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On the Hilbert 5-Class Field Towers of Some Pure Metacyclic Fields and Total Capitulation

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Abstract: In this paper, we study the length of the Hilbert 5-class field towers of some pure metacyclic fields of degree 20 over $\mathbb{Q}$, which are the normal closure of some pure quintic fields, by means of some results of 5-groups. The capitulation problem is investigated too.

Keywords: Groups of maximal class, 5-class field, Transfer, 5-class groups.

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1 Introduction

Let k be an algebraic number field and $C_{k,5}$ its 5-class group, that is the 5-Sylow subgroup of its class group C_k. Let $k_5^{(1)}$ be the Hilbert 5-class field of k, that is the unramified abelian maximal extension of k for finite and infinite primes. Put $k_5^{(0)} = k$ and by $k_5^{(i+1)}$ we denote the Hilbert 5-class field of $k_5^{(i)}$ for any natural $i \geq 0$. Then the sequence of fields

$$k = k_5^{(0)} \subset k_5^{(1)} \subset k_5^{(2)} \subset .... \subset k_5^{(i)} \subset .....$$

is called the 5-class field tower of k. If for all $i \geq 1$, $k_5^{(i)} \neq k_5^{(i+1)}$, the tower is said to be infinite, otherwise the tower is said to be finite, and the minimal natural i such that $k_5^{(i)} = k_5^{(i+1)}$ is called the length of the tower.
The task to determine whether or not the 5-class field tower of a number field k is finite, is until nowday a classical and difficult open problem of class field theory. Although, we have that if the rank of $C_{k_5^{(1)},5} \leq 2$, then by means of group theory, the length of the tower is at most 2 [4].
Our contribution in this paper is to determine the length of the 5-class field tower and to investigate the capitulation problem for some families of pure metacyclic fields, which are the normal closure of a pure quintic fields. The novelty in this procedure is the combination of some results of 5-group theory and triviality of 5-class numbers of some fields.
Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, where n is a 5^{th} power-free natural number and $k_0 = \mathbb{Q}(\zeta_5)$ be the 5^{th} cyclotomic field, then $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ is the normal closure of Γ, and a pure metacyclic field of absolute degree 20. By $C_{k,5}^{(\sigma)}$ we denote the subgroup of ambiguous ideal classes under the action of $Gal(k/k_0) = \langle\sigma\rangle$.
The aim of this paper is to investigate the 5-class field tower of k and the capitulation of the 5-ideal classes of k in its six cyclic quintic unramified extensions within the Hilbert 5-class field $k_5^{(1)}$ of k, whenever $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$. Our main theorem is the following.

Theorem 1.*Let k_0 be the 5^{th} cyclotomic field and k be the normal closure of a pure quintic field. Suppose that the 5-class group $C_{k,5}$ of k is of type $(5,5)$ and the rank of ambiguous ideal classes $C_{k,5}^{(\sigma)}$ under the action of $Gal(k/k_0) = \langle\sigma\rangle$ is 1, then the length of the 5-class field tower of k is 2. Furthermore there is a total capitulation of $C_{k,5}$ in the all unramified quintic cyclic extensions of k.*

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The computational number theory system PARI/GP [17] allowed us to underpin this results by numerical examples.

2 Some Results of Group Theory

Let G be p-groupe and $\gamma_2(G) = [G,G]$ be its commutator group. G is called metabelian if $\gamma_2(G)$ is abelian. The Frattini subgroup $\phi(G)$ of G is the intersection of all maximal subgroups of G [[3], page 25, Definition 1]. The subgroup G^p of G, generated by the p^{th} powers is contained in $\gamma_2(G)$, which as a result, coincides with the Frattini subgroup $\phi(G) = G^p\gamma_2(G) = \gamma_2(G)$. By the basis theorem of Burnside [[3], Theorem 1.12], every minimal system of generators of G contains exactly d elements with p^d is the order of $G/\phi(G)$. In the case of $G/\gamma_2(G)$ is of type (p,p), the group $G =< x,y >$ can be generated by two elements x and y. If we declare the lower central series of G by

$$\begin{cases} \gamma_1(G) = G \\ \gamma_j(G) = [\gamma_{j-1}(G), G] \text{ for } j \geq 2, \end{cases}$$

By [[5], Corollary 2] we have Kaloujnine's commutator relation $[\gamma_j(G), \gamma_l(G)] \subseteq \gamma_{j+l}(G)$, for $j,l \geq 1$, and for an index of nilpotence $c \geq 2$ the series

$$G = \gamma_1(G) \supset \gamma_2(G) \supset ..... \supset \gamma_c(G) \supset \gamma_{c+1}(G) = 1$$

becomes stationary.

The coclass $cc(G)$ of a p-group G of order p^n and nilpotency class c is defined as $cc(G) = n-c$. If $cc(G) = 1$, then G is called of maximal class.
The two-step centralizer

$$\chi_2(G) = \{g \in G \,|\, [g,u] \in \gamma_4(G) \text{ for all u} \in \gamma_2(G)\}$$

of the two-step factor group $\gamma_2(G)/\gamma_4(G)$, that is the largest subgroup of G such that $[\chi_2(G), \gamma_2(G)] \subset \gamma_4(G)$. It is characteristic and contains the commutator subgroup $\gamma_2(G)$. If G is of maximal class and order p^n, according to [[9], Proposition 3.1.4] and [[5], Lemma 2.5] we have that $\chi_2(G)$ is maximal normal subgroup of G for $n \geq 4$ Moreover $\chi_2(G)$ coincides with G if and only if $n = 3$, because if $n = 3$, then $\gamma_2(G)$ is central and $\gamma_4(G)$ is trivial, so $\chi_2(G) = G$. If G is not of maximal class then the "only if" clause fails, as clearly any group of class at most two will have $\gamma_2(G)$ central and $\gamma_4(G)$ trivial, so $\chi_2(G) = G$.
Let $k = k(G)$, the isomorphism invariant of G, be defined by $[\chi_2(G), \gamma_2(G)] = \gamma_{n-k}(G)$, where $k = 0$ for $n = 3$ and $0 \leq k \leq n-4$ if $n \geq 4$, also for $n \geq p+1$ we have $k = min\{n-4, p-2\}$ [[13], p.331].
k gives a measure for the deviation from the maximal degree of commutativity $[\chi_2(G), \gamma_2(G)] = 1$ and is called *defect of commutativity* of G.
With a further invariant e, it will be expressed, which factor $\gamma_j(G)/\gamma_{j+1}(G)$ is cyclic for the first time in the lower central series [15], and we have $e+1 = min\{3 \leq j \leq c \,|\, 1 \leq |\gamma_j(G)/\gamma_{j+1}| \leq p\}$.
In this definition of e, we exclude $\gamma_2(G)/\gamma_3(G)$, since is always cyclic. The value $e = 2$ is characteristic for a group G of maximal class. For $e \geq 3$, that is for G of coclass $cc(G) \geq 2$, we can also define $e = max\{3 \leq j \leq c-1 \,|\, |\gamma_j(G)/\gamma_{j+1}| > p\}$ [[11], Definition 2.2].

2.1 *On the 5-class group of maximal class*

Let G be a metabelian 5-group of order 5^n, and $cc(G) = 1$. Then G is of maximal class and the commutator factor $G/\gamma_2(G)$ is of type $(5,5)$ [13].
By $G \sim G_a^{(n)}(z,w)$, we denote the representative of an isomorphism class of metabelian 5-groups G, with a system of invariants z,w and a, which satisfy the theorem:

Theorem 2.*Let G be a metabelian 5-group of order 5^n where $n \geq 5$ and $k = k(G)$ its invariant defined before. Suppose that G is of maximal class, then G can be generated by two elements, $G =< x,y >$, be selected such that $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$.*
Let $s_2 = [y,x] \in \gamma_2(G)$ and $s_j = [s_{j-1}, x] \in \gamma_j(G)$ for $j \geq 3$. Then we have:

–$s_j^5 s_{j+1}^{10} s_{j+2}^{10} s_{j+3}^5 s_{j+4} = 1$ *for* $j \geq 2$.
–$x^5 = s_{n-1}^w$ *with* $w \in \{0,1,2,3,4\}$.
–$y^5 s_2^{10} s_3^{10} s_4^5 s_5 = s_{n-1}^z$ *with* $z \in \{0,1,2,3,4\}$.

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–$[y,s_2]=\prod_{i=1}^{k} s_{n-i}^{a_{n-i}}$ *with* $a=(a_{n-1},...a_{n-k})$ *exponents such that* $0\leq a_{n-i}\leq 4$.

*Proof.*See [[14], Theorem 1] for $p=5$.

Let $G=\langle x,y\rangle$ be a metabelian 5-group of maximal class and order 5^n, such that $G/\gamma_2(G)$ is of type $(5,5)$, then G admits six maximal normal subgroups $H_1,...,H_6$, which contain $\gamma_2(G)$ as a normal subgroup of index 5. We have that $\chi_2(G)$ is one of the groups H_i.
We keep that $x\in G\setminus\chi_2(G)$ and $y\in\chi_2(G)\setminus\gamma_2(G)$. The six normal maximal subgroups $H_1,...,H_6$ are arranged as follows: $H_1=\langle y,\gamma_2(G)\rangle=\chi_2(G)$, $H_i=\langle xy^{i-2},\gamma_2(G)\rangle$ for $2\leq i\leq 6$.
The order of the abelianization of each H_i, is given by the following theorem.

Theorem 3.*Let G, H_i and the invariant k be as before. Suppose that the commutator group $\gamma_2(G)$ is abelian. Then for $1\leq i\leq 6$, the order of the commutator factor groups of H_i is given by:*

–*If $n=2$ we have : $|H_i/\gamma_2(H_i)|=5$ for $1\leq i\leq 6$.*
–*If $n\geq 3$ we have : $|H_i/\gamma_2(H_i)|=5^2$ for $2\leq i\leq 6$, and $|H_1/\gamma_2(H_1)|=5^{n-k-1}$*

*Proof.*See [[11], Theorem 3.1] for $p=5$.

Lemma 1.*Let G be a 5-group of order $|G|=5^n$ where $n\geq 3$, with abelian commutator group $\gamma_2(G)$. Assume that $G/\gamma_2(G)$ is of type $(5,5)$. Then G is of maximal class if and only if G admits a normal maximal subgroup with factor commutator of order 5^2. Furthermore G admits at least five normal maximal subgroups with factor commutator of order 5^2.*

*Proof.*Suppose that G is of maximal class, then according to Theorem 3, we conclude that G has five maximal normal subgroups with the order of commutator factor is 5^2 if $n\geq 4$, and has six when $n=3$. Conversely, Assume that $cc(G)\geq 2$, the invariant e defined before is greater than 3, and since each maximal normal subgroup H of G verify $|H/\gamma_2(H)|\geq 5^e$ we get that $|H/\gamma_2(H)|>5^2$

2.2 *On the transfer concept*

Let G be a group and let H be a subgroup of G. The transfer from G to H, denoted $V_{G\to H}$, can be decomposed as follows:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & H/\gamma_2(H) \\ G/\gamma_2(G) \arrow[ur, "V_{G\to H}"'] & \end{tikzcd}$$

Definition 1.*Let G be a group, H be a normal subgroup of G, and let $g\in G$ such that, f is the order of gH in G/H, $r=\frac{[G:H]}{f}$ and $g_1,....g_r$ be a representative system of G/H modulo $\langle gH\rangle$, then the transfer from G to H is defined by:*

$$\begin{aligned} V_{G\to H}: G/\gamma_2(G) &\longrightarrow H/\gamma_2(H)\\ g\gamma_2(G) &\longrightarrow \textstyle\prod_{i=1}^{r} g_i^{-1}g^f g_i\gamma_2(H)\end{aligned}$$

according to [[2], p50].
In the special case that G/H is a cyclic group of order 5 and $G=\langle h,H\rangle$, then the transfer $V_{G\to H}$ is given as:

(1) If $g\in H$, we have $V_{G\to H}(g\gamma_2(G))=g^{1+h+h^2+h^3+h^4}\gamma_2(H)$

(2) $V_{G\to H}(h\gamma_2(G))=h^5\gamma_2(H)$

Let G be a 5-group of maximal class, with $G/\gamma_2(G)$ is of type $(5,5)$, we keep that $x\in G\setminus\chi_2(G)$ and $y\in\chi_2(G)\setminus\gamma_2(G)$. The six maximal normal subgroups $H_1,...,H_6$ are arranged as follows: $H_1=\langle y,\gamma_2(G)\rangle=\chi_2(G)$, $H_i=\langle xy^{i-2},\gamma_2(G)\rangle$ for $2\leq i\leq 6$. The image of the transfers from G to its six normal maximal subgroups H_i, $1\leq i\leq 6$, is given by the following theorem:

Theorem 4.*Let $G=\langle x,y\rangle$ be a metabelian 5-group of maximal class of order 5^n, where $n\geq 3$, and H_i, $1\leq i\leq 6$ are its six maximal normal subgroups. Assume that x and y are selected such that $x\in G\setminus\chi_2(G)$ and $y\in\chi_2(G)\setminus\gamma_2(G)$, and the relations of Theorem 2 with exponents w,z are satisfied. Suppose that the cosets $g\gamma_2(G)\in G/\gamma_2(G)$ are represented in the form $g\equiv x^jy^l\,(\bmod\,\gamma_2(G))$ with $0\leq j,l\leq 4$, then the images of the transfers $V_{G\to H_i}$ are given by:*

$$V_{G\to H_i}(x^jy^l\gamma_2(G))=s_{n-1}^{wj+zl}\gamma_2(H_i)\ for\ 1\leq i\leq 6.$$

*Proof.*See [[10], Theorem 2.2] for $p=5$.

2.3 *Invariants of metabelian 5-group of maximal class*

In this paragraph, we investigate the purely group theoretic results to determine the invariants of metabelian 5-group of maximal class developed in Theorem 2. We keep the same hypothesis of the group G, the generators x and y of the group $G=\langle x,y\rangle$, and the six normal maximal subgroups H_i, $1\leq i\leq 6$, of G.
In the case that the transfers from two subgroups H_i and H_j to $\gamma_2(G)$ are trivial, we can determine completely the 5-group G.

Proposition 1.*Let G be a metabelian 5-group of maximal class and order 5^n, $n\geq 5$. If the transfers $V_{\chi_2(G)\to\gamma_2(G)}$ and $V_{H_2\to\gamma_2(G)}$ are trivial, then $n\leq 6$ and $\gamma_2(G)$ is of exponent 5. Furthermore:*

– *If $n=6$ then $G\sim G_a^{(6)}(1,0)$ where $a=0$ or 1.*
– *If $n=5$ then $G\sim G_a^{(5)}(0,0)$ where $a=0$ or 1.*

*Proof.*Let s,z,w and a defined as Theorem 2. Suppose that $n\geq 7$, so $\gamma_5(G)=\langle s_5,\gamma_6(G)\rangle$, because G is of maximal class and $|\gamma_5(G)/\gamma_6(G)|=5$. By [[5], Lemma 3.3] we have $y^5s_5\in\gamma_6(G)$, thus $\gamma_5(G)=\langle s_5^4,\gamma_6(G)\rangle=\langle y^5s_5s_5^4,\gamma_6(G)\rangle=\langle y^5,\gamma_6(G)\rangle$, and since $V_{\chi_2(G)\to\gamma_2(G)}(y)=y^5=1$, because the transfers are trivial by hypothesis, we get that $\gamma_5(G)=\gamma_6(G)$, which is impossible, whence $n\leq 6$ and According to [[5], Theorem 3.2], $\gamma_2(G)$ is of exponent 5.
If $n=6$, we have $V_{\chi_2(G)\to\gamma_2(G)}$ and $V_{H_2\to\gamma_2(G)}$ are trivial, so by Theorem 2 we obtain $x^5=s_5^w=1$ which imply $w=0$, because $0\leq w\leq 4$. Since $\gamma_2(G)$ is of exponent 5, we have $s_2^5=1$ and by Theorem 2 the relation $s_4^5s_5^{10}s_6^{10}s_7^5s_8=1$ gives $s_4^5=1$, also $s_3^5s_4^{10}s_5^{10}s_6^5s_7=1$ gives $s_3^5=1$. We replace in $y^5s_2^{10}s_3^{10}s_4^5s_5=s_5^z$ and we get $s_5=s_5^z$, whence $z=1$. We have $[\chi_2(G),\gamma_2(G)]\subset\gamma_{6-k}(G)\subset\gamma_4(G)$ then $6-k\geq 4$, and $0\leq k\leq 2$, thus $[y,s_2]=s_4^{\alpha\beta}$, $a=(\alpha,\beta)$. If $k=0$, then $a=0$ and $G\sim G_0^{(6)}(1,0)$, if $k=1$ then $a=1$ and $G\sim G_1^{(6)}(1,0)$ and if $k=2$ then $G\sim G_a^{(6)}(1,0)$.
If $n=5$, we have $[\chi_2(G),\gamma_2(G)]\subset\gamma_{5-k}(G)\subset\gamma_4(G)$ then $5-k\geq 4$, and $0\leq k\leq 1$. We have $s_4^5=1$, $s_2^5=s_3^5=1$ and $[y,s_2]=s_4^a$. the relation $y^5s_2^{10}s_3^{10}s_4^5s_5=s_4^z$ imply $s_4^z=1$ so $z=0$. As $n=6$ we obtain $w=0$. If $k=0$ then $G\sim G_0^{(5)}(0,0)$ and if $k=1$ $G\sim G_a^{(5)}(0,0)$.

Proposition 2.*Let G be a metabelian 5-group of maximal class and order 5^n, $n\geq 5$. If the transfers $V_{H_2\to\gamma_2(G)}$ and $V_{H_i\to\gamma_2(G)}$, $3\leq i\leq 6$, are trivial, then we have:*

– *If $n=5$ or 6 then $G\sim G_a^{(n)}(0,0)$.*
– *If $n\geq 7$ then $G\sim G_0^{(n)}(0,0)$.*

*Proof.*Let s,z,w and a defined as Theorem 2. If $n=5$ or 6, by [[5], Theorem 1.6] we have $[\chi_2(G),\gamma_2(G)]=1$ and $[\chi_2(G),\gamma_2(G)]\subset\gamma_4(G)$ elementary, and $(\gamma_2(\chi_2(G)))^5=1$ and $\prod_{i=2}^{3}[\gamma_i(G),\gamma_{5-i}(G)]=1$, we conclude that $(xy)^5=x^5y^5s_2^{10}s_3^{10}s_4^5s_5$ and we have $y^5s_2^{10}s_3^{10}s_4^5s_5=s_{n-1}^z$ then $(xy)^5=x^5s_{n-1}^z$ and since $V_{H_2\to\gamma_2(G)}$ and $V_{H_3\to\gamma_2(G)}$ are trivial then $(xy)^5=x^5=s_{n-1}^z=s_{n-1}^w=1$, thus $z=w=0$. Since $[\chi_2(G),\gamma_2(G)]=\gamma_{n-k}(G)\subset\gamma_4(G)$ we have $n-k\geq 4$, whence $0\leq k\leq 2$ because $n=5$ or 6 then $G\sim G_a^{(n)}(0,0)$.
If $n\geq 7$, according to [[5], corollary 1 p.69] we have, $\gamma_j(G)^5=\gamma_{j+4}(G)$ for $j\geq 2$, and since $y^5s_2^{10}s_3^{10}s_4^5s_5=s_{n-1}^z$ we obtain:

$$y^5=s_{n-1}^zs_5^{-1}s_4^{-5}s_3^{-10}s_2^{-10}\equiv s_{n-1}^zs_5^{-1}\,\mathrm{mod}\gamma_6(\mathrm{G})$$

because $s_2^5\in\gamma_6(G)$, $s_3^5\in\gamma_6(G)$ and $s_4^5\in\gamma_6(G)$, and since $n\geq 7$ we have $s_{n-1}\in\gamma_6(G)$, therefore $V=V_{H_3\to\gamma_2(G)}(y)\equiv s_5^{-1}\,\mathrm{mod}\gamma_6(\mathrm{G})$. Thus $\mathrm{Im}(V)\subset\gamma_5(G)$, In fact $\mathrm{Im}(V)=\gamma_5(G)$, and also we have $y\notin\mathrm{Ker}(V)$ and $\forall f\geq 2\ y^ks_f^l\notin\mathrm{Ker}(V)$. The kernel of V is formed by elements of $\gamma_2(G)$ of exponent 5, its exactly $\gamma_{n-4}(G)$, and since G is of maximal class then the rank of $\gamma_2(G)$ is 2 and $\gamma_2(G)$ admits exactly 25 elements of exponent 5, these elements form

$\gamma_{n-4}(G)$. We conclude that $|\chi_2(G)/\gamma_2(\chi_2(G))| = |\gamma_{n-4}(G)| \times |\gamma_5(G)| = 5^4 \times 5^{n-5} = 5^{n-1} = |\chi_2(G)|$, whence $\chi_2(G)$ is abelian because $\gamma_2(\chi_2(G)) = 1$, consequently $[y, s_2] = 1$, thus $a = 0$. As the cases $n = 5$ or 6 we obtain $(xy)^5 = x^5 s_{n-1}^z$, therefore $z = w = 0$, hence $G \sim G_0^{(n)}(0,0)$.
In the case when $V_{H_2\to\gamma_2(G)}$ and $V_{H_i\to\gamma_2(G)}$, $4 \le i \le 6$ are trivial, according to [[5], Theorem 1.6] we have $(xy^\mu)^5 = x^5(y^5 s_2^{10} s_3^{10} s_4^5 s_5)^\mu = s_{n-1}^w s_{n-1}^{\mu z}$ with $\mu = 2,3,4$, then we can admit the same reasoning to prove the result.

Proposition 3. *Let G be a metabelian 5-group of maximal class of order 5^n, $n \ge 5$. If the transfers $V_{H_i\to\gamma_2(G)}$ and $V_{H_j\to\gamma_2(G)}$, where $i, j \in \{3,4,5,6\}$ and $i \ne j$, are trivial, then we have: $G \sim G_0^{(n)}(0,0)$.*

Proof. Let s, z, w and a defined as Theorem 2. Assume that $H_i = \langle xy^{\mu_1}, \gamma_2(G)\rangle$ and $H_j = \langle xy^{\mu_2}, \gamma_2(G)\rangle$ where $\mu_1, \mu_2 \in \{1,2,3,4\}$ and $\mu_1 \ne \mu_2$. According to [[5], Theorem 1.6] we have already proved that $(xy^{\mu_1})^5 = s_{n-1}^{w+\mu_1 z}$ and $(xy^{\mu_2})^5 = s_{n-1}^{w+\mu_2 z}$. Since $V_{H_i\to\gamma_2(G)}$ and $V_{H_j\to\gamma_2(G)}$ are trivial, we obtain $s_{n-1}^{w+\mu_1 z} = s_{n-1}^{w+\mu_2 z} = 1$ then $w + \mu_1 z \equiv w + \mu_2 z \equiv 0 \pmod 5$ and since 5 does not divide $\mu_1 - \mu_2$ we get $z = 0$ and at the same time $w = 0$. To prove $a = 0$ we admit the same reasoning as Proposition 2.

3 Realisation of Metabelian 5-Group of Maximal Class

Let k be the normal closure of a pure quintic field $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where n is a 5^{th} power-free natural number. Let $k_0 = \mathbb{Q}(\zeta_5)$ be the 5^{th} cyclotomic field. Since $k = \mathbb{Q}(\zeta_5, \sqrt[5]{n}) = k_0(\sqrt[5]{n})$, we have that k is a cyclic Kummer extension of degree 5 of k_0. By $k^* = (k/k_0)^*$ we denote the relative genus field of k/k_0, that is the maximal unramified extension of k and abelian on k_0. Let $C_{k,5}$ be the 5-class group of k. By class field theory, k^* is contained in the Hilbert 5-class field of k, and $[k^* : k] = 5^t$, where t is the rank of the subgroup $C_{k,5}^{(\sigma)}$ of ambiguous classes under the action of $Gal(k/k_0) = \langle\sigma\rangle$ [[7], Lemma 2.3].
According to [[8], Proposition 5.8], we have the explicit form of k^*, depending on the decomposition of n in k_0.
Let p and q be a prime numbers such that $p \equiv -1 \pmod 5$ and $q \equiv \pm 2 \pmod 5$. According to [[6], Theorem 1.1, Conjecture 4.1], if $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, we have three forms of the natural number n as follows:

– $n = 5^e p$ with $e \in \{1,2,3,4\}$ and $p \not\equiv -1 \pmod{25}$.
– $n = p^e q \equiv \pm 1 \pm 7 \pmod{25}$ with $e \in \{1,2,3,4\}$, $p \not\equiv -1 \pmod{25}$ and $q \not\equiv \pm 7 \pmod{25}$.
– $n = p^e \equiv \pm 1 \pm 7 \pmod{25}$ with $e \in \{1,2,3,4\}$ and $p \equiv -1 \pmod{25}$.

Under these hypothesis, the extension k^*/k_0 has order 25 and admits six sub-extensions, where k is one of them, determined explicitly as follows:

Proposition 4. *We keep the same hypothesis on k, k^*, $C_{k,5}$ and $C_{k,5}^{(\sigma)}$. Then we have:*

– *If $n = p \equiv -1 \pmod{25}$, then the six sub-extensions of $(k/k_0)^*/k_0$ are: k, $k_0(\sqrt[5]{\pi_1^{\alpha_1}\pi_2^{\alpha_2}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+1}\pi_2^{\alpha_2+1}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+2}\pi_2^{\alpha_2+2}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+3}\pi_2^{\alpha_2+3}})$ and $k_0(\sqrt[5]{\pi_1^{\alpha_1+4}\pi_2^{\alpha_2+4}})$.*

– *If $n = p^e q$ with $p \not\equiv -1 \pmod{25}$ and $q \not\equiv \pm 7 \pmod{25}$, then the six sub-extensions of $(k/k_0)^*/k_0$ are: k, $k_0(\sqrt[5]{\pi_1^{\alpha_1} q})$, $k_0(\sqrt[5]{\pi_1^{e+\alpha_1}\pi_2^e q^2})$, $k_0(\sqrt[5]{\pi_1^{2e+\alpha_1}\pi_2^e q^3})$, $k_0(\sqrt[5]{\pi_1^{3e+\alpha_1}\pi_2^e q^4})$ and $k_0(\sqrt[5]{\pi_1^{4e+\alpha_1}\pi_2^e})$.*

– *If $n = 5^e p$ with $p \not\equiv -1 \pmod{25}$, then the six sub-extensions of $(k/k_0)^*/k_0$ are: k, $k_0(\sqrt[5]{\lambda^e\pi_1^{\alpha_1}\pi_2^{\alpha_2}})$, $k_0(\sqrt[5]{\lambda^{2e}\pi_1^{\alpha_1+1}\pi_2^{\alpha_2+1}})$, $k_0(\sqrt[5]{\lambda^{3e}\pi_1^{\alpha_1+2}\pi_2^{\alpha_2+2}})$, $k_0(\sqrt[5]{\lambda^{4e}\pi_1^{\alpha_1+3}\pi_2^{\alpha_2+3}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+4}\pi_2^{\alpha_2+4}})$.*

Where $\alpha_1, \alpha_2, e \in \{1,2,3,4\}$ with $\alpha_1 \ne \alpha_2$, π_1, π_2 are primes of k_0 above p such that $p = \pi_1\pi_2$ and $\lambda = 1 - \zeta_5$ is the unique prime of k_0 above 5.

Proof..

– If $n = p \equiv -1 \pmod{25}$, according to the proof of [[8], Theorem 5.15] we have $k^* = k(\sqrt[5]{\pi_1^{\alpha_1}\pi_2^{\alpha_2}})$ with $\alpha_1 \ne \alpha_2$, then $k^* = k_0(\sqrt[5]{\pi_1\pi_2}, \sqrt[5]{\pi_1^{\alpha_1}\pi_2^{\alpha_2}})$, because $p = \pi_1\pi_2$ in k_0, hence the six sub-extensions given are proved.

–If $n = p^e q$ with $p \not\equiv -1 (\mod 25)$ and $q \not\equiv \pm 7 (\mod 25)$, according to [[8], Proposition 5.8] we have $k^* = k(\sqrt[5]{\pi_1^{\alpha_1} q})$ since $p = \pi_1 \pi_2$ and q is inert in k_0. Then $k^* = k_0(\sqrt[5]{\pi_1^e \pi_2^e q}, \sqrt[5]{\pi_1^{\alpha_1} q})$, so we get the six sub-extensions by calculus.

–If $n = 5^e p$ with $p \not\equiv -1 (\mod 25)$, by the proof of [[8], Theorem 5.15], we have $k^* = k(\sqrt[5]{\lambda^e \pi_1^{\alpha_1} \pi_2^{\alpha_2}})$, $\alpha_1 \neq \alpha_2$. Since $n = 5^e p = (\lambda^4)^e \pi_1 \pi_2 = \lambda^{e'} \pi_1 \pi_2$ in k_0, then $(k/k_0)^* = k_0(\sqrt[5]{\lambda^e \pi_1 \pi_2}, \sqrt[5]{\lambda^e \pi_1^{\alpha_1} \pi_2^{\alpha_2}})$, hence the six sub-extensions given are proved.

By means of the explicit forms of the six sub-extensions of k^*/k_0 given by Proposition 4, we can state the following theorem:

Theorem 5. *Let $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ be the normal closure of $\mathbb{Q}(\sqrt[5]{n})$. Let k_0 be the the 5^{th} cyclotomic field. Suppose that the 5-class group $C_{k,5}$ of k is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, then $Gal(k^*/k_0)$ is of type $(5,5)$, and two sub-extensions of k^*/k_0 admit a trivial 5-class number.*

Proof. Since $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, then by class field theory, we have $[k^* : k] = 5$, whence $Gal(k^*/k_0)$ is of type $(5,5)$.

–If $n = p \equiv -1 (\mod 25)$ the six sub-extensions of k^*/k_0 are k, $k_0(\sqrt[5]{\pi_1^{\alpha_1} \pi_2^{\alpha_2}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+1} \pi_2^{\alpha_2+1}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+2} \pi_2^{\alpha_2+2}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+3} \pi_2^{\alpha_2+3}})$ and $k_0(\sqrt[5]{\pi_1^{\alpha_1+4} \pi_2^{\alpha_2+4}})$, with $e, \alpha_1, \alpha_2 \in \{1,2,3,4\}$, and $\alpha_1 \neq \alpha_2$. For each values of α_1 and α_2, we can see that the extensions $L_1 = k_0(\sqrt[5]{\pi_1})$ and $L_2 = k_0(\sqrt[5]{\pi_2})$ are sub-extensions of k^*/k_0.
In [[8], section 5.1], we have a detailed investigation of the rank of ambigous classes of $k_0(\sqrt[5]{x})/k_0$ denoted t. Precisely, we have $t = d + q^* - 3$, where d is the number of prime divisors of x in k_0, and q^* is an index of units defined as [[8], section 5.1].
For the extensions L_i/k_0, $(i = 1,2)$ we have $d = 1$, and by [[8], Theorem 5.15] we have $q^* = 2$, hence $t = 0$.

–If $n = p^e q$ with $p \not\equiv -1 (\mod 25)$ and $q \not\equiv \pm 7 (\mod 25)$ then the six sub-extensions of k^*/k_0 are: k, $k_0(\sqrt[5]{\pi_1^{\alpha_1} q})$, $k_0(\sqrt[5]{\pi_1^{e+\alpha_1} \pi_2^e q^2})$, $k_0(\sqrt[5]{\pi_1^{2e+\alpha_1} \pi_2^e q^3})$, $k_0(\sqrt[5]{\pi_1^{3e+\alpha_1} \pi_2^e q^4})$, $k_0(\sqrt[5]{\pi_1^{4e+\alpha_1} \pi_2^e})$. For the four extensions k , $k_0(\sqrt[5]{\pi_1^{e+\alpha_1} \pi_2^e q^2})$, $k_0(\sqrt[5]{\pi_1^{2e+\alpha_1} \pi_2^e q^3})$ and $k_0(\sqrt[5]{\pi_1^{3e+\alpha_1} \pi_2^e q^4})$, we have $d = 3$ and by [[8], Theorem 5.14] we have $q^* = 1$, so $t = 1$. For the extensions $k_0(\sqrt[5]{\pi_1^{4e+\alpha_1} \pi_2^e})$ and $k_0(\sqrt[5]{\pi_1^{\alpha_1} q})$ we have $d = 2$ and also $q^* = 1$, so $t = 0$.

–If $n = 5^e p$ with $p \not\equiv -1 (\mod 25)$, then the six sub-extensions of k^*/k_0 are: k, $k_0(\sqrt[5]{\lambda^e \pi_1^{\alpha_1} \pi_2^{\alpha_2}})$, $k_0(\sqrt[5]{\lambda^{2e} \pi_1^{\alpha_1+1} \pi_2^{\alpha_2+1}})$, $k_0(\sqrt[5]{\lambda^{3e} \pi_1^{\alpha_1+2} \pi_2^{\alpha_2+2}})$, $k_0(\sqrt[5]{\lambda^{4e} \pi_1^{\alpha_1+3} \pi_2^{\alpha_2+3}})$, $k_0(\sqrt[5]{\pi_1^{\alpha_1+4} \pi_2^{\alpha_2+4}})$.
Since $\alpha_1 \neq \alpha_2$, we get that $k_0(\sqrt[5]{\pi_i^{\alpha_i}})$ and $k_0(\sqrt[5]{\lambda^e \pi_j^{\alpha_j}})$ with $i, j \in \{1,2,3,4\}$ and $i \neq j$, are among the six. We see that these two extensions admit the value $t = 0$.

In short, we just proved that if $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, then two sub-extensions, denoted L_1 and L_2, of k^*/k_0 admit 0 as value of rank of the subgroup of ambiguous classes.
By $h_5(L_i)$, $(i = 1,2)$, we denote the class number of L_i, then we have $h_5(L_1) = h_5(L_2) = 1$. Otherwise $h_5(L_i) \neq 1$, then there exists a cyclic unramified extension of L_i, denoted F. We have that F is abelian over k_0, as $[F : k_0] = 5^2$, then F is contained in $(L_i/k_0)^*$ the relative genus field of L_i/k_0. Since $[(L_i/k_0)^* : L_i] = 5^t = 1$, we get that $(L_i/k_0)^* = L_i$, which contradicts the existence of F. Hence the 5-class number of L_i, $(i = 1,2)$, is trivial.

In the sequel, we denote by L_1 and L_2 the two sub-extensions of k^*/k_0, which verify Theorem 5, and by $\tilde{L}$ the three remaining sub-extensions different to k.
Let $G = Gal((k^*)_5^{(1)}/k_0)$, we have $\gamma_2(G) = Gal((k^*)_5^{(1)}/k^*)$, then $G/\gamma_2(G) = Gal(k^*/k_0)$ is of type $(5,5)$, therefore G is metabelian 5-group with factor commutator of type $(5,5)$, thus G admits exactly six maximal normal subgroups as follows:

$$H = Gal((k^*)_5^{(1)}/k),\ H_{L_i} = Gal((k^*)_5^{(1)}/L_i),\ (i = 1,2),\ \tilde{H} = Gal((k^*)_5^{(1)}/\tilde{L}).$$

With $\chi_2(G)$ is one of them.

Theorem 6. *Let $G = Gal((k^*)_5^{(1)}/k_0)$ be a 5-group of order 5^n, $n \geq 5$, then G is metabelian of maximal class. Furthermore we have:*

- *If* $\chi_2(G) = H_{L_i}(i=1,2)$ *then:* $G \sim G_a^{(n)}(z,0)$ *with* $n \in \{5,6\}$ *and* $a,z \in \{0,1\}$.
- *If* $\chi_2(G) = \tilde{H}$ *then :* $G \sim G_a^{(n)}(0,0)$ *with* $n = 5$ *or* 6.

$$G \sim G_0^{(n)}(0,0) \text{ with } n \geq 7 \text{ such that } n = s+1 \text{ where } h_5(\tilde{L}) = 5^s.$$

*Proof.*Let $G = Gal((k^*)_5^{(1)}/k_0)$ and $H = Gal((k^*)_5^{(1)}/k)$ its maximal normal subgroup, then $\gamma_2(H) = Gal((k^*)_5^{(1)}/k_5^{(1)})$, therefore $H/\gamma_2(H) = Gal(k_5^{(1)}/k) \simeq C_{k,5}$, and as $C_{k,5}$ is of type $(5,5)$ by hypothesis we get that $|H/\gamma_2(H)| = 5^2$. Lemma 1 imply that G is a metabelian 5-group of maximal class, generated by two elements $G = \langle x,y \rangle$, such that, $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$. Since $\chi_2(G) = \langle y, \gamma_2(G) \rangle$, we have $\chi_2(G) \neq H$. Otherwise we get that $|H/\gamma_2(H)| = 5^2$, which contradict Theorem 3.
According to Theorem 5, we have $h_5(L_1) = h_5(L_2) = 1$, then the transfers $V_{H_{L_i} \to \gamma_2(G)}$ are trivial.
If $\chi_2(G) = H_{L_i}$ the results are nothing else than Proposition 1.
If $\chi_2(G) = \tilde{H}$ and $n = 4$ then $\gamma_4(G) = 1$ and $[\chi_2(G), \gamma_2(G)] = \gamma_2(\tilde{H})$, also $[\chi_2(G), \gamma_2(G)] = \gamma_4(G) = 1$ then $\chi_2(\tilde{H}) = 1$, whence $\tilde{H}$ is abelian. Consequently $\tilde{H}/\gamma_2(\tilde{H}) = C_{\tilde{L},5}$, so $h_5(\tilde{L}) = |\tilde{H}| = 5^3$ because its a maximal subgroup of G. Since $\tilde{L}$ and k have always the same conductor, we deduce that $h_5(k)$ and $h_5(\tilde{L})$ verify the relations $5^5 h_{\tilde{L}} = uh_\Gamma^4$ and $5^5 h_k = uh_\Gamma^4$, given by C. Parry in [16], where u is a unit index and a divisor of 5^6. Using the 5-valuation on these relations we get that $h_5(\tilde{L}) = 5^s$ where s is even, which contradict the fact that $h_5(\tilde{L}) = 5^3$, hence $n \geq 5$.
The results of the theorem are exactly application of Propositions 2, 3.
According to Proposition 2, if $n \geq 7$ we have $|\chi_2(G)| = 5^{n-1}$ and since $h_5(\tilde{L}) = |\tilde{H}/\gamma_2(\tilde{H})| = |\tilde{H}| = 5^{n-1} = 5^s$, we deduce that $n = s+1$.

4 Proof of Main Theorem

Let k be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ having elementary abelian bicyclic 5-class group $C_{k,5}$ of type $(5,5)$. the subgroup $C_{k,5}^{(\sigma)}$ of $C_{k,5}$, of ambiguous ideal classes under the action of $Gal(k/k_0) = \langle \sigma \rangle$ has rank 1 or 2. If rank $C_{k,5}^{(\sigma)} = 1$, then the relative genus field k^* is a quintic cyclic extension of k.
Let $M = Gal((k^*)_5^{(2)}/k_0)$ and let $R = Gal((k^*)_5^{(2)}/k)$ be a normal maximal subgroup of M, whose commutator $\gamma_2(R) = Gal((k^*)_5^{(2)}/k_5^{(1)})$. Then by Artin reciprocity low [1], we get that $R/\gamma_2(R) \simeq Gal(k_5^{(1)}/k) \simeq C_{k,5}$ of type $(5,5)$, which means that $|R/\gamma_2(R)| = 5^2$, hence by Lemma 1, M is a 5-group of maximal class .
We have that $\gamma_2(M) = Gal((k^*)_5^{(2)}/k^*)$ and $\gamma_3(M) = Gal((k^*)_5^{(2)}/(k^*)_5^{(1)})$, which imply that $M/\gamma_2(M) = Gal(k^*/k_0)$ of type $(5,5)$ and $\gamma_2(M)/\gamma_3(M) = Gal((k^*)_5^{(1)}/k^*)$. Since M is of maximal class, rank $\gamma_2(M)/\gamma_3(M) \leq 2$, consequently $\gamma_3(M)$ can be generated by two elements. By [[4], Theorem 2.1], we deduce that $\gamma_2(M)$ is abelian, therefore $\gamma_3(M) = 1$, hence $(k^*)_5^{(1)} = (k^*)_5^{(2)}$, and since $(k^*)_5^{(1)} \subset k_5^{(2)} \subset (k^*)_5^{(2)}$, we conclude that the Hilbert 5-class field tower of k must stop at the second stage, which means that the length of the tower is 2.
D. C. Mayer in [[12], Section 3.1] has studied the structure of the coclass graph on the set of all isomorphism classes $G \sim G_a^{(n)}(z,w)$ of finite p-groups G of maximal class. By means of the position of $G \sim G_a^{(n)}(z,w)$ in the coclass graph we can determine the possible cases of its maximal subgroups.
Since $(k^*)_5^{(1)} = k_5^{(2)}$ then the second 5-class group of k, $G_k^{(2)} = Gal(k_5^{(2)}/k)$, is a maximal subgroup of $G = Gal((k^*)_5^{(1)}/k_0) = Gal(k_5^{(2)}/k_0)$ of index 5. The results of Theorem 6, and the distribution of metabelian 5-groups of maximal class given in [[12], Figure 3.3], allow us to know all possible cases of $G_k^{(2)}$, as follows:

–If $G \sim G_a^{(n)}(z,0)$ with $n \in \{5,6\}$ and $a,z \in \{0,1\}$ then $G_k^{(2)} \sim G_0^{(n)}(0,0)$ with $n \in \{4,5\}$
–If $G \sim G_a^{(n)}(0,0)$ with $n \in \{5,6\}$ then $G_k^{(2)} \sim G_0^{(n-1)}(0,0)$
If $G \sim G_0^{(n)}(0,0)$ with $n \geq 7$ then $G_k^{(2)} \sim G_0^{(n-1)}(0,0)$

We see that all these cases of $G_k^{(2)}$ satisfy $w = z = 0$.
By the Galois correspondence $H_i = Gal(k_5^{(2)}/K_i)$, the six maximal normal subgroups $H_1, \ldots, H_6$ of $G_k^{(2)}$ are associated with the six unramified cyclic quintic extensions $K_1, \ldots, K_6$ of k, which are represented by the norm class groups $Norm_{K_i/k}(C_{K_i,5})$ as subgroups of index 5 in $C_{k,5}$ according to [1]. The abelianizations of the H_i are isomorphic to the 5-class groups of the K_i, because,

$$H_i/\gamma_2(H_i) = Gal(k_5^{(2)}/K_i)/Gal(k_5^{(2)}/(K_i)_5^{(1)}) \simeq Gal((K_i)_5^{(1)}/K_i) \simeq C_{K_i,5}$$

We are interested in the capitulation of ideal classes of $C_{k,5}$ in $C_{K_i,5}$. Therefore, we investigate the kernel of the class

extension homomorphism $j_{K_i/k} : C_{k,5} \longrightarrow C_{K_i,5}$.
We say that we have total capitulation if $Ker(j_{K_i/k}) = C_{k,5}$ and partial capitulation otherwise.
According to Artin [2], the following commutative diagram establishes the connection between the number theoretical extensions $j_{K_i/k}$ of 5-class groups and the group theoretical transfers $V_{G_k^{(2)} \to H_i} = V_i$,

$$\begin{tikzcd}
C_{k,5} \arrow[r, "j_{K_i/k}"] \arrow[d, leftrightarrow, "\text{Artin isomorphism}"'] & C_{K_i,5} \arrow[d, leftrightarrow, "\text{Artin isomorphism}"] \\
G_k^{(2)}/\gamma_2(G_k^{(2)}) \arrow[r, "V_i"'] & H_i/\gamma_2(H_i)
\end{tikzcd}$$

The commutativity of this diagram shows that $Ker(j_{K_i/k}) \simeq Ker(V_i)$ for $1 \leq i \leq 6$.
According to Theorem 4, the image of the transfer V_i is $V_i(g\gamma_2(G_k^{(2)})) = s_{n-1}^{wj+zl}\gamma_2(H_i)$, where $g \equiv x^j y^l (\mathrm{mod}\gamma_2(G_k^{(2)}))$ and $G_k^{(2)} = \langle x, y\rangle$. Since for all possible cases of $G_k^{(2)}$ we have $z = w = 0$, then all images of the transfers are trivial, $V_i(g\gamma_2(G_k^{(2)})) = 1$, for all $g \in G_k^{(2)}$ and $1 \leq i \leq 6$. Thus we have $Ker(V_i) = G_k^{(2)}/\gamma_2(G_k^{(2)})$, which means that $Ker(j_{K_i/k}) = C_{k,5}$. Hence we have total capitulation of $C_{k,5}$ in the six cyclic quintic unramified extensions of k.

5 Numerical Examples

For these numerical examples of the natural n, we have that $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, which means that k^* is cyclic quintic extension of k. Hence by Theorem 1, the length of the 5-class field tower of k is 2, and there is total capitulation of $C_{k,5}$ in the six cyclic quintic unramified extensions of k. We note that the absolute degree of $(k^*)_5^{(1)}$ surpass 100, then the task to determine the order of G is definitely far beyond the reach of computational algebra systems like MAGMA and PARI/GP.

Table 1: $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ with $C_{k,5}$ is of type $(5,5)$ and rank$C_{k,5}^{(\sigma)} = 1$.

n	$h_{k,5}$	$C_{k,5}$	rank $(C_{k,5}^{(\sigma)})$		n	$h_{k,5}$	$C_{k,5}$	rank $(C_{k,5}^{(\sigma)})$
118	25	$(5,5)$	1		1999	25	$(5,5)$	1
145	25	$(5,5)$	1		2007	25	$(5,5)$	1
449	25	$(5,5)$	1		2507	25	$(5,5)$	1
475	25	$(5,5)$	1		2725	25	$(5,5)$	1
559	25	$(5,5)$	1		6725	25	$(5,5)$	1
718	25	$(5,5)$	1		7375	25	$(5,5)$	1
818	25	$(5,5)$	1		7493	25	$(5,5)$	1
1018	25	$(5,5)$	1		28625	25	$(5,5)$	1
1195	25	$(5,5)$	1		55625	25	$(5,5)$	1
1249	25	$(5,5)$	1		168125	25	$(5,5)$	1
1499	25	$(5,5)$	1		149^2	25	$(5,5)$	1
1945	25	$(5,5)$	1		199^3	25	$(5,5)$	1

Declarations

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