Jordan Journal of Mathematics and Statistics (JJMS), 16(3), 2023, pp 585 - 597
DOI: https://doi.org/10.47013/16.3.13

AMALGAMATIONS OF POTENT, SEMIPOTENT, AND SEMISUITABLE RINGS

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Abstract. We investigate the transfer of the notion of semisuitable, potent, and semipotent rings in different settings of the amalgamated algebras along an ideal. We put the transfer results in use to provide examples subject to the involved ring theoretic notions as well as to recover some previous results related to the transfer of these notions in other constructions such as trivial ring extension.

1. Introduction

All rings considered in this paper are commutative with unity. Let A and B be rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

which is called the amalgamation of A with B along J with respect to f. This construction was introduced and studied by M. DAnna, C. A. Finocchiaro, and M. Fontana in [7, 9]. It is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by M. DAnna and M. Fontana in [10, 11, 12]). Let A be a ring, I an ideal of A, and $\mathrm{id}_A : A \to A$ the identity ring homomorphism. The amalgamated duplication of A along I, denoted by $A \bowtie I$ is the subring of $A \times A$ given by

$$A \bowtie I = A \bowtie^{\mathrm{id}_A} I := \{(a, a + i) \mid a \in A, i \in I\}.$$

2010 *Mathematics Subject Classification.* Primary 13C10, 13C11, 13E05, 13F05, 13H10; Secondary 16A30,16A50,16A52.

Key words and phrases. Amalgamation of a ring along an ideal, trivial ring extension, suitable rings, semi-suitable rings, potent rings, semi potent rings.

Received: Aug. 3, 2022 Accepted: Nov. 10, 2022 .

For more information about the amalgamation, we refer the reader to [4, 6, 7, 8, 9, 11, 12, 13]. The amalgamated algebras along an ideal also generalize the trivial ring extension (idealization) [7], which was introduced by Nagata in 1955. He defined it in the following way: Let A be a commutative ring and M an A-module. The trivial ring extension of R by M is the commutative ring $A \ltimes M = A \times M$ with component-wise addition and multiplication given by $(a, m)(a', m') = (aa', am' + a'm)$. For more about the trivial ring extension, we refer the reader to [2, 3, 5, 16, 14].

Let A be a ring and I be an ideal of A. We say that idempotents lift modulo I if for each x in A such that $x - x^2 \in I$, there is an idempotent e in A such that $e - x \in I$ [15]. A is called a suitable ring if idempotents lift modulo every left ideal of A [17]. Examples of suitable rings include clean rings (a ring is called clean if all of its elements are clean. Where the clean element is the element that can be written as a sum of a unit and an idempotent) [17]. Let $J(A)$ denote the Jacobson radical of A. A is called semisuitable ring if idempotents lift modulo $J(A)$ [1].

A is called semipotent ring if any ideal I that is not contained in $J(A)$ contains a non zero idempotent, equivalently; A is semipotent if and only if for any $a \in A \backslash J(A)$, there is a non zero $x \in A$ such that $xax = x$. A semipotent ring A is called potent if idempotents lift modulo $J(A)$ [15]. It is easy to see from the definitions that potent rings are semipotent and suitable rings are potent. Also, A is potent if and only if A is semipotent and semisuitable. Hence we have the following diagram of implications:

$$\begin{array}{c} \text{Clean rings} \Rightarrow \text{Suitable rings} \Rightarrow \text{Potent rings} \Rightarrow \text{Semipotent rings} \\ \Downarrow \\ \text{Semisuitable rings} \end{array}$$

The first implication is reversible in the commutative case [17, Proposition 1.8], while all the others are irreversible in general even in the commutative case. For counterexamples, we refer the reader to [17, Page 271], [18, Example 25], and [1, Example 2.2]. Once more, the ring of integers $\mathbb{Z}$ is semisuitable which is not semipotent [1, Example 2.2] and the existence of a semipotent ring which is not potent implies the existence of a semipotent ring which is not semisuitable. In summary, the following diagram of implications displays the relation among the distinct classes of clean, potent, semipotent and semisuitable rings in the commutative case:

Semisuitable rings $\Leftrightarrow$ Clean rings $\Rightarrow$ Potent rings $\Rightarrow$ Semipotent rings

In 2014, M. Chhiti, N. Mahdou, and M. Tamekkante gave a characterization for the amalgamation of A with B along J with respect to f ($A \bowtie^f J$) to be clean [6]. In 2019, K. Adarbeh investigated the transfer of the (Semi)suitable rings along with related concepts, such as potent and semipotent rings in the general context of the trivial ring extension [1].

Throughout, $J(A)$ denotes the Jacobson radical of A; $Nil(A)$ denotes the nilradical of A; $Id(A)$ denotes the set of all idempotents of A; $Max(A)$ denotes the set of all maximal ideals of A; $A \bowtie^f J$ denotes the amalgamation of A with B along J with respect to f as it is defined above.

In Section 2 of this paper, we establish necessary and sufficient conditions for $A \bowtie^f J$ to be semisuitable or (semi)potent ring in the commutative case under the condition, $J \subseteq J(B)$. Namely, If J is an ideal of B such that $J \subseteq J(B)$, then $A \bowtie^f J$ is semisuitable if and only if A is semisuitable. If in addition, $f(A)J = 0$, then $A \bowtie^f J$ is (semi)potent if and only if A is (semi)potent. In section 3, more transfer results are obtained by assuming conditions on $Id(B)$. For example, we prove that under the assumptions, f is surjective and $J \subseteq Id(B)$, $A \bowtie^f J$ is semisuitable if and only if A is semisuitable. In both sections, we illustrate the results by examples and counterexamples subject to the involved ring theoretic concepts.

2. Transfer Results Subject to conditions on $J(B)$

This section is devoted to study the transfer of Potent, Semipotent, and Semisuitable rings in amalgamations along ideals contained in the Jacobson radical. We start this section by recalling the following facts:

Lemma 2.1. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B.*

(1) *If I is an ideal of A, then $I \bowtie^f J := \{(i, f(i) + j) \mid i \in I, j \in J\}$ is an ideal of $A \bowtie^f J$ and $\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}$* [7].

(2) *$Max(A \bowtie^f J) = \{\mathrm{m} \bowtie^f J : \mathrm{m} \in Max(A)\} \cup \{\overline{Q}' : Q \in Max(B)$ and $J \nsubseteq Q\}$, where $\overline{Q}' = \{(a, f(a) + j) : a \in A, j \in J, f(a) + j \in Q\}$* [8, Corollary 2.5 and Corollary 2.7].

The following lemma describes the Jacobson radical of $A \bowtie^f J$, where $J \subseteq J(B)$. This lemma will be used frequently to prove the main results of this article.

Lemma 2.2. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If $J \subseteq J(B)$, then $J(A \bowtie^f J) = J(A) \bowtie^f J$.*

Proof. Since $J \subseteq J(B)$, then $J \subseteq Q$ for each $Q \in Max(B)$. So the set $\{\overline{Q}' : Q \in Max(B)$ and $J \nsubseteq Q\}$ is empty. Hence, by Lemma 2.1, $Max(A \bowtie^f J) = \{\mathrm{m} \bowtie^f J : \mathrm{m} \in Max(A)\}$ where $\mathrm{m} \bowtie^f J = \{(a, f(a) + j) \mid a \in \mathrm{m}, j \in J\}$. So that, $(a, f(a) + j) \in J(A \bowtie^f J)$ if and only if $a \in \mathrm{m}$ for each $\mathrm{m} \in Max(A)$, and $j \in J$. Consequently, $(a, f(a)+j) \in J(A \bowtie^f J)$ if and only if $a \in J(A)$ and $j \in J$. Therefore, $J(A \bowtie^f J) = J(A) \bowtie^f J$.

□

The following theorem provides a necessary and sufficient condition for $A \bowtie^f J$ to be semisuitable, when $J \subseteq J(B)$.

Theorem 2.1. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \subseteq J(B)$. Then $A \bowtie^f J$ is semisuitable if and only if A is semisuitable.*

Proof. Since $J \subseteq J(B)$, then by Lemma 2.2, $J(A \bowtie^f J) = J(A) \bowtie^f J$. Let $(a, f(a)+j) \in A \bowtie^f J$ be such that $(a, f(a)+j) - (a, f(a)+j)^2 \in J(A \bowtie^f J)$. Then $(a - a^2, (f(a) + j) - (f(a) + j)^2) \in J(A) \bowtie^f J$. So $a - a^2 \in J(A)$. Since A is semisuitable, there is an idempotent e in A such that $e - a \in J(A)$. Then $(e, f(e))$ is an idempotent in $A \bowtie^f J$ and $(e, f(e)) - (a, f(a) + j) = (e - a, f(e - a) - j) \in J(A) \bowtie^f J = J(A \bowtie^f J)$. Thus, idempotents lift modulo $J(A \bowtie^f J)$. Hence, $A \bowtie^f J$ is semisuitable.

Conversely, suppose that $A \bowtie^f J$ is semisuitable. Let $a \in A$ be such that $a - a^2 \in J(A)$. Then $(a, f(a)) - (a, f(a))^2 = (a - a^2, f(a - a^2)) \in J(A) \bowtie^f J = J(A \bowtie^f J)$. But $A \bowtie^f J$ is semisuitable, so there is an idempotent $(e, f(e)+j)$ in $A \bowtie^f J$ such that $(e, f(e) + j) - (a, f(a)) = (e - a, f(e - a) + j) \in J(A \bowtie^f J) = J(A) \bowtie^f J$. Then clearly, e is an idempotent in A and $e - a \in J(A)$. Therefore, A is semisuitable.

□

For the special case of B is local, we obtain the following corollary of Theorem 2.1.

Corollary 2.1. *Let $f : A \to B$ be a ring homomorphism, (B, m) be a local ring, and let J be a proper ideal of B. Then $A \bowtie^f J$ is semisuitable if and only if A is semisuitable.*

Proof. This follows from Theorem 2.1 and the fact that $J \subseteq \mathrm{m} = J(B)$.

□

As an application of Theorem 2.1, we recover [1, Theorem 3.1, part (2)]; which describes the transfer of semisuitable rings in the trivial ring extension.

Corollary 2.2. [1, Theorem 3.1, parts (1) and (2)] *Let A be a ring and M an A-module. Then $A \ltimes M$ is semisuitable if and only if A is semisuitable.*

Proof. Notice that $A \ltimes M = A \bowtie^{\iota_A} (0 \ltimes M)$, where $\iota_A : A \to A \ltimes M$ is the canonical embedding $a \mapsto (a, 0)$. Since $J = 0 \ltimes M \subseteq J(A \ltimes M)$, this corollary follows from Theorem 2.1. □

Next, we are interested in constructing an example of a semisuitable ring which is not semipotent. For this purpose we prove the following proposition, which will be used later to prove Proposition 2.2. On another hand, the following proposition insures that the homomorphic image through an ideal contained in the Jacobson radical of a (semi)potent ring is (semi)potent, and hence it is a recovery of [1, Proposition 2.4, Corollary 2.5].

Proposition 2.1. *Let A be a commutative ring and $I \subseteq J(A)$ be an ideal of A.*

(1) *If A is semipotent, then $\frac{A}{I}$ is a semipotent ring.*
(2) *If A is potent, then $\frac{A}{I}$ is a potent ring.*

Proof. (1) Assume that A is semipotent and I is a ideal of A such that $I \subseteq J(A)$. Let $\frac{K}{I}$ be an ideal of $\frac{A}{I}$ that is not contained in $J(\frac{A}{I})$. But $J(\frac{A}{I}) = \frac{J(A)}{I}$. So $K \nsubseteq J(A)$. Since A is semipotent, K contains a nonzero idempotent e. Then $e + I \in \frac{K}{I}$ is an idempotent (since $(e + I)^2 = e^2 + I = e + I$). It remains to show that $e + I \neq 0 + I$. Suppose not, that is, $e + I = 0 + I$. Then $e \in I \subseteq J(A)$. So $1 - e = 1 - ee$ is a unit in A but $1 - e$ is an idempotent (since $(1-e)^2 = 1-2e+e^2 = 1-2e+e = 1-e$). It follows that $1-e=1$ which implies $e=0$, a contradiction. Hence, $e+I \neq 0+I$. Therefore, $\frac{A}{I}$ is semipotent.

(2) Since A is potent, then A is semipotent and semisuitable. By Part (1), A is semipotent gives that $\frac{A}{I}$ is semipotent, for every ideal I subset of $J(A)$. And by [1, Theorem 2.3], A is semisuitable gives that $\frac{A}{I}$ is semisuitable, for every ideal I subset of $J(A)$. Therefore, $\frac{A}{I}$ is a potent ring, for every ideal I subset of $J(A)$.

□

The following proposition is a direct consequence of Lemma 2.2 and Proposition 2.1.

Proposition 2.2. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \subseteq J(B)$. If $A \bowtie^f J$ is (semi)potent, then so is A.*

Proof. Since $J \subseteq J(B)$, then by Lemma 2.2, $J(A \bowtie^f J) = J(A) \bowtie^f J$. So $0 \bowtie^f J \subseteq J(A \bowtie^f J)$. Hence, by Theorem 2.1, $A \cong \frac{A \bowtie^f J}{0 \bowtie^f J}$ is (semi)potent. □

Theorem 2.1 and Proposition 2.2 can be used to provide new examples of semisuitable rings which are not semipotent rings (consequently, not clean), as shown below.

Example 2.1. *Let $A = \mathbb{Z}$ and $B = \mathbb{Z}_{16}$. Then A is a semisuitable ring* [1] *and B is a local ring with maximal ideal $\langle \overline{2} \rangle$. Consider the ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}_{16}$, $x \mapsto \overline{x}$. Then by Corollary 2.1, $\mathbb{Z} \bowtie^f \langle \overline{4} \rangle$ is semisuitable. On the other hand, $\mathbb{Z}$ is not semipotent* [1]. *Since $\langle \overline{4} \rangle \subseteq J(\mathbb{Z}_{16}) = \langle \overline{2} \rangle$, then by Proposition 2.2, $\mathbb{Z} \bowtie^f \langle \overline{4} \rangle$ is not semipotent.*

Next, our goal is to introduce a new counterexample of a potent ring that is not clean. We first prove the following theorem which provides a necessary and sufficient condition for that $A \bowtie^f J$ to be (semi)potent, under the conditions $J \subseteq J(B)$ and $f(A)J = 0$.

Theorem 2.2. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \subseteq J(B)$. If $f(A)J = 0$, then:*

(1) *$A \bowtie^f J$ is semipotent if and only if A is semipotent.*

(2) *$A \bowtie^f J$ is potent if and only if A is potent.*

Proof. (1) Assume that A is semipotent. Let $(a, f(a)+j) \in A \bowtie^f J - J(A \bowtie^f J)$. But $J \subseteq J(B)$ implies that $J(A \bowtie^f J) = J(A) \bowtie^f J$ by Lemma 2.2. So $a \in A - J(A)$. Since A is semipotent, there exists $x \in A - \{0\}$ such that $ax^2 = x$. Consider $X = (x, f(x))$. Then $0 \neq X \in A \bowtie^f J$ and moreover, $(a, f(a) + j)X^2 = (a, f(a)+j)(x^2, f(x^2)) = (ax^2, f(ax^2)+f(x^2)j) = (x, f(x)) = X$ (note that since $f(A)J = 0$, then $f(x^2)j = 0$). Therefore, $A \bowtie^f J$ is semipotent. The other direction follows from Proposition 2.2.

(2) Follows immediately from Theorem 2.1, Part(1), and the fact that a ring is potent if and only if it is semipotent and semisuitable.

□

Recall that a division ring is the ring in which every nonzero element a has a multiplicative inverse (Fields in the commutative case). Let D be a division ring and S be a subring of D containing 1. Then

$$R(D, S) = \{(x_1, x_2, \cdots, x_n, s, s, s, \cdots) \mid n \geq 1, x_i \in D, s \in S\}$$

is a ring with component-wise operations [17]. In the following example, we use part(2) of Theorem 2.2 to provide a new counterexample of a potent ring that is not clean.

Example 2.2. *Let $A = R(\mathbb{Q}, \mathbb{Z})$ and $B = \mathbb{Z}_{10} \ltimes \mathbb{Z}_{10}$. Then A is a commutative potent ring which is not clean* [17]. *Consider the ring homomorphisms $g : A \to \mathbb{Z}$, $(x_1, x_2, \cdots, x_n, s, s, s, \cdots) \mapsto s$ and $h : \mathbb{Z}_4 \to \mathbb{Z}_{10} \ltimes \mathbb{Z}_{10}$, $x \mapsto (5x, 0)$. Let f be the ring homomorphism $A \to \mathbb{Z} \to \mathbb{Z}_4 \to B$, given by $(x_1, x_2, \cdots, x_n, s, s, s, \cdots) \mapsto (5\overline{s}, 0)$. Let $J = 0 \ltimes \langle 2 \rangle$. Then J is an ideal of B and $J \subset 0 \ltimes \mathbb{Z}_{10} = J(B)$. It is easy to check that $f(A)J = 0$. Thus, by part (2) of Theorem 2.2, $A \bowtie^f J$ is potent. But since A is not clean and $J \subset J(B)$, then by* [6, Corollary 2.6], *$A \bowtie^f J$ is not clean.*

3. Transfer results subject to conditions on $Id(B)$

More transfer results can be built by interacting J with $Id(B)$. We begin with the following proposition.

Proposition 3.1. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \cap Id(B) = 0$.*

(1) *If $A \bowtie^f J$ is semipotent, then A is semipotent.*
(2) *If $A \bowtie^f J$ is potent, then A is potent.*

Proof. (1) Assume that $A \bowtie^f J$ is semipotent. We claim that $J \subseteq J(B)$. On the contrary, suppose that $J \nsubseteq J(B)$. Then there is $j \in J$ such that $j \notin J(B)$. Then $(0, j) \notin J(A \bowtie^f J)$. So there exists $(x, f(x)+k) \in A \bowtie^f J - \{(0, 0)\}$ such that $(0, j)(x, f(x)+k)^2 = (x, f(x)+k)$. This implies that $x = 0$ and $jk^2 = k$. Since $(0, k) = (x, f(x)+k) \neq (0, 0)$, then $k \neq 0$. Now, $jk = j(jk^2) = (jk)^2$. So $jk \in J \cap Id(B) = 0$. Hence, $jk = 0$ which implies $k = jk^2 = 0$, a contradiction. Thus, $J \subseteq J(B)$. Therefore, by Proposition 2.2, A is semipotent

(2) Assume that $A \bowtie^f J$ is potent. Then it is semipotent and semisuitable. By part (1), A is semipotent. Also, by the proof of part (1), $J \subseteq J(B)$. But $A \bowtie^f J$ is semisuitable, so by Proposition 2.1, A is semisuitable. Therefore, A is potent.

□

The following proposition proves that $A \bowtie^f J$ inherits the semipotency from A and B, under the conditions $J \subseteq Id(B)$ and $Ann(J) = 0$.

Proposition 3.2. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \subseteq Id(B)$ and $Ann(J) = 0$. If A and B are semipotent rings, then so is $A \bowtie^f J$.*

Proof. First, note that since $J \subseteq Id(B)$, then $j^2 = j$ and $2j = 0$ for all $j \in J$. Now, let $(a, f(a) + j) \in A \bowtie^f J - J(A \bowtie^f J)$. Then there is a maximal ideal M of $A \bowtie^f J$ such that $(a, f(a) + j) \notin M$. By Lemma 2.1, $Max(A \bowtie^f J) = \{\mathrm{m} \bowtie^f J : \mathrm{m} \in Max(A)\} \cup \{\overline{Q}' : Q \in Max(B) \text{ and } J \nsubseteq Q\}$. If $M = \mathrm{m} \bowtie^f J$, where $\mathrm{m} \in Max(A)$, then $a \notin \mathrm{m}$ and so $a \notin J(A)$. Since A is semipotent, then there is $x \in A - \{0\}$ such that $ax^2 = x$. Then $(x, f(x) + f(x)j) \in A \bowtie^f J - \{(0, 0)\}$. Since $(1 + j)^2 = 1 + j$ and $j(1 + j) = 0$, then $(a, f(a) + j)(x, f(x) + f(x)j)^2 = (ax^2, (f(a) + j)f(x^2)(1 + j)^2) = (x, (f(a)+j)f(x^2)(1+j)) = (x, f(ax^2)(1+j)+f(x^2)j(1+j)) = (x, f(x)(1+j)+0) = (x, f(x) + f(x)j)$.

If $M = \overline{Q}'$, where $Q \in Max(B)$ and $J \nsubseteq Q$, then $(a, f(a) + j) \notin \overline{Q}'$ which implies $f(a) + j \notin Q$, so $f(a) + j \notin J(B)$. Since B is semipotent, then there is $y \in B - \{0\}$ such that $(f(a) + j)y^2 = y$. Since $Ann(J) = 0$ and $y \neq 0$, then there exists $k \in J$ such that $yk \neq 0$. So $(0, yk) \in A \bowtie^f J - \{(0, 0)\}$ and $(a, f(a) + j)(0, yk)^2 = (a, f(a)+j)(0, (yk)^2) = (a, f(a)+j)(0, y^2k) = (0, (f(a)+j)y^2k) = (0, yk)$. Therefore, $A \bowtie^f J$ is semipotent. □

The following proposition proves that $A \bowtie^f J$ inherits the potency from A, under the condition $J \subseteq Id(B)$.

Proposition 3.3. *Let $f : A \to B$ be a surjective ring homomorphism and let J be an ideal of B such that $J \subseteq Id(B)$. Then $A \bowtie^f J$ is semisuitable if and only if A is semisuitable.*

Proof. Let $(a, f(a)+j) \in A \bowtie^f J$ be such that $(a, f(a)+j)-(a, f(a)+j)^2 \in J(A \bowtie^f J)$. Then $(a - a^2, (f(a) + j) - (f(a) + j)^2) = (a - a^2, f(a) - f(a^2)) \in J(A \bowtie^f J)$ (since $J \subseteq Id(B)$ implies $(f(a) + j)^2 = f(a^2) + j$). For any $\mathrm{m} \in Max(A)$, $\mathrm{m} \bowtie^f J \in Max(A \bowtie^f J)$, so $(a - a^2, f(a) - f(a^2)) \in \mathrm{m} \bowtie^f J$ and hence $a - a^2 \in \mathrm{m}$. It follows that $a - a^2 \in J(A)$. Since A is semisuitable, there is an idempotent e in A such that $e - a \in J(A)$. Then $(e - a, f(e - a)) \in \mathrm{m} \bowtie^f J$, for each $\mathrm{m} \in Max(A)$. Also, since f is surjective and $e - a \in J(A)$, then $f(e - a) \in J(B)$. Hence, $(e - a, f(e - a)) \in \overline{Q}'$, for each $Q \in Max(B)$ with $J \nsubseteq Q$. It follows that $(e - a, f(e - a)) \in J(A \bowtie^f J)$. Since $e \in Id(A)$ and $J \subseteq Id(B)$, it is easy to check that $(e, f(e)+j)$ is an idempotent in $A \bowtie^f J$. Moreover, $(e, f(e) + j) - (a, f(a) + j) = (e - a, f(e - a)) \in J(A \bowtie^f J)$. Thus, idempotents lift modulo $J(A \bowtie^f J)$. Hence, $A \bowtie^f J$ is semisuitable.
Conversely, suppose that $A \bowtie^f J$ is semisuitable. Let $a \in A$ be such that $a - a^2 \in J(A)$. Again, since f is surjective, then as above, $(a, f(a))-(a, f(a))^2 = (a-a^2, f(a-a^2)) \in J(A \bowtie^f J)$. But $A \bowtie^f J$ is semisuitable, so there is an idempotent $(e, f(e)+j)$ in $A \bowtie^f J$ such that $(e, f(e) + j) - (a, f(a)) = (e - a, f(e - a) + j) \in J(A \bowtie^f J)$. Then clearly, e is an idempotent in A and moreover, $e - a \in J(A)$. Therefore, A is semisuitable. □

Propositions 3.2 and 3.3 can be used to prove that $A \bowtie^f J$ inherits the potent condition from A and B under the same conditions.

Corollary 3.1. *Let $f : A \to B$ be a surjective ring homomorphism and let J be an ideal of B such that $J \subseteq Id(B)$ and $Ann(J) = 0$. If A and B are potent rings, then so is $A \bowtie^f J$.*

Proof. This follows from Propositions 3.2, 3.3, and the fact that a ring is potent if and only if it is semipotent and semisuitable. □

Another application of Proposition 3.3 is the following corollary, which deals with the duplication as a special case of the amalgamation.

Corollary 3.2. *Let A be a ring and let I be an ideal of A such that $I \subseteq Id(A)$. Then $A \bowtie I$ is semisuitable if and only if A is semisuitable.*

Proof. Notice that $A \bowtie I = A \bowtie^{\mathrm{id}_A} I$, where $\mathrm{id}_A : A \to A$ the identity ring homomorphism. Since id_A is surjective and $I \subseteq Id(A)$, then by Proposition 3.3, $A \bowtie I$ is semisuitable if and only if A is semisuitable. □

The following is an illustrative example for Proposition 3.3.

Example 3.1. *Let $A = \mathbb{Z}$, $B = \mathbb{Z}_6$, and $J = \langle \overline{3} \rangle = \{\overline{0}, \overline{3}\}$. Clearly, $J \subseteq Id(B)$. Now, consider the surjective ring homomorphism $f : A \to B$, $x \mapsto \overline{x}$. Since A is semisuitable* [1]*, then by Proposition 3.3, $A \bowtie^f J = \mathbb{Z} \bowtie^f \langle \overline{3} \rangle$ is a semisuitable ring.*

The following proposition proves that $A \bowtie I$ inherits the semipotency from A, under the condition $I \subseteq Id(A)$.

Proposition 3.4. *Let A be a ring and let I be an ideal of A.*

(1) *If $A \bowtie I$ is semipotent, then so is A.*
(2) *If A is semipotent and $I \subseteq Id(A)$, then $A \bowtie I$ is semipotent.*

Proof. (1) Let $a \in A - J(A)$. Then $(a, a) \in A \bowtie I - J(A \bowtie I)$. But $A \bowtie I$ is semipotent, so there is $(0, 0) \neq (x, x + i) \in A \bowtie I$ such that $(a, a)(x, x + i)^2 = (x, x + i)$. So we have $ax^2 = x$ and $a(x + i)^2 = (x + i)$. If $x \neq 0$, then we are done. If $x = 0$, then $0 \neq i \in A$ and $ai^2 = i$. Therefore, A is semipotent.

(2) First, note that since $I \subseteq Id(A)$, then $i^2 = i$ and $2i = 0$ for all $i \in I$. Now, let $(a, a+i) \in A \bowtie I - J(A \bowtie I)$. Then there is a maximal ideal M of $A \bowtie I$ such

that $(a, a+i) \notin M$. If $M = \mathrm{m} \bowtie I$, where $\mathrm{m} \in Max(A)$, then $a \notin \mathrm{m}$ and so $a \notin J(A)$. Since A is semipotent, then there is $x \in A - \{0\}$ such that $ax^2 = x$. Since $(1+i)^2 = 1+i$ and $i(1+i) = 0$, then $(a, a+i)(x, x+xi)^2 = (ax^2, (a+i)x^2(1+i)^2) = (x, (a+i)x^2(1+i)) = (x, ax^2(1+i) + x^2 i(1+i)) = (x, x+xi)$.

If $M = \overline{Q}'$, where $Q \in Max(A)$ and $I \nsubseteq Q$, then $(a, a+i) \notin \overline{Q}'$ which implies $a+i \notin Q$, so $a+i \notin J(A)$. Again since A is semipotent, then there is $y \in A - \{0\}$ such that $(a+i)y^2 = y$. This implies $ay^2 = y - iy^2 = y - (iy)^2 = y - yi = (1-i)y$, but then $ay^2 i = 0$ and so $ayi = 0$. If $yi = 0$, then $ay^2 = y$. So $(y, y) \neq 0$ and $(a, a+i)(y, y)^2 = (a, a+i)(y^2, y^2) = (ay^2, (a+i)y^2) = (y, y)$. If $yi \neq 0$, then $(0, yi) \neq 0$ and $(a, a+i)(0, yi)^2 = (a, a+i)(0, (yi)^2) = (a, a+i)(0, yi) = (0, (a+i)yi) = (0, ayi + yi^2) = (0, yi)$. Therefore, $A \bowtie I$ is semipotent.

□

The following corollary proves that $A \bowtie I$ inherits the potency from A, under the condition $I \subseteq Id(A)$.

Corollary 3.3. *Let A be a ring and let I be an ideal of A such that $I \subseteq Id(A)$. Then $A \bowtie I$ is potent if and only if A is potent.*

Proof. This follows from Corollary 3.2 and Proposition 3.4. □

Next, we use Corollary 3.3; which is a consequence of Proposition 3.4; to construct the following example of potent ring.

Example 3.2. *Since $\mathbb{Z}_6$ is potent (since it's clean) and $\langle \overline{3} \rangle \subseteq Id(\mathbb{Z}_6)$, then by Corollary 3.3, $\mathbb{Z}_6 \bowtie \langle \overline{3} \rangle$ is a potent ring.*

Acknowledgement

We would like to thank the editor and the referees for their valuable efforts. Actually, the comments of the revision raised the level of the article and made it more interesting to the math community.

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