \forall \ x, \ y \in Q\},$$
$$N_\rho(Q) = \{a \in Q : \ xy \cdot a = x \cdot ya \ \ \forall \ x, \ y \in Q\}.$$

The intersection

$$N(Q) = N_\rho(Q) \cap N_\lambda(Q) \cap N_\mu(Q)$$

is called the nucleus of Q.

A triple of permutations, $(U,\ V,\ W)$ on a loop Q, is called an autotopism of Q provided that

$$xU \cdot yV = (xy)W. \tag{2.1}$$

for all $x,\ y \in Q$. The set Atp(Q) of all autotopisms of Q is a group under composition. It is easy to see that

$$a \in N_\lambda(Q) \Leftrightarrow (L(a),\ I,\ L(a)) \in Atp(Q) \tag{2.2}$$

$$a \in N_\mu(Q) \Leftrightarrow (R^{-1}(a),\ L(a),\ I) \in Atp(Q) \tag{2.3}$$

$$a \in N_\rho(Q) \Leftrightarrow (I,\ R(a),\ R(a)) \in Atp(Q) \tag{2.4}$$

Conjugacy closed loops (CC-loop) are loops satisfying the following two identities:

$$zy \cdot x = zx \ \cdot x\backslash(yx). \qquad (RCC)$$

$$x \cdot yz = (xy)/x \ \cdot xz. \qquad (LCC)$$

A loop $(Q, \cdot)$ is called an extra loop if it satisfies any of the following for all $x, y, z \in Q$:

$$xy \cdot xz = x(yx \cdot z). \tag{2.5}$$

$$yx \cdot zx = (y \cdot xz)x. \tag{2.6}$$

A loop is a weak inverse property loop if it satisfies any one of the equivalent identities:

$$x(yx)^\rho = y^\rho \quad or \quad (xy)^\lambda x = y^\lambda. \tag{2.7}$$

A loop $(Q,\ \cdot)$ is power associative if subloops generated by singletons are groups. This is easily seen to be equivalent to $x^{m+n} = x^m \cdot x^n$ for every $x \in Q$ and for all $m,\ n \in Z$.

An identity of length five is said to be of second Bol-Moufang type if:

(1) It has 3 distinct variables with one of them appearing 3 times.

(2) The variables appear in the same order on both sides.

Definition 2.1. [2] Let A be a group of automorphisms of a loop Q. Let $H = A \times Q$, define $'\circ'$ on H by

$$(\alpha,\ x) \circ (\beta,\ y) = (\alpha\beta,\ x\beta \cdot y) \tag{2.8}$$

for all $(\alpha,\ x),\ (\beta,\ y) \in H$, then $(H,\ \circ)$ is a loop, called the $A(Q)$ holomorph of Q.

Theorem 2.1. [15]. *A loop is WIP PACC if and only if it satisfies*

$$(xy \cdot x)\ \cdot xz = x((yx \cdot x)z). \qquad (LWPC)$$

$$zx\ \cdot (x \cdot yx) = (z(x \cdot xy))x. \qquad (RWPC)$$

3. Main Results

Proposition 3.1. *Let Q be a LWPC (RWPC) loop. Then $x^{\lambda} = x^{\rho} = x^{-1}$.*

Proof. Put $z = 1$ and $y = x^{\lambda}$ in LWPC (RWPC) identity. □

The following lemma gives an autotopism characterization of a WIP PACC loop.

Lemma 3.1. *Let $(Q,\ \cdot)$ be a loop.*
(a) $(Q,\ \cdot)$ is a LWPC loop $\Leftrightarrow A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x)) \in\ AtpQ$
(b) $(Q,\ \cdot)$ is a RWPC loop $\Leftrightarrow B(x) = (R(x),\ L^{-2}(x)R(x)L(x),\ R(x)) \in\ AtpQ$.

Proof. (a) Let Q be a LWPC loop, then Q satisfies the LWPC identity.
Put $y = (y/x)/x$ in the LWPC identity to obtain

$$(x((y/x)/x))x \cdot xz = x(yz),$$

$$\Rightarrow yR^{-2}(x)L(x)R(x) \cdot zL(x) = (yz)L(x).$$

Thus,

$$A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$$

is an autotopism of Q.
Conversely, if $A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$ is an autotopism of Q.
Applying this autotopism to the product yz, to obtain

$$yR^{-2}(x)L(x)R(x) \cdot zL(x) = (yz)L(x),$$

put $y = yx \cdot x$, to obtain the LWPC identity.

(b) Replace $y = x\backslash(x\backslash y)$ in the RWPC identity to obtain

$$zx \cdot x((x\backslash(x\backslash y))x) = zy \cdot x,$$

$$\Rightarrow zR(x) \cdot yL^{-2}(x)R(x)L(x) = (zy)R(x)$$

Thus, $B(x) = (R(x),\ L^{-2}(x)R(x)L(x),\ R(x))$ is an autotopism of Q.

Conversely, if $B(x) = (R(x),\ L^{-2}(x)R(x)L(x),\ R(x))$ is an autotopism of Q. Applying this autotopism to the product zy and substitute $y = x \cdot xy$ to get the RWPC identity. □

Theorem 3.1. *Let Q be a LWPC loop. The $A(Q)$ holomorph of $(Q,\ \cdot)$ is LWPC if and only if*

$$((x\beta)y \cdot (x\alpha)) \cdot xz = (x\beta)((y(x\alpha) \cdot x)z). \tag{3.1}$$

Proof. Suppose $A(Q)$ is a LWPC loop. Then by Lemma 3.1 (a), the triple

$$(R^{-2}(\bar{x})L(\bar{x})R(\bar{x})),\ L(\bar{x}),\ L(\bar{x}))$$

is an autotopism of $A(Q)$ for every $\bar{x} = (\alpha,\ x)$ in $A(Q)$.

Let $\bar{y} = (\beta,\ y)$ and $\bar{z} = (\delta,\ z)$ be in $A(Q)$.

Then we have

$$\bar{y}R^{-2}(\bar{x})L(\bar{x})R(\bar{x}) \cdot \bar{z}L(\bar{x}) = (\bar{y}\bar{z})L(\bar{x}) \tag{3.2}$$

We first calculate $\bar{y}R^{-2}(\bar{x})$.

Let $\bar{y}R^{-2}(\bar{x}) = \bar{t} = (\epsilon,\ t)$ for some $t \in Q$ and $\epsilon \in A$. Then

$$\bar{y} = \bar{t}R(\bar{x})R(\bar{x}) = \bar{t}(\bar{x})(\bar{x})$$

$$= [(\epsilon,\ t)(\alpha,\ x)](\alpha,\ x)$$

$$= (\epsilon\alpha^2,\ (t\alpha^2 \cdot x\alpha) \cdot x).$$

Then

$$(\beta,\ y) = (\epsilon\alpha^2,\ (t\alpha^2 \cdot x\alpha) \cdot x)$$

implies

$$\epsilon = \beta\alpha^{-2},$$

and

$$t = yR^{-1}(x)R^{-1}(x\alpha)\alpha^{-2}.$$

So that

$$\bar{y}R^{-2}(\bar{x}) = (\beta\alpha^{-2},\ ((y/x)/(x\alpha))\alpha^{-2}).$$

Now

$$\begin{aligned}
&\bar{y}R^{-2}(\bar{x})L(\bar{x})R(\bar{x}) \\
&\qquad = [(\alpha,\ x)(\beta\alpha^{-2},\ ((y/x)/(x\alpha))\alpha^{-2})](\alpha,\ x) \\
&\qquad = ((\alpha\beta\alpha^{-2},\ \ (x\beta\cdot((y/x)/(x\alpha))\alpha^{-2})(\alpha,\ x) \\
&\qquad = (\alpha\beta\alpha^{-1},\ \ (x\beta\cdot((y/x)/(x\alpha)))\alpha^{-1}\cdot x).
\end{aligned}$$

$$\bar{z}L(\bar{x}) = (\alpha\delta,\ x\delta\cdot z).$$

Therefore,

$$\begin{aligned}
&\bar{y}R^{-2}(\bar{x})L(\bar{x})R(\bar{x})\cdot\bar{z}L(\bar{x}) \\
&\qquad = (\alpha\beta\alpha^{-1},\ \ (x\beta\cdot((y/x)/(x\alpha)))\alpha^{-1}\cdot x)(\alpha\delta,\ x\delta\cdot z) \\
&\qquad = (\alpha\beta\delta,\ (x\beta\cdot((y/x)/(x\alpha)))\delta\cdot x\alpha\delta)\cdot(x\delta\cdot z).
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
(\bar{y}\bar{z})L(\bar{x}) &= (\alpha,\ x)[(\beta,\ y)(\delta,\ z)] \\
&= (\alpha,\ x)(\beta\delta,\ y\delta\cdot z) \\
&= (\alpha\beta\delta,\ x\beta\delta\cdot(y\delta\cdot z)).
\end{aligned} \tag{3.4}$$

From equations (3.3) and (3.4), we have

$$((x\beta((y/x)/(x\alpha)))\delta\cdot x\alpha\delta)\cdot(x\delta\cdot z)) = (x\beta\delta\cdot(y\delta\cdot z)),$$

or

$$((x\beta((y/x)/(x\alpha)))x\alpha)\cdot xz\delta^{-1} = (x\beta\cdot yz\delta^{-1}).$$

Set $y = y(x\alpha)\cdot x$ and $z = z\delta$ so that

$$((x\beta)y\cdot(x\alpha))\cdot xz = (x\beta)((y(x\alpha)\cdot x)z).$$

□

Corollary 3.1. *Let Q be a LWPC loop. Then the holomorph $A(Q)$ of Q is a LWPC loop if and only if*

$$(R^{-1}(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ \ L(x),\ \ L(x\beta)) \tag{3.5}$$

is an autotopism of Q, for all $x,\ y,\ z \in Q$ and $\alpha,\ \beta \in A$.

Theorem 3.2. *Let Q be a RWPC loop. The $A(Q)$ holomorph of $(Q,\ \cdot)$ is RWPC if and only if*

$$yx \cdot ((x\alpha) \cdot z(x\beta)) = (y(x \cdot (x\alpha)z))(x\beta) \tag{3.6}$$

Proof. Let Q be a RWPC loop, then from Lemma (3.1) (b),

$$(R(\bar{x}),\ L^{-2}(\bar{x})R(\bar{x})L(\bar{x}),\ R(\bar{x}))$$

is an autotopism of $A(Q)$ for every $\bar{x} = (\alpha,\ x)$ in $A(Q)$.
Let $\bar{y} = (\beta,\ y)$ and $\bar{z} = (\delta,\ z)$ be in $A(Q)$.
Then $\bar{y}R(\bar{x}) \cdot \bar{z}L^{-2}(\bar{x})R(\bar{x})L(\bar{x}) = (\bar{y}\bar{z})R(\bar{x})$.

$$\bar{y}R(\bar{x}) = (\beta,\ y)(\alpha,\ x) = (\beta\alpha,\ y\alpha \cdot x).$$
$$\bar{z}L^{-2}(\bar{x}) = (\alpha^{-2}\delta,\ (x\alpha^{-2}\delta)\backslash((x\alpha^{-1}\delta)\backslash z)).$$

$$\begin{aligned}\bar{z}L^{-2}(\bar{x})R(\bar{x})L(\bar{x}) &= (\alpha,\ x)(\alpha^{-2}\delta,\ ((x\alpha^{-2}\delta)\backslash((x\alpha^{-1}\delta)\backslash z))(\alpha,\ x)),\\ &= (\alpha,\ x)((\alpha^{-2}\delta\alpha,\ ((x\alpha^{-2}\delta)\backslash((x\alpha^{-1}\delta)\backslash z))\alpha \cdot x)),\\ &= (\alpha^{-1}\delta\alpha,\ x\alpha^{-2}\delta\alpha \cdot ((x\alpha^{-2}\delta)\backslash((x\alpha^{-1}\delta)\backslash z))\alpha \cdot x).\end{aligned}$$

$$\begin{aligned}&\bar{y}R(\bar{x}) \cdot \bar{z}L^{-2}(\bar{x})R(\bar{x})L(\bar{x}) \\ &= (\beta\alpha,\ y\alpha \cdot x)(\alpha^{-1}\delta\alpha,\ x\alpha^{-2}\delta\alpha \cdot ((x\alpha^{-2}\delta)\backslash((x\alpha^{-1}\delta)\backslash z))\alpha \cdot x),\\ &= (\beta\delta\alpha,\ (y\alpha \cdot x)\alpha^{-1}\delta\alpha \cdot (x\alpha^{-2}\delta\alpha \cdot ((x\alpha^{-2}\delta)\backslash((x\alpha^{-1}\delta)\backslash z))\alpha \cdot x)),\end{aligned} \tag{3.7}$$

$$= (\beta\delta\alpha,\ ((y\delta \cdot x\alpha^{-1}\delta) \cdot (x\alpha^{-2}\delta \cdot (x\alpha^{-2}\delta\backslash((x\alpha^{-1}\delta)\backslash z))\alpha \cdot x\alpha^{-1}))\alpha). \tag{3.8}$$

$$\bar{y}\bar{z}R(\bar{x}) = (\beta\delta\alpha,\ ((y\delta \cdot z) \cdot x\alpha^{-1})\alpha). \tag{3.9}$$

Compare equations (3.8) and (3.9) and set $y = y\delta^{-1}$, $x = x\delta^{-1}\alpha$ to get

$$yx \cdot (x\delta^{-1}\alpha^{-1}\delta)(((x\delta^{-1}\alpha^{-1}\delta)\backslash(x\backslash z))(x\delta^{-1})) = yz \cdot x\delta^{-1}.$$

Replacing $\delta^{-1}\alpha^{-1}\delta$ by θ, then $\delta^{-1} = \delta$ and $z = x \cdot (x\theta)z$, we have

$$yx \cdot (x\theta \cdot z(x\delta)) = (y(x \cdot (x\theta)z))x\delta.$$

□

Corollary 3.2. *Let Q be a RWPC loop. Then the holomorph $A(Q)$ is a RWPC loop if and only if*

$$(R(x),\ \ L^{-1}(x)L^{-1}(x\theta)R(x\delta)L(x\theta),\ \ R(x\delta)) \tag{3.10}$$

is an autotopism of Q, for all $x,\ y,\ z \in Q$ and $\theta,\ \delta \in A$.

Corollary 3.3. *Let Q be a loop. Then the holomorph $A(Q)$ is WIP PACC if and only if $((x\beta)y \cdot (x\alpha)) \cdot xz = (x\beta)((y(x\alpha) \cdot x)z)$ and $yx \cdot ((x\alpha) \cdot z(x\beta)) = (y(x \cdot (x\alpha)z))(x\beta)$.*

Theorem 3.3. *Let Q be a loop and let $A(Q)$ be the holomorph of Q. Then:*
(i) $A(Q)$ is left conjugacy closed if and only if

$$((x\beta)y)/x \cdot xz = (x\beta) \cdot yz \text{ for } x,\ y \in Q,\ \text{and } \beta \in A. \tag{3.11}$$

(ii) $A(Q)$ is right conjugacy closed if and only if

$$yx \cdot x\backslash(z(x\beta)) = (yz)(x\beta) \text{for } x,\ y \in Q,\ \text{and } \beta \in A. \tag{3.12}$$

Proof. (i) Let $(\alpha,\ x)$, $(\beta,\ y)$ and $(\gamma,\ z)$ be in $A(Q)$. From the definition of holomorph in (2.8), we see that

$$(\alpha,\ x)/(\beta,\ y) = (\alpha\beta^{-1},\ (x/y)\beta^{-1}). \tag{3.13}$$

Now using (2.8) and (3.13) on the right hand side of the LCC:

$$
\begin{aligned}
&(((\alpha,\ x)(\beta,\ y))/(\alpha,\ x))\cdot(\alpha,\ x)(\gamma,\ z),\\
&=(\alpha\beta,\ x\beta\cdot y)/(\alpha,\ x)\cdot(\alpha\gamma,\ x\gamma\cdot z),\\
&=(\beta,\ ((x\beta\cdot y)/x\cdot\alpha^{-1})\cdot(\alpha\gamma,\ x\gamma\cdot z),\\
&=(\beta,\ ((xy\beta^{-1})/x\beta^{-1})\beta\cdot\alpha^{-1})(\alpha\gamma,\ x\gamma\cdot z),\\
&=(\beta\alpha\gamma,\ (xy\beta^{-1})/x\beta^{-1})\beta\cdot\alpha^{-1})\alpha\gamma)\cdot x\gamma\cdot z),\\
&=(\beta\alpha\gamma,\ (xy\beta^{-1})/x\beta^{-1})\beta\cdot)\gamma)\cdot x\gamma\cdot z),
\end{aligned}
$$

$$=(\alpha\beta\gamma,\ (xy\beta^{-1})/x\beta^{-1})\beta\cdot(xz\gamma^{-1}))\gamma). \tag{3.14}$$

Using (2.8), the L.H.S of the LCC gives

$$(\alpha\beta\gamma,\ (x\beta\cdot yz\gamma^{-1})\gamma). \tag{3.15}$$

Comparing (3.14) and (3.15), we have

$$
\begin{aligned}
x\beta\cdot yz\gamma^{-1}&=(xy\beta^{-1})/x\beta^{-1})\beta\cdot(xz\gamma^{-1})),\\
&=(((x\beta)y)/x)\beta^{-1}\beta\cdot xz\gamma^{-1},\\
&=((x\beta\cdot y)/x)\cdot xz\gamma^{-1},\\
x\beta\cdot yz&=(x\beta\cdot y)/x\cdot xz.
\end{aligned}
$$

(ii) This can be proved in a similar way as 1. $\square$

Corollary 3.4. *Let Q be a loop. Then $A(Q)$ holomorph of Q is conjugacy closed if and only if*

$$((x\beta)y)/x\cdot xz=(x\beta)\cdot yz, \tag{3.16}$$

and

$$yx\cdot x\backslash(z(x\beta))=(yz)(x\beta) \tag{3.17}$$

for $x,\ y\in Q,$ and $\beta\in A$.

Lemma 3.2. *Let $A(Q)$ be the holomorph of a LWPC loop Q. Then the following identities hold:*

(i) $L(x\beta)R(x\alpha)R(x)=R(x\alpha)R(x)L(x\beta)$,

(ii) $R^{-1}(x\alpha)L(x\beta)R(x\alpha) = R(x)L(x\beta)R^{-1}(x)$,

(iii) $L(x\beta) = R(x\alpha)R(x)L(x\beta)R^{-1}(x)R^{-1}(x\alpha) = R^{-1}(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha)R(x)$.

Lemma 3.3. *Let $A(Q)$ be the holomorph of a RWPC loop Q. Then the following identities hold:*

(i) $R(x\beta)L(x\alpha)L(x) = L(x\alpha)L(x)R(x\beta)$,

(ii) $L^{-1}(x\alpha)R(x\beta)L(x\alpha) = L(x)R(x\beta)L^{-1}(x)$,

(iii) $R(x\beta) = L(x\alpha)L(x)R(x\beta)L^{-1}(x)R^{-1}(x\alpha) = L^{-1}(x)L^{-1}(x\alpha)R(x\beta)L(x\alpha)L(x)$.

Theorem 3.4. *Let Q be a loop and let $A(Q)$ be the holomorph of Q.*

(a) $A(Q)$ is a LWPC loop $\Leftrightarrow$ $A(Q)$ is a LCC loop and $((x\beta)y\cdot x\alpha)x = (x\beta)(y(x\alpha)\cdot x)$.

(b) $A(Q)$is a RWPC loop $\Leftrightarrow$ $A(Q)$ is a RCC loop and $x((x\alpha)\cdot y(x\beta)) = (x\cdot(x\alpha)y)(x\beta)$.

Proof. (a) Let $A(Q)$ be a LWPC loop, then by Lemma 3.2 (i), $A(Q)$ satisfies

$$((x\beta)y\cdot x\alpha)x = (x\beta)(y(x\alpha)\cdot x)$$

or

$$y(x\alpha)\cdot x = (x\beta)\backslash(((x\beta)y\cdot x\alpha)x),$$

using this, $A(Q)$ in (3.1) becomes

$$((x\beta)y\cdot x\alpha)\cdot xz = x\beta(((x\beta)\backslash(((x\beta)y\cdot x\alpha)x))z) \Rightarrow$$

$$y\cdot xz = x\beta(((x\beta)\backslash(yx))z) \Rightarrow$$

$$((x\beta)y)/x\ \cdot xz = x\beta(yz).$$

For the converse, suppose $A(Q)$ is an LCC-loop and $((x\beta)y\cdot x\alpha)x = (x\beta)(y(x\alpha)\cdot x)$. From (3.11), we have

$$y\cdot xz = x\beta(((x\beta)\backslash(yx))z) \Rightarrow$$

$$((x\beta)y\cdot x\alpha)\cdot xz = x\beta(((x\beta)\backslash(((x\beta)y\cdot x\alpha)x))z.$$

This last identity becomes LWPC since $A(Q)$ also satisfies $((x\beta)y\cdot x\alpha)x = (x\beta)((y(x\alpha)\cdot x)$.

(b) We can mirror the prove in (a) above. □

Corollary 3.5. *The loop $A(Q)$ is a WIP PACC-loop if and only if $A(Q)$ is a CC-loop and both $((x\beta)y \cdot x\alpha)x = (x\beta)(y(x\alpha) \cdot x)$ and $x(x\alpha \cdot y(x\beta)) = ((x \cdot (x\alpha)y))x\beta$ are satisfied.*

Definition 3.1. A loop Q is said to have the doubly weak inverse property if it satisfies the following equivalent identities

$$(xy)^\rho[(x^\rho)^\rho] = y^\rho, \quad (x^\lambda)^\lambda \cdot (yx)^\lambda = y^\lambda. \tag{3.18}$$

Theorem 3.5. *Let Q be a doubly weak inverse property loop and let A be the automorphisms group of Q. Then the holomorph $A(Q)$ is a doubly weak inverse property loop.*

Proof. Let $(\alpha,\ x),\ (\beta,\ y) \in A(Q)$.
First note that $(\alpha,\ x)^\rho = (\alpha^{-1},\ x^\rho\alpha^{-1})$.
Now

$$\begin{aligned}
[(\alpha,\ x)(\beta,\ y)]^\rho[(\alpha,\ x)^\rho]^\rho &= (\alpha\beta,\ x\beta \cdot y)^\rho(\alpha^{-1},\ x^\rho\alpha^{-1})^\rho \\
&= (\beta^{-1}\alpha^{-1},\ (x\beta \cdot y)^\rho\beta^{-1}\alpha^{-1})(\alpha,\ (x^\rho)^\rho) \\
&= (\beta^{-1}\alpha^{-1},\ (x\alpha^{-1} \cdot y\beta^{-1}\alpha^{-1})^\rho)(\alpha,\ (x^\rho)^\rho) \\
&= (\beta^{-1}\alpha^{-1},\ (x \cdot y\beta^{-1})^\rho\alpha^{-1})(\alpha,\ (x^\rho)^\rho) \\
&= (\beta^{-1},\ (x \cdot y\beta^{-1})^\rho \cdot (x^\rho)^\rho),\ \text{(since Q is doubly weak inverse)} \\
&= (\beta^{-1},\ (y\beta^{-1})^\rho) \\
&= (\beta^{-1},\ y^\rho\beta^{-1}) \\
&= (\beta,\ y)^\rho.
\end{aligned}$$

□

In a loop $A(Q)$ with identity element $(i,\ 1)$, for all $(\alpha,\ x) \in A(Q)$ there exist a unique left inverse $(\alpha,\ x)^\lambda$ and a unique right inverse $(\alpha,\ x)^\rho$ such that $(\alpha,\ x)(\alpha\ x)^\rho = (\alpha,\ x)^\lambda(\alpha,\ x) = (i,\ 1)$.
The following lemma is obvious from definition.

Lemma 3.4. *Let Q be a loop and let $A(Q)$ be the holomorph of Q. Then Q is a loop with two sided inverse if and only if $A(Q)$ is a loop with two sided inverse.*

Proposition 3.2. *Let $A(Q)$ be the holomorph of a loop with two sided inverse. Then $A(Q)$ is a weak inverse property loop if and only if $A(Q)$ is a doubly weak inverse property loop.*

Proof. Suppose $A(Q)$ is a WIP loop, then

$$[(\alpha,\ x)(\beta,\ y)]^{\lambda}(\alpha,\ x) = (\beta,\ y)^{\lambda},$$

$$[(\alpha,\ x)(\beta,\ y)]^{\lambda}[(\alpha,\ x)^{\lambda}]^{\rho} = (\beta,\ y)^{\lambda}, \tag{3.19}$$

then replace $(\beta,\ y)^{\lambda}$ with $(\beta,\ y)^{\rho}$ for every $(\beta,\ y)^{\lambda} \in A(Q)$, then

$$[(\alpha,\ x)(\beta,\ y)]^{\rho}[(\alpha,\ x)^{\rho}]^{\rho} = (\beta,\ y)^{\rho}.$$

Thus, $A(Q)$ has the doubly weak inverse property.
Conversely, suppose $A(Q)$ has the doubly weak inverse property, i.e,

$$[(\alpha,\ x)(\beta,\ y)]^{\rho}[(\alpha,\ x)^{\rho}]^{\rho} = (\beta,\ y)^{\rho}, \tag{3.20}$$

set $(\beta,\ y)^{\rho} = (\beta,\ y)^{\lambda}$ for every $(\beta,\ y)^{\rho} \in A(Q)$,

$$[(\alpha,\ x)(\beta,\ y)]^{\lambda}[(\alpha,\ x)^{\lambda}]^{\rho} = (\beta,\ y)^{\lambda},$$

$$[(\alpha,\ x)(\beta,\ y)]^{\lambda}(\alpha,\ x) = (\beta,\ y)^{\lambda}.$$

Thus, $A(Q)$ has WIP. □

Corollary 3.6. *Let $A(Q)$ be the holomorph of a WIP PACC loop Q. Then $A(Q)$ has the doubly weak inverse property.*

Definition 3.2. An automorphism θ of a loop Q is nuclear if and only if $x\theta \cdot x^{-1} \in N$ or $x^{-1} \cdot x\theta \in N$.

We now show that the set of automorphisms of a WIP PACC loop is nuclear.

Theorem 3.6. *Let Q be an LWPC loop and let A be the the group of automorphisms of Q. Then $A(Q)$ is left nuclear.*

Proof. Let Q be an LWPC loop, then from Lemma 3.1,

$$A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$$

is an autotopism of Q.

Also, from Corollary 3.1,

$$(R^{-1}(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ \ L(x),\ \ L(x\beta))$$

is an autotopism of Q.

Since the set of autotopisms of a loop forms a group under componentwise multiplication, then

$$\begin{aligned}&(R^{-2}(x)L(x)R(x),\ L(x),\ L(x))^{-1}(R^{-1}(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ \ L(x),\ \ L(x\beta))\\ &= (R^{-1}(x)L^{-1}(x)R(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ I,\ L^{-1}(x)L(x\beta)).\end{aligned}\tag{3.21}$$

Let $U = R^{-1}(x)L^{-1}(x)R(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha)$, then the autotopism in (3.21) becomes

$$(U,\ I,\ L^{-1}(x)L(x\beta)).\tag{3.22}$$

Applying the autotopism in (3.22) to the product ab,

$$aU \cdot b = (ab)L^{-1}(x)L(x\beta),$$

with $b = 1$, we see that $U = L^{-1}(x)L(x\beta)$, so that the autotopism in (3.22) becomes

$$(L^{-1}(x)L(x\beta),\ I,\ L^{-1}(x)L(x\beta))\tag{3.23}$$

for all $x \in Q$ and all $\beta \in A$.

Then

$$yL^{-1}(x)L(x\beta) \cdot z = (yz)L^{-1}(x)L(x\beta),$$

for all $y,\ z \in Q$.

In particular, for $y = 1$, we have

$$zL((x\beta) \cdot x^{\rho}) = zL^{-1}(x)L(x\beta)$$

$$zL(x\beta \cdot x^{-1}) = zL^{-1}(x)L(x\beta)$$

Then the autotopism

$$(L^{-1}(x)L(x\beta),\ I,\ L^{-1}(x)L(x\beta))$$

becomes

$$(L(x\beta \cdot x^{-1}),\ I,\ L(x\beta \cdot x^{-1})).$$

Thus, $A(Q)$ is left nuclear. □

Theorem 3.7. *Let Q be an RWPC loop and let A be the the group of automorphisms of Q. Then $A(Q)$ is right nuclear.*

Proof. Let Q be an RWPC loop, then from Lemma 3.1 and Corollary 3.2, we have

$$(R(x),\ L^{-2}(x)R(x)L(x),\ R(x))^{-1}(R(x),\ L^{-1}(x)L^{-1}(x\theta)R(x\delta)L(x\theta),\ R(x\delta))$$

$$= (I,\ L^{-1}(x)R^{-1}(x)L(x)L^{-1}(x\theta)R(x\delta)L(x\theta),\ R^{-1}(x)R(x\delta))$$

$$= (I,\ R^{-1}(x)R(x\delta),\ R^{-1}(x)R(x\delta)).$$

Applying this autotopism on the product yz and setting $z = 1$ shows that $x^{-1} \cdot x\delta \in N_\rho$.
Thus, $A(Q)$ is right nuclear. □

Corollary 3.7. *Let Q be a WIP PACC loop and let $A(Q)$ be the holomorph of Q. Then $A(Q)$ is nuclear.*

Theorem 3.8. *Let Q be a loop and let A be the automorphism group of Q. Then the $A(Q)$ holomorph of Q is a WIP PACC loop if and only if*

$$((x\beta)((y/x)/(x\alpha)))x\alpha = (x\beta)/(xy^\rho) \tag{3.24}$$

and

$$x\alpha(((x\alpha)\backslash(x\backslash z))(x\beta)) = (z^\lambda x)\backslash(x\beta) \tag{3.25}$$

for $x,\ y \in Q,\ \ \alpha, \beta \in A$.

Proof. Suppose the holomorph $A(Q)$ of Q is WIP PACC, then Q satisfies equation (3.1), set $y = (y/x)/(x\alpha)$, and then $z = y^\rho$ in (3.1) to get

$$((x\beta)((y/x)/(x\alpha)))x\alpha \cdot (xy^\rho) = (x\beta)$$

or

$$((x\beta)((y/x)/(x\alpha)))x\alpha = (x\beta)/(xy^\rho).$$

Also, set $z = (x\alpha) \setminus (x \setminus z)$ and $y = z^\lambda$ in equation (3.6) to get

$$x\alpha(((x\alpha)\backslash(x\backslash z))(x\beta)) = (z^\lambda x)\backslash(x\beta).$$

□

The next theorem shows how the holomorph of an arbitrary loop and its automorphism group produces a WIP PACC loop.

Theorem 3.9. *Let A be the group of automorphisms of a loop Q and let $x \in N_\mu(Q)$. Then the $A(Q)$ holomorph of Q is WIP PACC loop if and only if the following conditions hold:*

(1) $(x\alpha \cdot x)x^\rho = x\alpha$ *and* $x^\lambda(x \cdot x\alpha) = x\alpha$,
(2) $L(x\beta)R(x\alpha) = R(x\alpha)L(x\beta)$,
(3) *each* $\alpha \in A$ *is nuclear for* $x \in Q$, *and* $\alpha, \beta \in A$.

Proof. Suppose $A(Q)$ is a WIP PACC loop, we show that (1), (2) and (3) hold in $A(Q)$.
Since $A(Q)$ is WIP PACC, then $A(Q)$ satisfies (3.1) and (3.6) by Corollary 3.3. Set $z = x^\rho,\ y = 1$ in (3.1), and $y = x^\lambda,\ z = 1$ in (3.6), to get (1).

Again, suppose $A(Q)$ is WIP PACC, then it is LWPC and by Theorem 3.1, then

$$(R^{-1}(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ \ L(x),\ \ L(x\beta))$$

is an autotopism of $A(Q)$.
Now since $x \in N_\mu$, then

$$\begin{aligned}(R(x),\ \ L^{-1}(x),\ I)&(R^{-1}(x)R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ L(x),\ L(x\beta))\\ &= (R^{-1}(x\alpha)L(x\beta)R(x\alpha),\ I,\ L(x\beta)).\end{aligned}$$

Applying the autotopism to the product yz and set $y = y(x\alpha)$ and $z = 1$ gives (2)
The same result can be obtained using the RWPC autotopism and the fact that $x \in N_\mu$.
(3) follows from Theorems 3.6 and 3.7 .

Conversely, suppose (1), (2), and (3) hold in $A(Q)$, we show that $A(Q)$ is WIP PACC. Now

$$(x\alpha) = (x\alpha \cdot x)x^{\rho},$$

$$y(x\alpha) = y((x\alpha \cdot x)x^{\rho}),$$

$$(x\beta) \cdot y(x\alpha) = (x\beta)(y((x\alpha \cdot x)x^{\rho})),$$

$$yR(x\alpha)L(x\beta) = (x\beta)yR((x\alpha \cdot x)x^{\rho}) \tag{3.26}$$

Using (2) on the left side of equation (3.26) and (3) on the right side of equation (3.26), we have

$$(x\beta)y \cdot x\alpha = (x\beta)((y(x\alpha) \cdot x)x^{\rho}),$$

$$((x\beta)y \cdot x\alpha) \cdot xx^{\rho} = (x\beta)((y(x\alpha) \cdot x)x^{\rho}),$$

set $x^{\rho} = z$, to get

$$((x\beta)y \cdot x\alpha) \cdot xz = (x\beta)((y(x\alpha) \cdot x)z).$$

Also, if $x^{\lambda}(x \cdot x\alpha) = x\alpha$,

$$(x^{\lambda}(x \cdot x\alpha))y = (x\alpha)y,$$

$$((x^{\lambda}(x \cdot x\alpha)y)(x\beta) = ((x\alpha)y) \cdot (x\beta). \tag{3.27}$$

Using (2) and (3) respectively on the right and left side of equation (3.27),

$$(x\alpha) \cdot y(x\beta) = (x^{\lambda}(x \cdot (x\alpha)y))(x\beta),$$

$$x^{\lambda}x \cdot ((x\alpha) \cdot y(x\beta)) = (x^{\lambda}(x \cdot (x\alpha)y))(x\beta),$$

and with $x^{\lambda} = z$, we have

$$zx \cdot ((x\alpha) \cdot y(x\beta)) = (z(x \cdot (x\alpha)y))(x\beta).$$

$\square$

Corollary 3.8. *Let Q be an LWPC (RWPC) loop with $x \in N_{\mu}(Q)$ and let $A(Q)$ be holomorph of Q, then $A(Q)$ is an extra loop.*

Let Q and R be abelian groups and $f : Q \times Q \to R$ be a mapping. We call f zero preserving if $f(x, 0) = 0 = f(0, x)$ for all $x \in Q$. Say that f is additive on the right if $f(x, y + z) = f(x, y) + f(x, z)$ for all $x, y, z \in Q$. Say that f is additive if it is both right and left additive. Define the radical $\mathrm{Rad}(f)$ as the set of all $x \in Q$ such that $f(x, y) = 0 = f(y, x)$ for all $y \in Q$.

Theorem 3.10. *Let Q be an abelian group and R a subgroup of Q. Let $f : Q \times Q \to R$ be such that $Rad(f) \leq R$, f is zero preserving and right additive. Then the operation defined by*

$$x \cdot y = x + y + f(x, y) \tag{3.28}$$

is an LWPC loop if and only if

$$f(2x + y, x) = 2f(x, x) + f(y, x) \text{ for all } x, y \in Q.$$

The operation is an RWPC loop if and only if

$$f(y,\ x) + f(z,\ x) + f(y + x,\ z) = f(x,\ z) + f(y,\ z) + f(y + z,\ x) \text{ for all } x, y, z \in Q$$

and

$$3f(x, x) + f(y, x) = f(x, x) + f(2x + y, x) \text{ for all } x, y \in Q.$$

Proof. By Theorem 3.4 (a), a loop is LWPC if and only if it is LCC and satisfies $(xy \cdot x)x = x(yx \cdot x)$ with $\alpha,\ \beta = I$.
Q is LCC by [5, Theorem 2.1] and satisfies $(xy \cdot x)x = x(yx \cdot x)$ follows from [7, Theorem 3.1].
Also, by Theorem 3.4 (b), a loop is RWPC if and only if it is RCC and satisfies $x(x \cdot yx) = (x \cdot xy)x$.
Now, using (3.28),

$$yz \cdot x = y + z + f(y,\ z) + x + f(y + z, x),$$

$$yx \cdot x\backslash(zx) = y + x + f(y, x) + z + f(z, x) - f(x, z) + f(y + x, z).$$

Thus, Q is an RCC if and only if $f(y,\ x)+f(z,\ x)+f(y+x,\ z) = f(x\ z)+f(y,\ z)+f(y+z,\ x)$.
Lastly, $x(x\cdot yx) = (x\cdot xy)x$ if and only if $3f(x,x)+f(y,x) = f(x,x)+f(2x+y,x)$. □

Corollary 3.9. *The loop $(Q,\cdot)$ is WIP PACC if and only if*

$$(i)\ f(2x+y,x) = 2f(x,x)+f(y,x) \text{ for all } x,y \in Q,$$

$$(ii)\ f(y,\ x)+f(z,\ x)+f(y+x,\ z) = f(x,\ z)+f(y,\ z)+f(y+z,\ x) \text{ for all } x,y,z \in Q$$

and

$$(iii)\ 3f(x,x)+f(y,x) = f(x,x)+f(2x+y,x) \text{ for all } x,y \in Q.$$

.

Example 3.1. *Here is a WIP PACC loop of order 16 with $N(Q) = \{1,2,3,4\}$.*

TABLE 1. A WIP PACC loop of order 16.

$\star$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	8	7	6	5	12	11	10	9	15	16	13	14
4	4	3	2	1	7	8	5	6	11	12	9	10	16	15	14	13
5	5	6	7	8	1	2	3	4	15	16	13	14	12	11	10	9
6	6	5	8	7	2	1	4	3	16	15	14	13	11	12	9	10
7	7	8	5	6	4	3	2	1	14	13	16	15	10	9	12	11
8	8	7	6	5	3	4	1	2	13	14	15	16	9	10	11	12
9	9	10	11	12	13	14	15	16	1	2	3	4	6	5	8	7
10	10	9	12	11	14	13	16	15	2	1	4	3	5	6	7	8
11	11	12	9	10	16	15	14	13	4	3	2	1	8	7	6	5
12	12	11	10	9	15	16	13	14	3	4	1	2	7	8	5	6
13	13	14	15	16	10	9	12	11	7	8	5	6	4	3	2	1
14	14	13	16	15	9	10	11	12	8	7	6	5	3	4	1	2
15	15	16	13	14	11	12	9	10	6	5	8	7	2	1	4	3
16	16	15	14	13	12	11	10	9	5	6	7	8	1	2	3	4

This example has been verified using the Groups, Algorithms, Programming (GAP Package) [13] and Library of GAP-LOOPS Package [12].

Acknowledgement

I would like to thank the editor and the referees for the valuable suggestions and recommendations which have improved the presentation and structural arrangements of this work.

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