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Some Inequalities Involving Csiszár Divergence Using A Montgomery Identity and Green Function Via Diamond integrals

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Abstract: In the present paper, some results involving Csiszár divergence are formulated via diamond integrals by using Montgomery ´ identity and Green function. By selecting various kinds of convex functions, bound of different entropies and divergence measures are obtained. Further, new bounds of different divergence measures are deduced.

Keywords: divergence measures; Montgomery identity; diamond integral; Green's function.

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1 Introduction

Convexity is a fundamental concept in geometry, although it is also frequently employed in other branches of mathematics. Convexity is also useful in fields other than mathematics, such as chemistry, physics, biology and other sciences. The research related to convex functions has received a rapid progress. This is because utilization of convex functions in modern analysis and a large number of significant inequalities are results of convex functions (see [22]).
Divergences have been proposed in order to assess the differences between probability distributions. Divergences of several kinds exist, for instance the f -divergence and Rényi divergence etc. (see [18,30,15,14,26]). In 1948, American mathematician, Shannon founded information theory with a well-known paper, "A Mathematical Theory of Communication". The entropy of a variable means "amount of information" for the variable.
Suppose that X is a continuous random variable with density u. Consider the set $\Theta = [b_1, b_2]$ with $b_1 < b_2$ and the set of all probability densities is defined by

$$S =: \{l | l : \theta \to \mathbb{R}, l(\vartheta) > 0, \int_{\Theta} l(\vartheta) d\vartheta = 1\}. \tag{1}$$

The differential entropy of X can be defined on Θ by

$$h_{\bar{c}}(X) := \int_{\Theta} u(\vartheta) \log \frac{1}{u(\vartheta)} d\vartheta, \quad u \in S. \tag{2}$$

The aim of the mathematical theory related to time scales is to merge discrete and continuous analysis presented by S. Hilger in 1988 (see [4,5]). This theory is developed very rapidly in last three decades. Several authors have established time scale versions of inequalities. Ansari *et al.* [1] have presented few inequalities containing Csiszár divergence for delta integrals on time scales. Bilal *et al.* [11] have extended Jensen's inequality for multiple integrals via diamond integrals. They have used obtained extensions to construct Hardy–type inequalities on time scales for the function of

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several variables via diamond–integral formalism. In [12], author have utilized Hermite interpolation, to extend Jensen's functional on time scales for n–convex functions using diamond integrals. Recently, Smoljak Kalamir [17] have extended certain Steffensen-type inequalities by utilizing diamond-α integrals. In [28], authors have introduced the quantum analogue of the dual Simpson type integral inequalities for the class of q-differentiable convex functions through a new identity. In [29], authors have extended Ostrowski type inequalities for the function whose first derivatives' absolute value are s–type p–convex. In [6], Bohner *et. al.* provided Montgomery identity using delta integrals and discussed it for some fixed time scales. In [27], Sarikaya *et. al.* have established weighted Montgomery identity on time scales.
The main motivation behind this work is to generalize the results involving different divergences by using approximate symmetric integrals (called diamond integrals) via Montgomery identity and Green function. These results extend the results of [15,1,20]. For this purpose, Shannon entropy and different divergence measures are reformulated by diamond integrals and their bounds are derived with the help of Montgomery identity and Green function involving diamond–integral formalism. By choosing set of real numbers as time scale in the obtained results, we get improvements of classical results already proved in literature [1,20]. Moreover, by choosing set of natural numbers including zero as time scale in the proved results, improvements of existing discrete classical results are obtained [15].
The structure of the manuscript is as follows: In Section 2, first of all some basics of time scales calculus are given. After that Montgomery identity and some of its related findings are recalled. Section 3 contains results containing Csiszár divergence via diamond integrals for $n-$convex functions. In Section 4, bounds of different divergence measures are estimated. Lastly, manuscript is concluded in Section 5.

2 Preliminaries

The present section contains some fundamental definitions and findings related to the mathematical theory of time scales. The time scale, indicated by $\mathbb{T}$, is a nonempty closed subset of real numbers. Its examples include $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Z}$. Let $r \in \mathbb{T}$, forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are given as

$$\sigma(r) := inf\{v \in \mathbb{T} : v > r\},$$

and

$$\rho(r) := sup\{v \in \mathbb{T} : v < r\},$$

respectively.
The development of time scales theory start with the ideas, the forward (delta) calculus and the backward (nabla) calculus. The Δ (delta) derivative of a mapping l is denoted by l^{Δ} and Δ integral is represented by $\int_{\mathbb{T}} l(\eta)\Delta\eta$. Likely, ∇ (nabla) derivative of a mapping l is denoted by l^{∇} and ∇ integral is denoted by $\int_{\mathbb{T}} l(\eta)\nabla\eta$. For more information related to time scales see [4].
In the paper [25] from 2006, Sheng, Fadag, Henderson and Davis have provided diamond-alpha integral given as follows: Consider $h : [c_1, c_2]_{\mathbb{T}} \to \mathbb{R}$ is a continuous mapping and $c_1, c_2 \in \mathbb{T}(c_1 < c_2)$. The diamond alpha integral of h is given as

$$\int_{c_1}^{c_2} h(\eta)\diamondsuit_{\alpha}\eta := \int_{c_1}^{c_2} \alpha h(\eta)\triangle\eta + \int_{c_1}^{c_2} (1-\alpha)h(\eta)\nabla\eta, \quad 0 \le \alpha \le 1,$$

if γh is $\triangle$ and $(1-\gamma)h$ is ∇ integrable on $[c_1, c_2]_{\mathbb{T}}$.
In case $\alpha = 0$, we have nabla-integral and for $\alpha = 1$, we have delta-integral.

In [7], the function γ given as follows:

$$\gamma(v) = \lim_{t \to v} \frac{\sigma(v) - t}{\sigma(v) + 2v - 2t - \rho(v)}.$$

Clearly,

$$\gamma(v) = \begin{cases} \frac{1}{2}, & \text{if } v \text{ is dense;} \\ \frac{\sigma(v)-v}{\sigma(v)-\rho(v)}, & \text{if } v \text{ is not dense} \end{cases}$$

and $0 \le \gamma(v) \le 1$.
Now we recall diamond integral which was proposed to provide a genuine symmetric integral on time scales. This integral provides better approximation than delta, nabla and diamond– alpha integrals. In [8], an "approximate" symmetric integral on time scales which is called diamond integral defined as follows:

Diamond Integral

Suppose $h:\mathbb{T}\to\mathbb{R}$ is a continuous mapping and $d_1,d_2\in\mathbb{T}(d_1<d_2)$. The diamond integral of h from d_1 to d_2 is given as

$$\int_{d_1}^{d_2} h(\xi)\Diamond\xi=\int_{d_1}^{d_2}\gamma(\xi)h(\xi)\triangle\xi+\int_{d_1}^{d_2}(1-\gamma(\xi))h(\xi)\nabla\xi,\quad 0\le\gamma\le 1,$$

with the condition that γh is $\triangle$ and $(1-\gamma)h$ is ∇ integrable on $[d_1,d_2]_{\mathbb{T}}$.
For more information related to $\diamond$-integral see [8].
In literature, the Montgomery identity is well known. It has been studied extensively since it was established. Montgomery is used to get a number of subversive inequalities like Ostrowski type inequality, trapezoid inequality and Grüss type inequality.
In [19], Mitrinović *et. al*. provided that if $\phi:[\rho,\ \varsigma]\to\mathbb{R}$ be differentiable on $[\rho,\ \varsigma]$ and $\phi':[\rho,\ \varsigma]\to\mathbb{R}$ be integrable on $[\rho,\ \varsigma]$ then Montgomery identity is given as follows:

$$\phi(r)=\frac{1}{\varsigma-\rho}\int_\rho^\varsigma\phi(\mathfrak{t})d\mathfrak{t}+\int_{v_1}^{v_2}\mathfrak{P}(r,\mathfrak{t})\phi'(\mathfrak{t})d\mathfrak{t},\tag{3}$$

where

$$\mathfrak{P}(r,\mathfrak{t})=\begin{cases}\frac{\mathfrak{t}-\rho}{\varsigma-\rho},\ \rho\le\mathfrak{t}\le r,\\ \frac{\mathfrak{t}-\varsigma}{\varsigma-\rho},\ r<\mathfrak{t}\le\varsigma.\end{cases}\tag{4}$$

The following Montgomery identity is presented in [2,3] and utilized to prove main results.

Theorem 1. *Suppose that U is an open interval in $\mathbb{R}$ and $\rho,\ \varsigma\in U$ s.t. $\rho<\varsigma$. Assume a function $\phi:U\to\mathbb{R}$, s.t., $\phi^{(n-1)}$ is absolutely continuous for $n\in\mathbb{N}$, then*

$$\phi(r)=\frac{1}{\varsigma-\rho}\int_\rho^\varsigma\phi(\mathfrak{t})d\mathfrak{t}+\sum_{\mathfrak{z}=0}^{n-2}\frac{\phi^{(\mathfrak{z}+1)}(\rho)}{\mathfrak{z}!(\mathfrak{z}+2)}\frac{(r-\rho)^{\mathfrak{z}+2}}{\varsigma-\rho}$$
$$-\sum_{\mathfrak{z}=0}^{n-2}\frac{\phi^{(\mathfrak{z}+1)}(\varsigma)}{\mathfrak{z}!(\mathfrak{z}+2)}\frac{(r-\varsigma)^{\mathfrak{z}+2}}{\varsigma-\rho}+\frac{1}{(n-1)!}\int_\rho^\varsigma R_n(r,\mathfrak{t})\phi^{(n)}(\mathfrak{t})d\mathfrak{t},\tag{5}$$

where

$$R_n(r,\mathfrak{t})=\begin{cases}\frac{-(r-\mathfrak{t})^n}{n(\varsigma-\rho)}+\frac{r-\rho}{\varsigma-\rho}(r-\mathfrak{t})^{n-1},\ \rho\le\mathfrak{t}\le r,\\ \frac{-(r-\mathfrak{t})^n}{n(\varsigma-\rho)}+\frac{r-\varsigma}{\varsigma-\rho}(r-\mathfrak{t})^{n-1},\ r<\mathfrak{t}\le\varsigma.\end{cases}\tag{6}$$

For $n=1$, sums in (5) are empty, therefore (5) is reduced to Montgomery identity (3). In this paper we assumed that:
A1: $\Theta:=[b_1,b_2]_{\mathbb{T}}$, with $b_1,b_2\in\mathbb{T}$ and $b_1<b_2$.
A2: The set of all probability densities is denoted by $E=:\{l|l:\Theta\to\mathbb{R},l(\vartheta)>0,\int_\Theta l(\vartheta)\Diamond\vartheta=1\}$.
Consider the Green function $G:[\rho,\ \varsigma]\times[\rho,\ \varsigma]\to\mathbb{R}$ defined as

$$G(u,r)=\begin{cases}\frac{(u-\varsigma)(r-\rho)}{\varsigma-\rho},\ \rho\le r\le u,\\ \frac{(r-\varsigma)(u-\rho)}{\varsigma-\rho},\ u\le r\le\varsigma.\end{cases}\tag{7}$$

It is understood that (see [21,31,24,16]) any mapping $\phi\in C^2([\rho,\varsigma],\mathbb{R})$ can be expressed by

$$\phi(u)=\frac{\varsigma-u}{\varsigma-\rho}\phi(\rho)+\frac{u-\rho}{\varsigma-\rho}\phi(\varsigma)+\int_\rho^\varsigma G(u,r)\phi''(r)dr,\tag{8}$$

where G is given in (7).

Theorem 2. *Assume that the mapping $\phi:[0,\infty)\to(-\infty,\infty)$ is convex on $[\rho,\varsigma]\subset[0,\infty)$ and $\rho\le 1\le\varsigma$. If $l_1,l_2\in E$ and*

$$\rho\le\frac{l_2(\vartheta)}{l_1(\vartheta)}\le\varsigma,\quad\forall\,\vartheta\in\mathbb{T},$$

then

$$\int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\Diamond\vartheta\le\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma).\tag{9}$$

Proof. Since ϕ is convex on $[\rho, \varsigma]$, therefore

$$\phi\left(u\rho + (1-u)\varsigma\right) \leq u\phi(\rho) + (1-u)\phi(\varsigma), \tag{10}$$

for every $u \in [0,1]$. Put $u = \frac{\varsigma - v}{\varsigma - \rho}$, $1 - u = 1 - \frac{\varsigma - v}{\varsigma - \rho} = \frac{v - \rho}{\varsigma - \varsigma_1}$ in (10) to obtain

$$\phi(v) \leq \frac{\varsigma - v}{\varsigma - \rho}\phi(\rho) + \frac{v - \rho}{\varsigma - \rho}\phi(\varsigma). \tag{11}$$

Use $v = \frac{l_2(\zeta)}{l_1(\zeta)}$, in (11) to obtain

$$\phi\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right) \leq \frac{\varsigma - \frac{l_2(\zeta)}{l_1(\zeta)}}{\varsigma - \rho}\phi(\rho) + \frac{\frac{l_2(\zeta)}{l_1(\zeta)} - \rho}{\varsigma - \rho}\phi(\varsigma). \tag{12}$$

Multiply (12) by $l_1(\zeta)$ to obtain

$$l_1(\zeta)\phi\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right) \leq \frac{\varsigma l_1(\zeta) - l_2(\zeta)}{\varsigma - \rho}\phi(\rho) + \frac{l_2(\zeta) - \rho l_1(\zeta)}{\varsigma - \rho}\phi(\varsigma). \tag{13}$$

Integrate (13) over Θ and since $l_2, l_1 \in \Lambda$, therefore

$$\int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \leq \frac{\varsigma - 1}{\varsigma - \rho}\phi(\rho) + \frac{1 - \rho}{\varsigma - \rho}\phi(\varsigma),$$

which is the desired result.

3 Main Results

If suppositions of Theorem 2 remain valid, then it is possible to define the following functional $\digamma_1(\phi)$ involving Csiszár divergence for diamond integrals:

$$\digamma_1(\phi) = \frac{\varsigma - 1}{\varsigma - \rho}\phi(\rho) + \frac{1 - \rho}{\varsigma - \rho}\phi(\varsigma) - \int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\diamond\vartheta, \tag{14}$$

where ϕ is defined on $[\rho, \varsigma]$.

Remark. If suppositions of Theorem 2 remain valid, then $\digamma_1(\phi) \geq 0$.

Motivated by (9), we start with the next theorem.

Theorem 3. *Let the assumptions of Theorem 2 remain valid and $\phi \in C^2([\rho, \varsigma], \mathbb{R})$, then*

$$\int_\Theta l_2(\vartheta) G\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}, r\right)\diamond\vartheta \leq \frac{\varsigma - 1}{\varsigma - \rho}G(\rho, r) + \frac{1 - \rho}{\varsigma - \rho}G(\varsigma, r), \tag{15}$$

and (9) are equivalent for each $r \in [\rho, \varsigma]$, where G is defined as in (7).

Proof. Suppose that (9) is valid. Since, the function $G(\cdot, r)$ is convex and continuous for each $r \in [\rho, \varsigma]$, consequently (15) holds.
Conversely, assume that (15) is valid and $\phi \in C^2([\rho, \varsigma], \mathbb{R})$. Then, use of (8) gives

$$\begin{aligned}
&\frac{\varsigma - 1}{\varsigma - \rho}\phi(\rho) + \frac{1 - \rho}{\varsigma - \rho}\phi(\varsigma) - \int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\diamond\vartheta \\
&= \frac{\varsigma - 1}{\varsigma - \rho}\left[\phi(\rho) + \int_\rho^\varsigma G(\rho, r)\phi''(r)dr\right] \\
&+ \frac{1 - \rho}{\varsigma - \rho}\left[\phi(\varsigma) + \int_\rho^\varsigma G(\varsigma, r)\phi''(r)dr\right] \\
&- \int_\Theta l_2(\vartheta)\left[\frac{\varsigma - \frac{l_1(\vartheta)}{l_2(\vartheta)}}{\varsigma - \rho}\phi(\rho) + \frac{\frac{l_1(\vartheta)}{l_2(\vartheta)} - \rho}{\varsigma - \rho}\phi(\varsigma)\right. \\
&\left. + \int_\rho^\varsigma G\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}, r\right)\phi''(r)dr\right]\diamond\vartheta.
\end{aligned} \tag{16}$$

If we apply Fubini's theorem and choose $l_1, l_2 \in E$, then (16) takes the form

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)-\int_{\Theta} l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\Diamond\vartheta$$
$$=\frac{\varsigma-1}{\varsigma-\rho}\int_{\rho}^{\varsigma} G(\rho,r)\phi''(r)dr$$
$$+\frac{1-\rho}{\varsigma-\rho}\int_{\rho}^{\varsigma} G(\varsigma,r)\phi''(r)dr$$
$$-\int_{\rho}^{\varsigma}\left[\int_{\Theta} G(\frac{l_1(\vartheta)}{l_2(\vartheta)},r)\Diamond\vartheta\right]\phi''(r)dr.$$

Since $\phi''(r) \geq 0$ for each $r \in [\rho,\varsigma]$, therefore (9) is valid.

By using Montgomery identity and (9), following new result is established.

Theorem 4. *Let the suppositions of Theorem 3 are valid,* $(n \in \mathbb{N}, n > 2)$ *and the function* $\phi : [\rho,\varsigma] \to \mathbb{R}$ *with* $\phi^{(n-1)}$ *is absolutely continuous then*

$$\digamma_1(\phi)=\frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr+\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr \times\left(\sum_{\mathfrak{z}=0}^{n-1}\frac{\mathfrak{z}\left[\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}\right]}{(\mathfrak{z}-1)!}\right)dr + \frac{1}{(n-3)!}\int_{\rho}^{\varsigma}\phi^{(n)}(r)\left(\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))\tilde{R}_{n-2}(r,\mathfrak{t})\right)dr, \quad (17)$$

where

$$\tilde{R}_{n-2}(r,\mathfrak{t})=\begin{cases}\frac{(r-\mathfrak{t})^{n-2}}{(n-2)(\varsigma-\rho)}+\frac{r-\rho}{\varsigma-\rho}(r-\mathfrak{t})^{n-3}, & \rho\leq\mathfrak{t}\leq r,\\ \frac{(r-\mathfrak{t})^{n-2}}{(n-2)(\varsigma-\rho)}+\frac{r-\varsigma}{\varsigma-\rho}(r-\mathfrak{t})^{n-3}, & r<\mathfrak{t}\leq\varsigma,\end{cases} \quad (18)$$

$$\digamma_1\big(G(\cdot,r)\big)=\frac{\varsigma-1}{\varsigma-\rho}G(\rho,r)+\frac{1-\rho}{\varsigma-\rho}G(\varsigma,r)-\int_{\Theta} l_2(\vartheta)G\left(\frac{l_1(\vartheta)}{l_2(\vartheta)},r\right)\Diamond\vartheta, \quad (19)$$

and

$$\digamma_1(\phi)=\frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr+\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr \times\left(\sum_{\mathfrak{z}=0}^{n-1}\frac{\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}}{(\mathfrak{z}-3)!(\mathfrak{z}-1)}\right)dr + \frac{1}{(n-3)!}\int_{\rho}^{\varsigma}\phi^{(n)}(r)\left(\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))R_{n-2}(r,\mathfrak{t})\right)d\mathfrak{t}, \quad (20)$$

where

$$R_{n-2}(r,\mathfrak{t})=\begin{cases}\frac{-(r-\mathfrak{t})^{n-2}}{(n-2)(\varsigma-\rho)}+\frac{r-\rho}{\varsigma-\rho}(r-\mathfrak{t})^{n-3}, & \rho\leq\mathfrak{t}\leq r,\\ \frac{-(r-\mathfrak{t})^{n-2}}{(n-2)(\varsigma-\rho)}+\frac{r-\varsigma}{\varsigma-\rho}(r-\mathfrak{t})^{n-3}, & r<\mathfrak{t}\leq\varsigma.\end{cases} \quad (21)$$

Proof. From (5), second derivative of the function ϕ can be written as

$$\phi''(r)=\frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}+\sum_{\mathfrak{z}=0}^{n-2}\left(\frac{\mathfrak{z}}{(\mathfrak{z}-1)!}\right) \times\left(\frac{\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}}{\varsigma-\rho}\right)+\frac{1}{(n-3)!}\int_{\rho}^{\varsigma}\tilde{R}_{n-2}(r,\mathfrak{t})\phi^{(n)}(\mathfrak{t})d\mathfrak{t}. \quad (22)$$

Utilize (22) in Theorem 3 and rearrange the indices to get (17).
Use (5) on the function ϕ'' with n replace by $n-2$ $(n \geq 3)$ and rearrange the indices to obtain

$$\phi''(r) = \frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho} + \sum_{\mathfrak{z}=0}^{n-2}\left(\frac{1}{(\mathfrak{z}-3)!(\mathfrak{z}-1)}\right) \times \left(\frac{\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}}{\varsigma-\rho}\right) + \frac{1}{(n-3)!}\int_{\rho}^{\varsigma} R_{n-2}(r,\mathfrak{t})\phi^{(n)}(\mathfrak{t})d\mathfrak{t}. \quad (23)$$

Use (23) in Theorem 3 to obtain (20).

Remark. Choose set of real numbers as time scale in Theorem 4, to obtain the same result we can get from [13, (2.1)] by using (8) and (5).

The following result provides a sublime generalization of inequality containing Csiszár divergence via diamond integrals.

Theorem 5. *Let the suppositions of Theorem 4 remain valid. Consider $n \in \mathbb{N}$ and $G : [\rho,\ \varsigma]\times[\rho,\ \varsigma] \to \mathbb{R}$ be specified in (7).*
If

$$\int_{\rho}^{\varsigma} \digamma_1(G(u,r))\tilde{R}_{n-2}(r,\mathfrak{t})dr \geq 0, \quad \mathfrak{t}\in[\rho,\ \varsigma], \quad (24)$$

then,

$$\digamma_1(\phi) \geq \frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr + \frac{1}{(\varsigma-\rho)}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{z}=2}^{n-1}\frac{\mathfrak{z}\left[\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}\right]}{(\mathfrak{z}-1)!}\right)dr, \quad (25)$$

and if

$$\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))R_{n-2}(r,\mathfrak{t})dr \geq 0, \quad for\ \ all\quad \mathfrak{t}\in[\rho,\ \varsigma], \quad (26)$$

then,

$$\digamma_1(\phi) \geq \frac{\phi'(\varsigma)-\phi'(\rho)}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr + \frac{1}{(\varsigma-\rho)}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{z}=3}^{n-1}\frac{\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}}{(\mathfrak{z}-3)!(\mathfrak{z}-1)}\right)dr. \quad (27)$$

Proof. Since $\phi : [\rho,\ \varsigma] \to \mathbb{R}$ is n–convex, which implies $\phi^n(v) \geq 0$, $\forall v \in [\rho,\ \varsigma]$; this fact together with (24) gives

$$\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))\tilde{R}_{n-2}(r,\mathfrak{t})dr\phi^n(\mathfrak{t}) \geq 0, \quad for\ \ all\quad \mathfrak{t}\in[\rho,\ \varsigma]. \quad (28)$$

$$\Rightarrow \frac{1}{(n-3)!}\int_{\rho}^{\varsigma}\phi^n(\mathfrak{t})\left(\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))\tilde{R}_{n-2}(r,\mathfrak{t})dr\right)d\mathfrak{t} \geq 0. \quad (29)$$

Using (29) in (17), we get

$$\digamma_1(\phi) - \frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr - \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(u,r))\left(\sum_{\mathfrak{z}=2}^{n-1}\frac{\mathfrak{z}\left[\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}\right]}{(\mathfrak{z}-1)!}\right)dr \geq 0. \quad (30)$$

Linearity of $\digamma_1(\cdot)$ yields

$$\digamma_1(\phi) - \digamma_1\Bigg(\frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_\rho^\varsigma G(\cdot,r)dr + \frac{1}{\varsigma-\rho}\int_\rho^\varsigma G(\cdot,r)\sum_{\mathfrak{z}=2}^{n-1}\frac{\mathfrak{z}\big[\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}\big]}{(\mathfrak{z}-1)!}dr\Bigg) \geq 0. \quad (31)$$

Which is desired inequality.
Similarly by utilizing (20), one can get (27).

Theorem 6. *Assume that the suppositions of Theorem 4 are valid and*
(i) If $n \geq 4$ (n be even), then (25) and (27) hold.
(ii) If (25) holds then, $\forall r \in [\rho,\ \varsigma]$

$$\phi'(\rho)-\phi'(\varsigma)+\sum_{\mathfrak{z}=2}^{n-1}\frac{1}{(\mathfrak{z}-1)!}\left(\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}\right) \geq 0. \quad (32)$$

Or
(27) is satisfied then $\forall r \in [\rho,\ \varsigma]$

$$\phi'(\varsigma)-\phi'(\rho)+\sum_{\mathfrak{z}=3}^{n-1}\frac{\phi^{(\mathfrak{z})}(\rho)(r-\rho)^{\mathfrak{z}-1}-\phi^{(\mathfrak{z})}(\varsigma)(r-\varsigma)^{\mathfrak{z}-1}}{(\mathfrak{z}-3)!(\mathfrak{z}-1)} \geq 0. \quad (33)$$

In both situations, we have

$$\digamma_1(\phi(\cdot)) \geq 0. \quad (34)$$

Proof. Since $\digamma_1(G(\cdot,r)) \geq 0$ by means of Remark 3.
(i) As $\tilde{R}_{n-2}(r,\mathrm{t}), R_{n-2}(r,\mathrm{t}) \geq 0$ for for even n ($n \geq 4$),therefore (24) and (26) hold. Also ϕ is n-convex, consequently, utilizing Theorem 5, one obtains (25) and (27).
(ii) Use (32) in (25) and (33) in (27), to get (34).

4 Estimation of divergence measures

In this section, bound of different divergence measures are estimated. In [9] authors have introduced differential entropy $h_c(Z)$ via diamond integral formalism.

Definition 1. *The differential entropy of continuous random variable Z for diamond integral can be defined as follows:*

$$h_{\bar{c}}(Z) := \int_\Theta l_2(z)\log\frac{1}{l_2(z)}\diamondsuit z, \quad (35)$$

where $l_2 \in E$ and the base of '$\log$' is $\bar{c}$ for some fixed $\bar{c} > 1$.

In the following results, the generalization obtained in Theorem 5 by Montgomery identity is utilized to provide the following new bounds of the divergence measures via diamond integrals:

Theorem 7. *Assume that suppositions of Theorem 5 are true. If $n = 4, 6, \ldots$ and $\bar{c} > 1$, then*

$$h_{\bar{c}}(Z) \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho)+\frac{1-\rho}{\varsigma-\rho}\log(\varsigma) - \int_\Theta l_2(\vartheta)\log\big(l_1(\vartheta)\big)\diamondsuit\vartheta + \frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr + \frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big) \times\left(\sum_{\mathfrak{z}=2}^{n-1}(-1)^{\mathfrak{z}-1}\mathfrak{z}\left[-\frac{(r-\rho)^{\mathfrak{z}-1}}{\rho^{\mathfrak{z}}}+\frac{(r-\varsigma)^{\mathfrak{z}-1}}{\varsigma^{\mathfrak{z}}}\right]\right)dr, \quad (36)$$

and

$$h_{\bar{c}}(Z) \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho)+\frac{1-\rho}{\varsigma-\rho}\log(\varsigma) - \int_{\Theta} l_2(\vartheta)\log\left(l_1(\vartheta)\right)\Diamond\vartheta - \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right)dr + \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right) \times\left(\sum_{\mathfrak{z}=3}^{n-1}(-1)^{\mathfrak{z}-1}(\mathfrak{u}-2)\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr, \quad (37)$$

where $h_{\bar{c}}(Z)$ and $F_1\left(G(\cdot,r)\right)$ are defined in (35) and (19) respectively.

Proof. Since the function $\phi(z) = -\log z$ is n-convex for even n. Use $\phi(z) = -\log z$ in Theorem 5, then we obtain (36) and (37) from (25) and (27) respectively.

Definition 2. *In [10] authors have introduced KL divergence via diamond integrals given as follows:*

$$D(l_1,l_2) := \int_{\Theta} l_1(\vartheta)\ln\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)\Diamond\vartheta. \quad (38)$$

Theorem 8. *Assume that suppositions of Theorem 5 are true. If $n = 4,6,\ldots$), then*

$$D(l_1,l_2) \leq \frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma) - \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right)dr - \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right) \times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}-2}\mathfrak{u}}{\mathfrak{u}-1}\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr, \quad (39)$$

and

$$D(l_1,l_2) \leq \frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma) + \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right)dr - \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right) \times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}-2}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr, \quad (40)$$

where $D(l_1,l_2)$ and $F_1\left(G_n(\cdot,r)\right)$ are defined in (38) and (19) respectively.

Proof. Since the function $\phi(z) = z\ln z$ is n-convex for even n. Use $\phi(z) = z\ln z$ in Theorem 5, then (25) and (27) become (39) and (40) respectively.

Definition 3. *In [10] authors have introduced the Jeffrey's distance via diamond integral can be defined as follows:*

$$D_J(l_1,l_2) := \int_{\Theta}\left(l_1(\vartheta)-l_2(\vartheta)\right)\ln\left[\frac{l_1(\vartheta)}{l_2(\vartheta)}\right]\Diamond\vartheta. \quad (41)$$

Theorem 9. *Assume that suppositions of Theorem 5 are true. If $n = 4,6,\ldots$, then*

$$D_J(l_1,l_2) \leq \frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma) - \left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right)dr - \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\left(G(\cdot,r)\right) \times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}}\mathfrak{u}}{\mathfrak{u}-2}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr, \quad (42)$$

and

$$D_J(l_1,l_2) \leq \frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma) - \left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_\rho^\varsigma \digamma_1(G(\cdot,r))dr - \frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1(G(\cdot,r)) \times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr, \quad (43)$$

where $D_J(l_1,l_2)$ *and* $\digamma_1\big(G_n(\cdot,r)\big)$ *are defined in* (41) *and* (19) *respectively.*

Proof. Since the function $\phi(z)=(z-1)\ln z$ is n-convex for even n. Use $\phi(z)=(z-1)\ln z$ in Theorem 5, then (25) and (27) become (42) and (43) respectively.

Definition 4. *In [10] authors have introduced Triangular discrimination via diamond integral can be defined as follows:*

$$D_\triangle(l_1,l_2) := \int_\Theta \frac{\big(l_2(\vartheta)-l_1(\vartheta)\big)^2}{l_2(\vartheta)+l_1(\vartheta)}\diamondsuit\vartheta. \quad (44)$$

Theorem 10. *Assume that suppositions of Theorem 5 are true. If* $n=4,6,\ldots$, *then*

$$D_\triangle(l_1,l_2) \leq \frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1} - \frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr - \frac{4}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)\times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}^2\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr, \quad (45)$$

and

$$D_\triangle(l_1,l_2) \leq \frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1} - \frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr - \frac{4}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)\times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}(\mathfrak{u}-2)\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr, \quad (46)$$

where $D_\triangle(l_1,l_2)$ *and* $\digamma_1\big(G_n(\cdot,r)\big)$ *are defined in* (44) *and* (19) *respectively.*

Proof. Since the function $\phi(z)=\frac{(z-1)^2}{z+1}$ is n-convex for even n. Use $\phi(z)=\frac{(z-1)^2}{z+1}$ in Theorem 5, then (25) and (27) become (45) and (46) respectively.

4.1 Bounds of Divergence Measures in classical calculus

Now, we estimate different divergence measures in classical calculus.
If one chooses set of real numbers as time scale in Theorem 5, then (25) and (27) provide new inequalities containing Csiszár divergence in classical calculus:

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma) - \int_\Theta l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)d\vartheta \geq \frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1(G(\cdot,r))dr + \frac{1}{(\varsigma-\rho)}\int_\rho^\varsigma \digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{\mathfrak{u}\left[\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}\right]}{(\mathfrak{u}-1)!}\right)dr,$$

and

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma) - \int_{\Theta} l_2(\vartheta)\phi\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)d\vartheta \geq \frac{\phi'(\varsigma)-\phi'(\rho)}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1(G(\cdot,r))dr + \frac{1}{(\varsigma-\rho)}\int_{\rho}^{\varsigma} F_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}}{(\mathfrak{u}-3)!(\mathfrak{u}-1)}\right)dr.$$

where

$$F_1\big(G(\cdot,r)\big) = \frac{\varsigma-1}{\varsigma-\rho}G(\rho,r) + \frac{1-\rho}{\varsigma-\rho}G(\varsigma,r) - \int_{\Theta} l_2(\vartheta)G\left(\frac{l_1(\vartheta)}{l_2(\vartheta)},r\right)d\vartheta. \quad (47)$$

If one chooses set of real numbers as time scale in Theorem 7–10 then (36), (37), (39), (40), (42), (43), (45) and (46) provide new bounds in classical calculus for differential entropy, KL divergence, Jeffrey distance and Triangular discrimination:

$$\int_{\Theta} l_2(\vartheta)\log\frac{1}{l_2(\vartheta)}d\vartheta \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho)+\frac{1-\rho}{\varsigma-\rho}\log(\varsigma) - \int_{\Theta} l_2(\vartheta)\log\big(l_1(\vartheta)\big)d\vartheta + \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big)dr + \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big) \times \left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}-1}\mathfrak{u}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\int_{\Theta} l_2(\vartheta)\log\frac{1}{l_2(\vartheta)}d\vartheta \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho)+\frac{1-\rho}{\varsigma-\rho}\log(\varsigma) - \int_{\Theta} l_2(\vartheta)\log\big(l_1(\vartheta)\big)d\vartheta - \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big)dr + \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big) \times \left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}-1}(\mathfrak{u}-2)\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\int_{\Theta} l_1(\vartheta)\ln\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)d\vartheta \leq \frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma) - \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big)dr - \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big) \times \left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}-2}\mathfrak{u}}{\mathfrak{u}-1}\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\int_{\Theta} l_1(\vartheta)\ln\left(\frac{l_1(\vartheta)}{l_2(\vartheta)}\right)d\vartheta \leq \frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma) + \frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big)dr - \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big) \times \left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}-2}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\int_{\Theta}\big(l_1(\vartheta)-l_2(\vartheta)\big)\ln\left[\frac{l_1(\vartheta)}{l_2(\vartheta)}\right]d\vartheta \leq \frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}}\mathfrak{u}}{\mathfrak{u}-2}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

$$\int_{\Theta}\big(l_1(\vartheta)-l_2(\vartheta)\big)\ln\left[\frac{l_1(\vartheta)}{l_2(\vartheta)}\right]d\vartheta \leq \frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

$$\int_{\Theta}\frac{\big(l_2(\vartheta)-l_1(\vartheta)\big)^2}{l_2(\vartheta)+l_1(\vartheta)}d\vartheta \geq\leq \frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}$$
$$-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)dr$$
$$-\frac{4}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)\times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}^2\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

and

$$\int_{\Theta}\frac{\big(l_2(\vartheta)-l_1(\vartheta)\big)^2}{l_2(\vartheta)+l_1(\vartheta)}d\vartheta \leq \frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}$$
$$-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)dr$$
$$-\frac{4}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1\big(G(\cdot,r)\big)\times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}(\mathfrak{u}-2)\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

where $\digamma_1\big(G_n(\cdot,r)\big)$ is defined in (47).

4.2 Bounds of Divergence Measures in h-discrete calculus

Now, bounds of different divergence measures in h-discrete calculus are estimated in this section. Moreover, bounds of some divergence measures in discrete calculus are estimated.
If one selects set $h\mathbb{Z}$ as time scale, where $h>0$, in Theorem 5, then $\vartheta = hy \in h\mathbb{Z}$ for some $y\in\mathbb{Z}$. Therefore, (25) and (27) take the form:

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)-\frac{h}{2}\left[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}l_2(h\vartheta)\phi\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\right.$$
$$\left.+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}l_2(h\vartheta)\phi\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\right]\geq\frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))dr$$
$$+\frac{1}{(\varsigma-\rho)}\int_{\rho}^{\varsigma}\digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{\mathfrak{u}\left[\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}\right]}{(\mathfrak{u}-1)!}\right)dr,$$

and

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)-\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)\phi\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right) + \sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)\phi\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\Bigg] \geq \frac{\phi'(\varsigma)-\phi'(\rho)}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1(G(\cdot,r))dr + \frac{1}{(\varsigma-\rho)}\int_\rho^\varsigma \digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}}{(\mathfrak{u}-3)!(\mathfrak{u}-1)}\right)dr,$$

where

$$\digamma_1\big(G(\cdot,r)\big)=\frac{\varsigma-1}{\varsigma-\rho}G(\rho,r)+\frac{1-\rho}{\varsigma-\rho}G(\varsigma,r) - \frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)G\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)},r\right)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)G\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)},r\right)\Bigg]. \tag{48}$$

If one chooses set $h\mathbb{Z}$ as time scale in Theorem 7–10 then (36), (37), (39), (40), (42), (43), (45) and (46) provide following new bounds in h-discrete calculus for differential entropy, KL divergence, Jeffrey distance and Triangular discrimination:

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)\log\frac{1}{l_2(h\vartheta)}+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)\log\frac{1}{l_2(h\vartheta)}\Bigg] \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho) + \frac{1-\rho}{\varsigma-\rho}\log(\varsigma)-\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(\vartheta)\log\big(l_1(\vartheta)\big)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(\vartheta)\log\big(l_1(\vartheta)\big)\Bigg] + \frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr+\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big) \times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}-1}\mathfrak{u}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(h\vartheta)\log\frac{1}{l_2(h\vartheta)}+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(h\vartheta)\log\frac{1}{l_2(h\vartheta)}\Bigg] \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho) + \frac{1-\rho}{\varsigma-\rho}\log(\varsigma)-\frac{h}{2}\Bigg[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_2(\vartheta)\log\big(l_1(\vartheta)\big)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_2(\vartheta)\log\big(l_1(\vartheta)\big)\Bigg] - \frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr+\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big) \times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}-1}(\mathfrak{u}-2)\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\frac{h}{2}\left[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_1(h\vartheta)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_1(h\vartheta)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\right]\leq\frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma)-\frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}-2}\mathfrak{u}}{\mathfrak{u}-1}\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\frac{h}{2}\left[\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1} l_1(h\vartheta)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}} l_1(h\vartheta)\ln\left(\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right)\right]\leq\frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma)+\frac{1}{\rho\varsigma}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}-2}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\frac{h}{2}\left\{\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\big(l_1(h\vartheta)-l_2(h\vartheta)\big)\ln\left[\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right]+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\big(l_1(h\vartheta)-l_2(h\vartheta)\big)\ln\left[\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right]\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}}\mathfrak{u}}{\mathfrak{u}-2}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

$$\frac{h}{2}\left\{\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\big(l_1(h\vartheta)-l_2(h\vartheta)\big)\ln\left[\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right]+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\big(l_1(h\vartheta)-l_2(h\vartheta)\big)\ln\left[\frac{l_1(h\vartheta)}{l_2(h\vartheta)}\right]\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

$$\frac{h}{2}\left\{\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\frac{\big(l_2(h\vartheta)-l_1(h\vartheta)\big)^2}{l_2(h\vartheta)+l_1(h\vartheta)}+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\frac{\big(l_2(h\vartheta)-l_1(h\vartheta)\big)^2}{l_2(h\vartheta)+l_1(h\vartheta)}\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}\right.$$
$$\left.-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)dr-\frac{4}{\varsigma-\rho}\int_{\rho}^{\varsigma}F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}^2\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

and

$$\frac{h}{2}\left\{\sum_{\vartheta=\frac{b_1}{h}}^{\frac{b_2}{h}-1}\frac{\big(l_2(h\vartheta)-l_1(h\vartheta)\big)^2}{l_2(h\vartheta)+l_1(h\vartheta)}+\sum_{\vartheta=\frac{b_1}{h}+1}^{\frac{b_2}{h}}\frac{\big(l_2(h\vartheta)-l_1(h\vartheta)\big)^2}{l_2(h\vartheta)+l_1(h\vartheta)}\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}\right.$$
$$\left.-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr-\frac{4}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}(\mathfrak{u}-2)\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

where $\digamma_1\big(G(\cdot,r)\big)$ is defined in (48).

Remark. Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (25) and (27) to obtain new inequalities containing Csiszár divergence:

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)-\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta\phi\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)+\sum_{\vartheta=1}^{p}(l_2)_\vartheta\phi\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\right]$$
$$\geq\frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1(G(\cdot,r))dr$$
$$+\frac{1}{(\varsigma-\rho)}\int_\rho^\varsigma \digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{\mathfrak{u}\left[\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}\right]}{(\mathfrak{u}-1)!}\right)dr,$$

and

$$\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)-\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta\phi\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)+\sum_{\vartheta=1}^{p}(l_2)_\vartheta\phi\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\right]$$
$$\geq\frac{\phi'(\varsigma)-\phi'(\rho)}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1(G(\cdot,r))dr$$
$$+\frac{1}{(\varsigma-\rho)}\int_\rho^\varsigma \digamma_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}}{(\mathfrak{u}-3)!(\mathfrak{u}-1)}\right)dr,$$

where

$$\digamma_1\big(G(\cdot,r)\big)=\frac{\varsigma-1}{\varsigma-\rho}G(\rho,r)+\frac{1-\rho}{\varsigma-\rho}G(\varsigma,r)$$
$$-\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta G\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta},r\right)+\sum_{\vartheta=1}^{p}(l_2)_\vartheta G\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta},r\right)\right]. \quad (49)$$

Remark. Use $h=1, b_1=0, b_2=p, l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (36) and (37) to obtain new bounds for discrete Shannon entropy:

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta\log\frac{1}{(l_2)_\vartheta}+\sum_{\vartheta=1}^{p}(l_2)_\vartheta\log\frac{1}{(l_2)_\vartheta}\right]\geq\frac{\varsigma-1}{\varsigma-\rho}\log(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\log(\varsigma)-\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}l_2(\vartheta)\log\big(l_2(\vartheta)\big)+\sum_{\vartheta=1}^{p}l_2(\vartheta)\log\big(l_2(\vartheta)\big)\right]$$
$$+\frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr+\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}-1}\mathfrak{u}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

and

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_2)_\vartheta \log\frac{1}{(l_2)_\vartheta}+\sum_{\vartheta=1}^{p}(l_2)_\vartheta \log\frac{1}{(l_2)_\vartheta}\right]\geq\frac{\varsigma-1}{\varsigma-\rho}\log(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\log(\varsigma)-\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}l_2(\vartheta)\log\big(l_2(\vartheta)\big)+\sum_{\vartheta=1}^{p}l_2(\vartheta)\log\big(l_2(\vartheta)\big)\right]$$
$$-\frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr+\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}-1}(\mathfrak{u}-2)\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

where $\digamma_1\big(G(\cdot,r)\big)$ is defined in (49).

Remark. Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (39) and (40) to obtain following new bounds for discrete Kullback–Leibler divergence:

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_1)_\vartheta \ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)+\sum_{\vartheta=1}^{p}(l_1)_\vartheta \ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\right]\leq\frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma)-\frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}-2}\mathfrak{u}}{\mathfrak{u}-1}\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

and

$$\frac{1}{2}\left[\sum_{\vartheta=0}^{p-1}(l_1)_\vartheta \ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)+\sum_{\vartheta=1}^{p}(l_1)_\vartheta \ln\left(\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right)\right]\leq\frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma)+\frac{1}{\rho\varsigma}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}-2}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

where $\digamma_1\big(G(\cdot,r)\big)$ is defined in (49).

Remark. Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (42) and (43) to obtain following new bounds for discrete Jeffrey distance:

$$\frac{1}{2}\left\{\sum_{\vartheta=0}^{p-1}\big((l_1)_\vartheta-(l_2)_\vartheta\big)\ln\left[\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right]+\sum_{\vartheta=1}^{p}\big((l_1)_\vartheta-(l_2)_\vartheta\big)\ln\left[\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right]\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \digamma_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}}\mathfrak{u}}{\mathfrak{u}-2}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

and

$$\frac{1}{2}\left\{\sum_{\vartheta=0}^{p-1}\left((l_1)_\vartheta-(l_2)_\vartheta\right)\ln\left[\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right]+\sum_{\vartheta=1}^{p}\left((l_1)_\vartheta-(l_2)_\vartheta\right)\ln\left[\frac{(l_1)_\vartheta}{(l_2)_\vartheta}\right]\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

where $F_1\big(G(\cdot,r)\big)$ is defined in (49).

Remark. Use $h=1$, $b_1=0$, $b_2=p$, $l_1(\vartheta)=(l_1)_\vartheta$ and $l_2(\vartheta)=(l_2)_\vartheta$, in (45) and (46) to obtain new bounds for discrete triangular discrimination:

$$\frac{1}{2}\left\{\sum_{\vartheta=0}^{p-1}\frac{\left((l_2)_\vartheta-(l_1)_\vartheta\right)^2}{(l_2)_\vartheta+(l_1)_\vartheta}+\sum_{\vartheta=1}^{p}\frac{\left((l_2)_\vartheta-(l_1)_\vartheta\right)^2}{(l_2)_\vartheta+(l_1)_\vartheta}\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}\right.$$
$$\left.-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{4}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}^2\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

and

$$\frac{1}{2}\left\{\sum_{\vartheta=0}^{p-1}\frac{\left((l_2)_\vartheta-(l_1)_\vartheta\right)^2}{(l_2)_\vartheta+(l_1)_\vartheta}+\sum_{\vartheta=1}^{p}\frac{\left((l_2)_\vartheta-(l_1)_\vartheta\right)^2}{(l_2)_\vartheta+(l_1)_\vartheta}\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}\right.$$
$$\left.-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{4}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}(\mathfrak{u}-2)\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

where $F_1\big(G(\cdot,r)\big)$ is defined in (49).

4.3 Bounds of Divergence Measures in q-calculus

Now, we estimate bounds of different divergence measures in q-calculus.
If one chooses set $q^{\mathbb{N}_0}$, $q>1$ as time scale, in Theorem 5, then $\vartheta=q^y\in q^{\mathbb{N}_0}$ for some $y\in\mathbb{N}_0$. Further if $b_1=q$ and

$b_2 = q^p$, then (25) and (27) take the form:

$$\begin{aligned}
&\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)\\
&\qquad -\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(q^m)\phi\left(\frac{l_1(q^m)}{l_2(q^m)}\right)+\sum_{m=2}^{p} q^{m-1} l_2(q^m)\phi\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right]\\
&\qquad \geq \frac{\phi'(\rho)-\phi'(\varsigma)}{\varsigma-\rho}\int_\rho^\varsigma \mathcal{F}_1(G(\cdot,r))dr\\
&\qquad +\frac{1}{(\varsigma-\rho)}\int_\rho^\varsigma \mathcal{F}_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{\mathfrak{u}\left[\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}\right]}{(\mathfrak{u}-1)!}\right)dr,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\varsigma-1}{\varsigma-\rho}\phi(\rho)+\frac{1-\rho}{\varsigma-\rho}\phi(\varsigma)\\
&\qquad -\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(q^m)\phi\left(\frac{l_1(q^m)}{l_2(q^m)}\right)+\sum_{m=2}^{p} q^{m-1} l_2(q^m)\phi\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right]\\
&\qquad \geq \frac{\phi'(\varsigma)-\phi'(\rho)}{\varsigma-\rho}\int_\rho^\varsigma \mathcal{F}_1(G(\cdot,r))dr\\
&\qquad +\frac{1}{(\varsigma-\rho)}\int_\rho^\varsigma \mathcal{F}_1(G(\cdot,r))\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{\phi^{(\mathfrak{u})}(\rho)(r-\rho)^{\mathfrak{u}-1}-\phi^{(\mathfrak{u})}(\varsigma)(r-\varsigma)^{\mathfrak{u}-1}}{(\mathfrak{u}-3)!(\mathfrak{u}-1)}\right)dr,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1\big(G(\cdot,r)\big) &= \frac{\varsigma-1}{\varsigma-\rho}G(\rho,r)+\frac{1-\rho}{\varsigma-\rho}G(\varsigma,r)\\
&\quad -\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(q^m)G\left(\frac{l_1(q^m)}{l_2(q^m)},r\right)+\sum_{m=2}^{p} q^{m-1} l_2(q^m)G\left(\frac{l_1(q^m)}{l_2(q^m)},r\right)\right]. \qquad (50)
\end{aligned}$$

If one chooses set $q^{\mathbb{N}_0}$, $q>1$ as time scale in Theorem 7–10 then (36), (37), (39), (40), (42), (43), (45) and (46) provide new bounds in q-calculus for differential entropy, KL divergence, Jeffrey distance and Triangular discrimination:

$$\begin{aligned}
&\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(q^m)\log\frac{1}{l_2(q^m)}+\sum_{m=2}^{p} q^{m-1} l_2(q^m)\log\frac{1}{l_2(q^m)}\right] \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho)\\
&\qquad +\frac{1-\rho}{\varsigma-\rho}\log(\varsigma)-\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(\vartheta)\log\big(l_2(\vartheta)\big)+\sum_{m=2}^{p} q^{m-1} l_2(\vartheta)\log\big(l_2(\vartheta)\big)\right]\\
&\qquad +\frac{1}{\rho\varsigma}\int_\rho^\varsigma \mathcal{F}_1\big(G(\cdot,r)\big)dr+\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \mathcal{F}_1\big(G(\cdot,r)\big)\\
&\qquad \times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}-1}\mathfrak{u}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,
\end{aligned}$$

$$\begin{aligned}
&\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(q^m)\log\frac{1}{l_2(q^m)}+\sum_{m=2}^{p} q^{m-1} l_2(q^m)\log\frac{1}{l_2(q^m)}\right] \geq \frac{\varsigma-1}{\varsigma-\rho}\log(\rho)\\
&\qquad +\frac{1-\rho}{\varsigma-\rho}\log(\varsigma)-\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_2(\vartheta)\log\big(l_2(\vartheta)\big)+\sum_{m=2}^{p} q^{m-1} l_2(\vartheta)\log\big(l_2(\vartheta)\big)\right]\\
&\qquad -\frac{1}{\rho\varsigma}\int_\rho^\varsigma \mathcal{F}_1\big(G(\cdot,r)\big)dr+\frac{1}{\varsigma-\rho}\int_\rho^\varsigma \mathcal{F}_1\big(G(\cdot,r)\big)\\
&\qquad \times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}-1}(\mathfrak{u}-2)\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,
\end{aligned}$$

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_1(q^m)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)+\sum_{m=2}^{p} q^{m-1} l_1(q^m)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right] \leq \frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma)-\frac{1}{\rho\varsigma}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}-2}\mathfrak{u}}{\mathfrak{u}-1}\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\frac{q-1}{q+1}\left[\sum_{m=1}^{p-1} q^{m+1} l_1(q^m)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)+\sum_{m=2}^{p} q^{m-1} l_1(q^m)\ln\left(\frac{l_1(q^m)}{l_2(q^m)}\right)\right] \leq \frac{\varsigma-1}{\varsigma-\rho}\rho\ln(\rho)$$
$$+\frac{1-\rho}{\varsigma-\rho}\varsigma\ln(\varsigma)+\frac{1}{\rho\varsigma}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}-2}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[-\frac{(r-\rho)^{\mathfrak{u}-1}}{\rho^{\mathfrak{u}}}+\frac{(r-\varsigma)^{\mathfrak{u}-1}}{\varsigma^{\mathfrak{u}}}\right]\right)dr,$$

$$\frac{q-1}{q+1}\left\{\sum_{m=1}^{p-1} q^{m+1}\big(l_1(q^m)-l_2(q^m)\big)\ln\left[\frac{l_1(q^m)}{l_2(q^m)}\right]+\sum_{m=2}^{p} q^{m-1}\big(l_1(q^m)-l_2(q^m)\big)\ln\left[\frac{l_1(q^m)}{l_2(q^m)}\right]\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}\frac{(-1)^{\mathfrak{u}}\mathfrak{u}}{\mathfrak{u}-2}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

$$\frac{q-1}{q+1}\left\{\sum_{m=1}^{p-1} q^{m+1}\big(l_1(q^m)-l_2(q^m)\big)\ln\left[\frac{l_1(q^m)}{l_2(q^m)}\right]+\sum_{m=2}^{p} q^{m-1}\big(l_1(q^m)-l_2(q^m)\big)\ln\left[\frac{l_1(q^m)}{l_2(q^m)}\right]\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}(\rho-1)\ln(\rho)+\frac{1-\rho}{\varsigma-\rho}(\varsigma-1)\ln(\varsigma)$$
$$-\left(\frac{\ln\rho-\ln\varsigma}{\varsigma-\rho}-\frac{1}{\rho\varsigma}\right)\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{1}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}\frac{(-1)^{\mathfrak{u}}(\mathfrak{u}-2)}{\mathfrak{u}-1}\left[\left(\frac{\mathfrak{u}-1}{\rho^{\mathfrak{u}}}+\frac{1}{\rho^{\mathfrak{u}-1}}\right)(r-\rho)^{\mathfrak{u}-1}-\left(\frac{\mathfrak{u}-1}{\varsigma^{\mathfrak{u}}}+\frac{1}{\varsigma^{\mathfrak{u}-1}}\right)(r-\varsigma)^{\mathfrak{u}-1}\right]\right)dr,$$

$$\frac{q-1}{q+1}\left\{\sum_{m=1}^{p-1} q^{m+1}\frac{\big(l_2(q^m)-l_1(q^m)\big)^2}{l_2(q^m)+l_1(q^m)}+\sum_{m=2}^{p} q^{m-1}\frac{\big(l_2(q^m)-l_1(q^m)\big)^2}{l_2(q^m)+l_1(q^m)}\right\}$$
$$\leq\frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}\right.$$
$$\left.-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)dr-\frac{4}{\varsigma-\rho}\int_\rho^\varsigma F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=2}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}^2\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

and

$$\frac{q-1}{q+1}\left\{\sum_{m=1}^{p-1} q^{m+1}\frac{\left(l_2(q^m)-l_1(q^m)\right)^2}{l_2(q^m)+l_1(q^m)}+\sum_{m=2}^{p} q^{m-1}\frac{\left(l_2(q^m)-l_1(q^m)\right)^2}{l_2(q^m)+l_1(q^m)}\right\}$$
$$\leq \frac{\varsigma-1}{\varsigma-\rho}\frac{(\rho-1)^2}{\rho+1}+\frac{1-\rho}{\varsigma-\rho}\frac{(\varsigma-1)^2}{\varsigma+1}-\frac{1}{\varsigma-\rho}\left(\frac{(\varsigma-1)(\varsigma+3)}{(\varsigma+1)^2}\right.$$
$$\left.-\frac{(\rho-1)(\rho+3)}{(\rho+1)^2}\right)\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big)dr-\frac{4}{\varsigma-\rho}\int_{\rho}^{\varsigma} F_1\big(G(\cdot,r)\big)$$
$$\times\left(\sum_{\mathfrak{u}=3}^{n-1}(-1)^{\mathfrak{u}}\mathfrak{u}(\mathfrak{u}-2)\left[\frac{(r-\rho)^{\mathfrak{u}-1}}{(1+\rho)^{\mathfrak{u}+1}}-\frac{(r-\varsigma)^{\mathfrak{u}-1}}{(1+\varsigma)^{\mathfrak{u}+1}}\right]\right)dr,$$

where $F_1\big(G(\cdot,r)\big)$ is defined in (50).

5 Conclusion

In the present work, Montgomery identity and Green function are utilized to prove some inequalities containing divergence measures for diamond integrals. Bounds of different divergence measures are obtained by utilizing particular convex functions. The obtained new findings also provide new bounds of divergence measures for fixed time scales. The new demonstrated bounds are the improvements of bounds given in [1,15,20]. If one chooses $\gamma = 1$ then above all proved results give improvements of results given in [1]. Furthermore, one can fix time scale, to get continuous and discrete bounds different divergence measures which are already given in [15,20]. Possible future work includes study of Rényi entropy using diamond–integral formalism. Which may be included in future tasks.

Declarations

Competing interests: The author declares that there are no competing interests.
Authors' contributions: MB initiated the work and made calculations. KAK supervised and validated the draft. AN deduced the existing results and finalized the draft. JP dealt with the formal analysis and investigation. All the authors read and approved the final manuscript.
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