Jordan Journal of Mathematics and Statistics.
*Yarmouk University*

DOI:https://doi.org/10.47013/18.1.3

# The Continuous Classical Optimal Control for a Couple Fourth Order Linear Elliptic Equations

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Received: Sept. 18, 2023 Accepted : April. 16, 2024

**Abstract:** In this paper, the finite element Galerkin method (FEGM) with piecewise cubic Hermite basis function is applied to prove the existence and uniqueness of a couple state vector solution for a system of fourth-order linear partial differential equations (PDEs) of elliptic type with Dirichlet-Neumann boundary conditions (DNBCs), when the continuous classical couple control vector (CCCPCV) is considered. An existence theorem for a coupled continuous classical optimal control vector associated with the fourth-order linear PDEs of elliptic type is formulated and proved under appropriate conditions. The paper also discusses the existence and uniqueness of the solution to the coupled adjoint equations involving the couple state vector, when the classical couple optimal control vector is given. Finally, the derivation of the Fréchet derivative (FrD) of the cost function (CFn) to establish the theorem of the necessary condition (NEC) for optimality of the considered problem is demonstrated.

**Keywords:** A continuous classical couple optimal control; The finite element Galerkin method ;The necessary condition for optimality; Fourth order linear elliptic PDEs.

**2010 Mathematics Subject Classification**. 26A25; 26A35.

## 1 Introduction

The continuous classical optimal control problems (CCOCPs) were developed in the beginning of this century, governing by either partial differential equations (PDEs) [2][4] or ordinary differential equations (ODEs) [1]. The use of optimal control problems (OCPs) has become widespread in many real-life such as economic [11], biology [10], electric power[12], and aircraft [9], and many others field. In the last century, many researchers were interested in studying OCPs governing by either ODEs [8] or linear PDEs [7]. In the resent years, the importance of OCPs has led to increased attention from researchers, who are now studying and developing OCPs involving second-order nonlinear PDEs of elliptic type [3], hyperbolic type [6] or parabolic type[5]. the OCP considered in this work is governing by coupled fourth-order linear elliptic PDEs with DNBCs, In this paper, the FEGM with piecewise cubic Hermite (PCH) basis function is applied to prove the existence and uniqueness of a couple state vector (CPSV) solution for a systen of coupled fourth-order linear elliptic PDEs (LEPDEs) with DNBCs, when the CCCPCV is considered. Under appropriate assumptions, the paper develops and proves an existence theorem of a continuous classical couple optimal control vector (CCCPOCV) associated with the couple fourth-order linear PDEs of elliptic type. Additionally it discusses the existence and uniqueness of the solution of the couple adjoint equations involving the CPSV, when the CCCPOCV is given. Furthermore, it derives the FrD of the cost function (CFn). Finally, the paper establishes the theorem of the NEC for optimality of the considered problem.

## 2 Problem Statement

Let $\Omega$ be an open and bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ in $R^2$. Consider the CCOCP of a coupled fourth order LEPDEs with DNBCs:

$$\Delta^2 y_1 - \Delta y_1 + y_1 - y_2 = f_1(X) + q_1, on\Omega \tag{1}$$

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$$\Delta^2 y_2 - \Delta y_2 + y_2 + y_1 = f_2(X) + q_2, on\Omega \tag{2}$$

$$y_1 = 0, on\Gamma \tag{3}$$

$$\frac{\partial y_1}{\partial n} = 0, on\Gamma \tag{4}$$

$$y_2 = 0, on\Gamma \tag{5}$$

$$\frac{\partial y_2}{\partial n} = 0, on\Gamma \tag{6}$$

where $\overrightarrow{y} = (y_1, y_2) = (y_1(x_1, x_2), y_2(x_1, x_2)) \in (H_0^4(\Omega))^2$is the CPSV corresponding to CCCPCV $(q_1, q_2) = (q_1(x_1, x_2), q_2(x_1, x_2)) \in (L^2(\Omega))^2$ and $(f_1, f_2) = (f_1(X), f_2(X)) \in (L^2(\Omega))^2$ is a vector of a given function defined on $\Omega \times \Omega$ for all $X = (x_1, x_2) \in \Omega$.
The set of admissible CCCPCV is
$\overrightarrow{Q_a} \subset L^2(\Omega) \times L^2(\Omega)$
$= \{\overrightarrow{q} = (q_1, q_2) \in (L^2(\Omega))^2 | (q_1, q_2) \in Q_1 \times Q_2 = \overrightarrow{Q} \subset R^2 a.e.in\Omega\}$ with $\overrightarrow{Q} \subset R^2$ is convex and bounded
The CFn is given by

$$Min.J_0(\overrightarrow{q}) = \frac{1}{2} \parallel y_1 - y_{1d} \parallel^2 + \frac{1}{2} \parallel y_2 - y_{2d} \parallel^2 + \frac{1}{2} \parallel q_1 \parallel^2 + \frac{1}{2} \parallel q_2 \parallel^2, (q_1, q_2) \in \overrightarrow{Q_a} \tag{7}$$

where $(y_{1d}, y_{2d}) = (y_{1d}(x_1, x_2), y_{2d}(x_1, x_2))$ is the desired state and $(y_1, y_2) = (y_{1q_1}, y_{2q_2})$ is the solution of CPSV (1-6) corresponding to the CCCPCV $\overrightarrow{q} = (q_1, q_2)$.
The continuous classical couple optimal control problem (CCCPOCP) is to minimize the CFn (7) subject to $(q_1, q_2) \in \overrightarrow{Q_a}$ where the notations $(u, u)$ and $(\overrightarrow{u}, \overrightarrow{u})_{(L^2(\Omega))^2}$ denote the inner product in $L^2(\Omega)$ and $(L^2(\Omega))^2$ respectively, by $\parallel u \parallel$ and $\parallel \overrightarrow{u} \parallel_{(L^2(\Omega))^2} = \sum_{i=1}^n \parallel u_i \parallel_{L^2(\Omega)}$ denote the norm in $L^2(\Omega)$ and $(L^2(\Omega))^2$ respectively, by $\parallel u \parallel_{H_0^2(\Omega)}$ and $\parallel \overrightarrow{u} \parallel_{(H_0^2(\Omega))^2} = \sum_{i=1}^n \parallel u_i \parallel_{H_0^2(\Omega)}$ denote the norm in $H_0^2(\Omega)$ and $(H_0^2(\Omega))^2$ respectively, , also the notations$\rightharpoonup$and $\longrightarrow$will refer to the weak convergence and strong convergence of a sequence respectively.

## 3 Solution of the CPSV Equations (CPSVEs)

In order to find the classical solution of problem (1-6), we shall first obtain their weak forms (WFs). Let
$\overrightarrow{S} = H_0^2(\Omega) \times H_0^2(\Omega)$
$= \{\overrightarrow{v} : \overrightarrow{v} = (v_1, v_2) = (v_1(x_1, x_2), v_2(x_1, x_2)) \in (H_0^2(\Omega))^2, \forall (x_1, x_2) \in \Omega$, with $v_1 = v_2 = 0$ and $\frac{\partial y_1}{\partial n} = \frac{\partial y_2}{\partial n} = 0$ on $\Gamma\}$
The WF of the CPSVEs (1-6), when $\overrightarrow{y} \in (H_0^2(\Omega))^2$ are obtained by multiplying both sides of equations (1) and (2) by $v_1 \in H_0^2(\Omega)$ and $v_2 \in H_0^2(\Omega)$ respectively, integrating both sides of the obtained equations over $\Omega$and then using the generalized Green's theorem twice for the first term which have the fourth order derivatives and once for the second term which have the second order derivatives, we obtain the WFs.

$$(\Delta y_1, \Delta v_1) + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) = (f_1(X), v_1) + (q_1, v_1), \forall v_1 \in H_0^2(\Omega) \tag{8}$$

and

$$(\Delta y_2, \Delta v_2) + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_1, v_2) = (f_2(X), v_2) + (q_2, v_2), \forall v_2 \in H_0^2(\Omega) \tag{9}$$

By adding (8) and (9), we find $\forall \overrightarrow{y} \in (H_0^2(\Omega))^2$

$$E(\overrightarrow{y}, \overrightarrow{v}) = l(\overrightarrow{v}), \forall \overrightarrow{v} = (v_1, v_2) \in \overrightarrow{S} \tag{10}$$

Where the symmetric BLF $E(\overrightarrow{y}, \overrightarrow{v})$ and the continuous linear form $l(\overrightarrow{v})$ are defined when $\overrightarrow{q} \in (L^2(\Omega)^2$ is fixed by

$$E(\overrightarrow{y}, \overrightarrow{v}) = (\Delta y_1, \Delta v_1) + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (\Delta y_2, \Delta v_2) + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_1, v_2) \tag{11}$$

$$l(\overrightarrow{v}) = (f_1(X), v_1) + (q_1, v_1) + (f_2(X), v_2) + (q_2, v_2), \forall (v_1, v_2) \in \overrightarrow{S} \tag{12}$$

## *3.1 Assumptions*

1.The bilinear form $E(.,.)$ is satisfied the following properties:

a) $E(\overrightarrow{y},\overrightarrow{v})$ is coercive, i.e. $\forall\overrightarrow{y}\in\overrightarrow{S}\,\exists c_0>0$ such that $E(\overrightarrow{y},\overrightarrow{v})\geq c_0\parallel\overrightarrow{y}\parallel^2_{(H_0^2(\Omega))^2}$.

b) $E(\overrightarrow{y},\overrightarrow{v})$s continuous, i.e.$\exists\, c_1>0$ such that $\mid E(\overrightarrow{y},\overrightarrow{v})\mid\leq c_1\parallel\overrightarrow{y}\parallel_{(H_0^2(\Omega))^2}\parallel\overrightarrow{v}\parallel_{(H_0^2(\Omega))^2},\forall\overrightarrow{y},\overrightarrow{v}\in\overrightarrow{S}$

2.$l(\overrightarrow{v})$is a bounded functional on $\overrightarrow{S}$ where $\overrightarrow{q}$ is bounded, i.e. $\exists c_2>0$ such that $\mid l(\overrightarrow{v})\mid\leq c_2\parallel\overrightarrow{v}\parallel_{(H_0^2(\Omega))^2},\forall\overrightarrow{v}\in\overrightarrow{S}$

To find the solution of the general classical problem (10), the GFEM is used by choosing an approximation subspace $\overrightarrow{S_n}\subset\overrightarrow{S}$ (which has a finite dimension $n$) and the problem (10) reduce to the discrete Galerkin WF: find $\overrightarrow{y_n}\in\overrightarrow{S_n}$ such that

$$E(\overrightarrow{y_n},\overrightarrow{v})=l(\overrightarrow{v}),\forall\overrightarrow{v}\in\overrightarrow{S_n}\tag{13}$$

**Theorem 1.***For every fixed CCCPCV* $\overrightarrow{q}=(q_1,q_2)\in(L^2(\Omega))^2$ *there exists a unique approximation solution* $\overrightarrow{y_n}=(y_{1n},y_{2n})\in\overrightarrow{S_n}$ *for problem (13).*

*Proof.*For each $n$, $\overrightarrow{S_n}$ be the set of continuous and PCH type polynomials functions in $\Omega$, since continuity in $(C^1(\Omega))^2$ is required. We will define two Hermite basis functions namely $\overrightarrow{\varphi_j}$ and $\overrightarrow{\bar{\varphi}_j}$, i.e. $\{\overrightarrow{\varphi_1},\overrightarrow{\varphi_2},...,\overrightarrow{\varphi_n},\overrightarrow{\bar{\varphi}_1},\overrightarrow{\bar{\varphi}_2},...,\overrightarrow{\bar{\varphi}_n}\}$be a finite basis of$\overrightarrow{S_n},\forall n$. We now express $\overrightarrow{y_n}=\overrightarrow{y_n}(x_1,x_2)$ as finite linear combination as

$$\overrightarrow{y_n}=\sum_{i=1}^{n}(\overrightarrow{c_j}\overrightarrow{\varphi_j}+\overrightarrow{\bar{c}_j}\overrightarrow{\bar{\varphi}_j})\equiv(\sum_{j=1}^{n}c_{1j}\varphi_{1j}+\bar{c}_{1j}\bar{\varphi}_{1j},\sum_{j=1}^{n}c_{2j}\varphi_{2j}+\bar{c}_{2j}\bar{\varphi}_{2j})\tag{14}$$

where $\overrightarrow{c_j}$ ,$\overrightarrow{\bar{c}_j}$are unknown constant vector,$\forall j=1,2,...,n$.
By substituting the solution $\overrightarrow{y_n}$in equation (13) and $\overrightarrow{v_j}=\overrightarrow{\varphi}_j+\overrightarrow{\bar{\varphi}}_j$, then can be rewriting in matrix notation

$$K\bar{c}=b\tag{15}$$

where $K=(k_{ij})_{n\times n}$, $k_{ij}=E(\overrightarrow{\varphi}_j+\overrightarrow{\bar{\varphi}}_j,\overrightarrow{\varphi}_i+\overrightarrow{\bar{\varphi}}_i)$, $b=(b_i)_{n\times1}$, $b_i=l(\overrightarrow{\varphi}_i+\overrightarrow{\bar{\varphi}}_i)$ and $\overrightarrow{c}=(\overrightarrow{c_1},...,\overrightarrow{c_n},\overrightarrow{\bar{c}_1},...,\overrightarrow{\bar{c}_n})^T$
By using assumption (3.1(1- a)), then equation (15) has a unique solution.

*Remark.*$\forall\overrightarrow{v}\in(H_0^2(\Omega))^2$,there exists a sequence $\{\overrightarrow{\varphi}_n\}$ with $\overrightarrow{\varphi}_n\in\overrightarrow{S_n},\forall n$ and $\overrightarrow{\varphi}_n\longrightarrow\overrightarrow{v}$ in $\overrightarrow{S}$,problem (13) has a unique solution $\overrightarrow{y_n}$ ,hence corresponding to the sequence $\{\overrightarrow{y_n}\}_{n=1}^{\infty}$we have a sequence of approximation problem (13), for each $n=1,2,...,$ i.e., $\overrightarrow{\varphi}_n\in\overrightarrow{S_n}$ such that

$$E(\overrightarrow{y_n},\overrightarrow{\varphi_n})=l(\overrightarrow{\varphi_n}),\forall\overrightarrow{\varphi_n}\in\overrightarrow{S_n},\forall n\tag{16}$$

which has a sequence of$\{\overrightarrow{y_n}\}_{n=1}^{\infty}$

**Theorem 2(Existence of solution of the CPSVEs).** *The sequence of solution*$\{\overrightarrow{y_n}\}_{n=1}^{\infty}$*(of the sequence of WF (16)) converges to* $\overrightarrow{y}$*(solution of (12)).*

*Proof.*Since $\overrightarrow{y_n}$ is a solution of (16), then using assumptions (3.1(1- a)) and (3.1-2), we find $\|\overrightarrow{y_n}\|_{(H_0^2(\Omega))^2}\leq c_2,c_2>0,\forall n$
From Alaoglu theorem [3], there exists a subsequence of $\{\overrightarrow{y_n}\}$ (say again $\{\overrightarrow{y_n}\}$ such that $\overrightarrow{y_n}\rightharpoonup\overrightarrow{y}$ in $\overrightarrow{S}$
We want to show that the sequence $\{\overrightarrow{y_n}\}_{n=1}^{\infty}$ of the solutions of (16) converges to the solution $\overrightarrow{y}$ of (12)
First, to prove the L.H.S. of (16)$\longrightarrow$the L.H.S. of (12)
Since $\overrightarrow{y_n}\rightharpoonup\overrightarrow{y}$ in $\overrightarrow{S}$ and $\overrightarrow{\varphi_n}\longrightarrow\overrightarrow{v}$ in $\overrightarrow{S}$, we obtain
$|E(\overrightarrow{y_n},\overrightarrow{\varphi_n})-E(\overrightarrow{y},\overrightarrow{v})|=|E(\overrightarrow{y_n},\overrightarrow{\varphi_n}-\overrightarrow{v})-E(\overrightarrow{y_n}-\overrightarrow{y},\overrightarrow{v})|$
$\leq c_1\|\overrightarrow{y_n}\|_{(H_0^2(\Omega))^2}\|\overrightarrow{\varphi_n}-\overrightarrow{v}\|_{(H_0^2(\Omega))^2}+c_1\|\overrightarrow{y_n}-\overrightarrow{y}\|_{(H_0^2(\Omega))^2}\|\overrightarrow{v}\|_{(H_0^2(\Omega))^2}\longrightarrow0$
then $E(\overrightarrow{y_n},\overrightarrow{\varphi_n})\longrightarrow E(\overrightarrow{y},\overrightarrow{v})$.
Second, to prove the R.H.S. of (16) $\longrightarrow$ the R.H.S. of (12).
since $\overrightarrow{\varphi_n}\longrightarrow\overrightarrow{v}$ in $\overrightarrow{S}$ and $\overrightarrow{\varphi_n}\rightharpoonup\overrightarrow{v}$ in $\overrightarrow{S}$, and for fixed $\overrightarrow{v}\in\overrightarrow{S}$

$$l_{\overrightarrow{v}}(\overrightarrow{\varphi})=E(\overrightarrow{\varphi},\overrightarrow{v})\text{is linear with respect to}\overrightarrow{S}\tag{17}$$

Then $l(\overrightarrow{\varphi_n}) \longrightarrow l(\overrightarrow{v})$
This gives $E(\overrightarrow{y}, \overrightarrow{v}) = l(\overrightarrow{v}), \forall \overrightarrow{v} \in \overrightarrow{S}$
Therefore $\overrightarrow{y}$ is a solution of (12)
From assumption (3.1(1- a)) and (17), it follows that
$c_0 \|\overrightarrow{y} - \overrightarrow{y_n}\|_{(H_0^2(\Omega))^2} \leq E(\overrightarrow{y} - \overrightarrow{y_n}, \overrightarrow{y} - \overrightarrow{y_n}) = E(\overrightarrow{y} - \overrightarrow{y_n}, \overrightarrow{y}) - E(\overrightarrow{y} - \overrightarrow{y_n}, \overrightarrow{y_n})$
$= E(\overrightarrow{y} - \overrightarrow{y_n}, \overrightarrow{y}) - E(\overrightarrow{y}, \overrightarrow{y_n}) + E(\overrightarrow{y_n}, \overrightarrow{y_n})$
$= l_{\overrightarrow{y}}(\overrightarrow{y} - \overrightarrow{y_n}) \longrightarrow 0$
Therefore $\|\overrightarrow{y} - \overrightarrow{y_n}\|_{(H_0^2(\Omega))^2}$
Hence $\{\overrightarrow{y_n}\}$converges strongly to $\overrightarrow{y}$ with respect to $\|.\|_{(H_0^2(\Omega))^2}$
**Uniqueness of the solution**
Let $\overrightarrow{y_1}, \overrightarrow{y_2}$be two solutions of (12), then
$E(\overrightarrow{y_1}, \overrightarrow{v}) = l(\overrightarrow{v}), \forall \overrightarrow{v} \in \overrightarrow{S}$
$E(\overrightarrow{y_2}, \overrightarrow{v}) = l(\overrightarrow{v}), \forall \overrightarrow{v} \in \overrightarrow{S}$
The a above two equation give

$$E(\overrightarrow{y_1} - \overrightarrow{y_2}, \overrightarrow{v}) = 0, \forall \overrightarrow{v} \in \overrightarrow{S} \tag{18}$$

Now, by inserting $\overrightarrow{v} = \overrightarrow{y_1} - \overrightarrow{y_2}$ in (18) and using assumption (I(1- a)), we find that
$\overrightarrow{y_1} = \overrightarrow{y_2}$,i.e. the solution is unique.

## 4 Existence of a couple optimal classical control

In this section, the following lemmas are important in the proof of the existence of a couple optimal classical control theorem

**Lemma 1.***The operator $\overrightarrow{q} \mapsto \overrightarrow{y}_{\overrightarrow{q}}$ from $\overrightarrow{Q_a}$ to $(L^2(\Omega))^2$is Lipschitz continuous, i.e.*
$\|\overrightarrow{\delta y}\|_{(H_0^2(\Omega))^2} \leq k\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2}$ *, for $k > 0$*

*Proof.*Let $\overrightarrow{q}', \overrightarrow{q} \in \overrightarrow{Q_a}$ are two vectors of controls of the WFs (10) respectively, $\overrightarrow{y}'$ and $\overrightarrow{y}$ be their corresponding vectors of state solutions, subtracting the two obtained WFs, and Substituting $\overrightarrow{\delta y} = \overrightarrow{y}' - \overrightarrow{y}$, $\overrightarrow{\delta q} = \overrightarrow{q}' - \overrightarrow{q}$in the above obtained equation, with inserting $v_1 = \delta y_1$ and $v_2 = \delta y_2$ we obtain

$$E(\delta y, \delta y) = (\delta q_1, \delta y_1) + (\delta q_2, \delta y_2) \tag{19}$$

Taking the absolute value of (19) with using assumption (3.1(1-a)) and the Cauchy-Schwarz inequality (C-SI), we deduce that

$$c_0\|\overrightarrow{\delta y}\|_{(H_0^2(\Omega))^2} \leq \|\delta q_1\|\|\delta y_1\| + \|\delta q_2\|\|\delta q_2\| \tag{20}$$

Since $\|\delta y_i\| \leq \|\overrightarrow{\delta y}\|_{(L^2(\Omega))^2} \leq \|\overrightarrow{\delta y}\|_{(H_0^2(\Omega))^2}$ and $\|\delta q_i\| \leq \|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2}, \forall i = 1, 2$, then (20) becomes

$$\|\overrightarrow{\delta y}\|_{(H_0^2(\Omega))^2} \leq k\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2}, \text{with}(k = \frac{2}{c_0}) \tag{21}$$

Then the operator$\overrightarrow{q} \mapsto \overrightarrow{y}_{\overrightarrow{q}}$ is Lipschitz continuous on $(L^2(\Omega))^2$

**Lemma 2.***The norm $\|.\|$is weakly lower semicontinuous (WLSC).*

**Lemma 3.***The norm $\|.\|$ is strictly convex.*

**Theorem 3.***Consider the CFn is given by (7), we assume $Q_i, \forall i = 1, 2$ is convex. If $J_0(\overrightarrow{q})$is coercive, then there exists a couple classical optimal control for the problem.*


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*Proof.*since $Q_i$, for each $i = 1,2$ is convex, hence $\overrightarrow{Q_a}$ is convex. Since $J_0(\overrightarrow{q}) \geq 0$ and $J_0(\overrightarrow{q})$ is coercive, then there exists a minimizing sequence $\{\overrightarrow{q_n}\} = \{(q_{1n}, q_{2n})\} \in \overrightarrow{Q_a}$ such that
$\lim_{n\longrightarrow\infty} J_0(\overrightarrow{q_n}) = \inf_{\overrightarrow{w}\in\overrightarrow{Q_a}} J_0(\overrightarrow{w})$
Therefore, there exists a constant $C > 0$ such that

$$\|\overrightarrow{q_n}\|_{(L^2(\Omega)^2} \leq C, \forall n, \text{then}\|\overrightarrow{q_{1n}}\|_{(L^2(\Omega)^2} \leq C_1, \text{and}\|\overrightarrow{q_{2n}}\|_{(L^2(\Omega)^2} \leq C_2, \forall n \tag{22}$$

From Alaoglu theorem, there exists a subsequence of $\{\overrightarrow{q_n}\}$ (say again $\{\overrightarrow{q_n}\}$) such that $\overrightarrow{q_n} \rightharpoonup \overrightarrow{q}$ in $(L^2(\Omega))^2$
Since for each control vector $\overrightarrow{q_n} = (\overrightarrow{q_{1n}}, \overrightarrow{q_{1n}})$, the state equation has a unique solution $\overrightarrow{y_n} = \overrightarrow{y}_{\overrightarrow{q_n}}$(for each $n$ by Theorem 1).
We need to prove $\overrightarrow{y_n}$ is bounded in $\overrightarrow{S_n}$ By using assumptions (3.1 (1- a)) and (3.1 (2)), using the C-SI and the bounded of the control vector, yields
$c_0\|\overrightarrow{y_n}\|^2_{(H^2_0(\Omega))^2} \leq E(\overrightarrow{y_n}, \overrightarrow{y_n}) \leq \|f_1\|\|y_{1n}\| + \|q_{1n}\|\|y_{1n}\| + \|f_2\|\|y_{2n}\| + \|q_{2n}\|\|y_{2n}\|$
$\leq l_1\|y_{1n}\| + C_1\|y_{1n}\| + l_2\|y_{2n}\| + C_2\|y_{2n}\|$
$\leq (r_1 + r_2)\|\overrightarrow{y_n}\|_{(H^2_0(\Omega))^2} = \tau\|\overrightarrow{y_n}\|_{(H^2_0(\Omega))^2}$
where $r_1 = \max\{l_1, C_1\}$ and $r_2 = \max\{l_2, C_2\}$ and $\tau = \max\{r_1, r_2\}$, then
$\|\overrightarrow{y_n}\|_{(H^2_0(\Omega))^2} \leq K$ , where $K = \frac{\tau}{c_0}$, $k \geq 0$
Then there exists a subsequence of $\{\overrightarrow{y_n}\}$ (say again $\{\overrightarrow{y_n}\}$) such that $\overrightarrow{q_n} \rightharpoonup \overrightarrow{q}$ in $\overrightarrow{S}$ (by Alaoglu theorem)
Since for each $n$, $\overrightarrow{y_n} = (\overrightarrow{y_{1n}}, \overrightarrow{y_{1n}})$satisfies the WF (13) , we have

$$E(\overrightarrow{y_n}, \overrightarrow{v}) = (f_1, v_1) + (q_{1n}, v_1) + (f_2, v_2) + (q_{2n}, v_2), \forall (v_1, v_2) \in \overrightarrow{S}, \forall n \tag{23}$$

To show that (23) converges to

$$E(\overrightarrow{y}, \overrightarrow{v}) = (f_1, v_1) + (\bar{q}_1, v_1) + (f_2, v_2) + (\bar{q}_2, v_2), \forall (v_1, v_2) \in \overrightarrow{S} \tag{24}$$

First, since
$y_{in} \rightharpoonup \bar{y}_i$ in $H^2_0(\Omega)$ , $\forall i = 1,2$, then $y_{in} \rightharpoonup \bar{y}_i$ in $L^2(\Omega)$,$\nabla y_{in} \rightharpoonup \nabla\bar{y}_i$ in $L^2(\Omega)$, and $\Delta y_{in} \rightharpoonup \Delta\bar{y}_i$ in $L^2(\Omega)$,
And a use of the C-SI, one gets
$|(\Delta y_{1n}, \Delta v_1) + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) + (\Delta y_{2n}, \Delta v_2) + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_{1n}, v_2) - (\Delta\bar{y}_1, \Delta v_1) - (\nabla\bar{y}_1, \nabla v_1) - (\bar{y}_1, v_1) + (\bar{y}_2, v_1) - (\Delta\bar{y}_2, \Delta v_2) - (\nabla\bar{y}_2, \nabla v_2) - (\bar{y}_2, v_2) - (\bar{y}_1, v_2)|$
$\leq \|\Delta y_{1n} - \Delta\bar{y}_1\|\|\Delta v_1\| + \|\nabla y_{1n} - \nabla\bar{y}_1\|\|\nabla v_1\| + \|y_{1n} - \bar{y}_1\|\|v_1\| + \|y_{2n} - \bar{y}_2\|\|v_1\| + \|\Delta y_{2n} - \Delta\bar{y}_2\|\|\Delta v_2\| +$
$\|\nabla y_{2n} - \nabla\bar{y}_2\|\|\nabla v_2\| + \|y_{2n} - \bar{y}_2\|\|v_2\| + \|y_{1n} - \bar{y}_1\|\|v_2\| \longrightarrow 0$
Second, since $q_{1n} \rightharpoonup \bar{q}_1$ in $L^2(\Omega)$ and $q_{2n} \rightharpoonup \bar{q}_2$ in $L^2(\Omega)$
then the R.H.S. of (23) converges to the R.H.S. of (24)
Since $J_0(\overrightarrow{q})$ is WLSC (from lemma 2), and since $\overrightarrow{q_n} \rightharpoonup \overrightarrow{q}$ in $(L^2(\Omega))^2$, we observe that
$J_0(\overrightarrow{q}) \leq \lim_{n\longrightarrow\infty} \inf J_0(\overrightarrow{q_n}) = \lim_{n\longrightarrow\infty} J_0(\overrightarrow{q_n}) = \inf_{\overrightarrow{w}\in\overrightarrow{Q_a}} J_0(\overrightarrow{w})$, then $J_0(\overrightarrow{q_n}) = \inf_{\overrightarrow{w}\in\overrightarrow{Q_a}}$
Hence $\overrightarrow{q}$ is a couple classical optimal control
To prove $\overrightarrow{q}$ is unique
From strict convexity of $J_0(\overrightarrow{q})$ (by lemma 3), we conclude the uniqueness of $\overrightarrow{q}$.

## 5 The NECs for optimality

In order to state the NECs for a couple classical optimal control, we drive the FrD of the Hamiltonian to establish the NECs for optimality.

**Theorem 4.***Consider the CFn which is given by (7), and the adjoint $(z_1, z_2) = (z_{1q_1}, z_{2q_1})$ equations of the couple state equations (1-6) are given by*

$$\Delta^2 z_1 - \Delta z_1 + z_1 + z_2 = y_1 - y_{1d}, on\Omega \tag{25}$$

$$\Delta^2 z_2 - \Delta z_2 + z_2 - z_1 = y_2 - y_{2d}, on\Omega \tag{26}$$

$$z_1 = 0, on\Gamma \tag{27}$$

$$\frac{\partial z_1}{\partial n} = 0, on\Gamma \tag{28}$$

$$z_2 = 0, on\Gamma \tag{29}$$

$$\frac{\partial z_2}{\partial n} = 0, on\Gamma \tag{30}$$

*Then the FrD of $J_0$ is given by*
$(J_0'(\overrightarrow{q}), \overrightarrow{\delta q}) = (\overrightarrow{z} + \overrightarrow{q}, \overrightarrow{\delta q})$

*Proof.*Rewriting the couple of the adjoint equations (CPAEs) (25-30) by their WFs, we get

$$(\Delta z_1, \Delta v_1) + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) = (y_1 - y_{1d}, v_1), \forall v_1 \in H_0^2(\Omega) \tag{31}$$

$$(\Delta z_2, \Delta v_2) + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) = (y_2 - y_{2d}, v_2), \forall v_2 \in H_0^2(\Omega) \tag{32}$$

By adding (31) and (32), we get for fixed couple classical control vector $\overrightarrow{q} = (q_1, q_2) \in (L^2(\Omega))^2$ the WF of the CPAEs has a uniqueness and existence solution $(z_1, z_2) = (z_{1q_1}, z_{2q_2}) \in \overrightarrow{S}$ (by applying the similar ways of the theorem (1) and theorem (2)), we have

$$(\Delta z_1, \Delta v_1) + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) + (\Delta z_2, \Delta v_2) + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) = (y_1 - y_{1d}, v_1) + (y_2 - y_{2d}, v_2) \tag{33}$$

By substituting the solution $y_1$ once and $y_1 + \delta y_1$ once again in (10), subtracting the obtained equations one from the other, with substituting $v_1 = z_1$ , we obtain

$$(\Delta \delta y_1, \Delta z_1) + (\nabla \delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) = (\delta q_1, z_1), \forall z_1 \in H_0^2(\Omega) \tag{34}$$

Also substituting $v_1 = \delta y_1$ in (31), then subtracting the obtained equation with (34), we get

$$(\delta y_2, z_1) + (z_2, \delta y_1) = -(\delta q_1, z_1) + (y_1 - y_{1d}, \delta y_1) \tag{35}$$

By substituting the solution $y_2$ once and $y_2 + \delta y_2$ once again in (11), subtracting the obtained equations one from the other, with substituting $v_2 = z_2$, we obtain

$$(\Delta \delta y_2, \Delta z_2) + (\nabla \delta y_2, \nabla z_2) + (\delta y_2, z_2) + (\delta y_1, z_2) = (\delta q_2, z_2), \forall z_2 \in H_0^2(\Omega) \tag{36}$$

Also $v_2 = \delta y_2$ in (32), then subtracting the obtained equation with (36), we get

$$-(\delta y_1, z_2) - (z_1, \delta y_2) = -(\delta q_2, z_2) + (y_2 - y_{2d}, \delta y_2) \tag{37}$$

Adding (35) and (37), we get

$$(\delta q_1, z_1) + (\delta q_2, z_2) = (y_1 - y_{1d}, \delta y_1) + (y_2 - y_{2d}, \delta y_2) \tag{38}$$

Then for the CFn, we have
$J_0(\overrightarrow{q} + \overrightarrow{\delta q}) = \frac{1}{2} \iint_\Omega (y_1 + \delta y_1 - y_{1d})^2 dx_1 dx_2 + \frac{1}{2} \iint_\Omega (y_2 + \delta y_2 - y_{2d})^2 dx_1 dx_2 + \frac{1}{2} \iint_\Omega (q_1 + \delta q_1)^2 dx_1 dx_2$
$+ \frac{1}{2} \iint_\Omega (q_2 + \delta q_2)^2 dx_1 dx_2$
$= \frac{1}{2} \iint_\Omega [(y_1 - y_{1d})^2 + 2(y_1 - y_{1d})\delta y_1 + (\delta y_1)^2] dx_1 dx_2 + \frac{1}{2} \iint_\Omega [(y_2 - y_{2d})^2 + 2(y_2 - y_{2d})\delta y_2 + (\delta y_2)^2] dx_1 dx_2 +$
$\frac{1}{2} \iint_\Omega [(q_1)^2 + 2q_1 \delta q_1 + (\delta q_1)^2] dx_1 dx_2 + \frac{1}{2} \iint_\Omega [(q_2)^2 + 2q_1 \delta q_2 + (\delta q_2)^2] dx_1 dx_2$
But by using (38), we have
$J_0(\overrightarrow{q} + \overrightarrow{\delta q}) - J_0(\overrightarrow{q}) = (\delta q_1, z_1) + (q_1 \delta q_1) + (\delta q_2, z_2) + (q_2 \delta q_2) + \frac{1}{2} \|\overrightarrow{\delta y}\|^2_{(L^2(\Omega))^2} + \frac{1}{2} \|\overrightarrow{\delta q}\|^2_{(L^2(\Omega))^2}$

$$= (\overrightarrow{z} + \overrightarrow{q}, \overrightarrow{\delta q}) + \frac{1}{2} \|\overrightarrow{\delta y}\|^2_{(L^2(\Omega))^2} + \frac{1}{2} \|\overrightarrow{\delta q}\|^2_{(L^2(\Omega))^2} \tag{39}$$

From Lemma 1, we get that

$$\frac{1}{2} \|\overrightarrow{\delta y}\|^2_{(L^2(\Omega))^2} \le 2 \|\overrightarrow{\delta q}\|^2_{(L^2(\Omega))^2} = \varepsilon_1(\overrightarrow{\delta q}) \|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \tag{40}$$

where $\varepsilon_1(\overrightarrow{\delta q}) \longrightarrow 0$ as $\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \longrightarrow 0$, and $\varepsilon_1(\overrightarrow{\delta q}) = 2\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2}$

$$\frac{1}{2} \|\overrightarrow{\delta q}\|^2_{(L^2(\Omega))^2} = \frac{1}{2} \|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} = \varepsilon_2(\overrightarrow{\delta q}) \|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \tag{41}$$

where $\varepsilon_2(\overrightarrow{\delta q}) \longrightarrow 0$ as $\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \longrightarrow 0$, and $\varepsilon_2(\overrightarrow{\delta q}) = \frac{1}{2}\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2}$
Hence the FrD of $J_0$ is

$$J_0(\overrightarrow{q}+\overrightarrow{\delta q}) - J_0(\overrightarrow{q}) = (J_0'(\overrightarrow{q}),\overrightarrow{\delta q}) + \varepsilon(\overrightarrow{\delta q})\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \tag{42}$$

where $\varepsilon(\overrightarrow{\delta q}) \longrightarrow 0$ as $\|\overrightarrow{\delta q}\|_{(L^2(\Omega))^2} \longrightarrow 0$ Finally, form (39) and (42), one gets
$(J_0'(\overrightarrow{q}),\overrightarrow{\delta q}) = (\overrightarrow{z}+\overrightarrow{q},\overrightarrow{\delta q})$.

**Theorem 5.***The CCCPOC of the considered problem is $J_0'(\overrightarrow{q}) = \overrightarrow{z}+\overrightarrow{q}$ with $\overrightarrow{y} = \overrightarrow{y}_{\overrightarrow{q}}$ and $\overrightarrow{z} = \overrightarrow{z}_{\overrightarrow{q}}$.*

*Proof.*If $\overrightarrow{q}$ is an optimal control vector of the problem
$J_0'(\overrightarrow{q}) = \min J_0'(\overrightarrow{w}), \forall \overrightarrow{w} \in \overrightarrow{Q_a}$, i.e.
$J_0'(\overrightarrow{q}) =$ implies $\overrightarrow{z}+\overrightarrow{q} = 0$
$q_1 = -z_1$ and $q_2 = -z_2$ with $\overrightarrow{\delta q} = \overrightarrow{w} - \overrightarrow{q}$
Hence the NEC of the optimality is $(J_0'(\overrightarrow{q}),\overrightarrow{\delta q}) \geq 0$
Therefore $(\overrightarrow{z}+\overrightarrow{q},\overrightarrow{w}) \geq (\overrightarrow{z}+\overrightarrow{q},\overrightarrow{q}), \forall \overrightarrow{w} \in (L^2(\Omega))^2$.

## Conclusion

The FEGM with PCH basis function is suitable to prove the existence and uniqueness of a couple state vector solution for a coupled fourth order LEPDEs with DNBCs, when the CCCPCV is considered. Under appropriate conditions, the existence theorem of a CCCPOCV associated with a couple fourth order LEPDEs is formulated and proved. The existence and uniqueness of the solution of the couple adjoint equations which involves to the CPSV is discussed, when the continuous classical couple optimal control vector is given. Finally, the derivation of the FrD for the CFn is demonstrated and applied to establish the theorem of the NEC for optimality of the considered problem.

## Declarations

**Competing interests**: The author have no competing interests to declares
**Authors' contributions**: The author read and approved the final manuscript
**Funding**: No outside funding was used to support this work
**Availability of data and materials**: Not applicable
**Acknowledgments**: I would like to thank Mustansiriyah University (www.uomustansiriyah.com) Baghdad -Iraq for its support in the present work. Also I wish to thank the reviewers and the editors for their constructive and invaluable comments and suggestions for improving the paper

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