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On the Well-Posedness For The Incompressible Hall-Magnetohydrodynamic System in Critical Fourier-Besov-Morrey Spaces

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Abstract: In this work, we establish the local well-posedness of the incompressible Hall-magnetohydrodynamic system in critical framework Fourier-Besov-Morrey spaces, which can be seen as a complement to the corresponding results of the Hall-magnetohydrodynamics equations in Fourier-Besov spaces.

Keywords: Well-posedness; Hall-magnetohydrodynamic system; Fourier-Besov-Morrey spaces.

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1 Introduction

In this article, we study the following three-dimensional incompressible Hall- magnetohydrodynamic system:

$$\begin{cases} \partial_t a - \nu\Delta a + (a\cdot\nabla)a + \nabla p = (\nabla\times b)\times b, & t>0,\ x\in\mathbb{R}^3, \\ \partial_t b - \rho_D\Delta b + \rho_H\nabla\times((\nabla\times b)\times b) = \nabla\times(a\times b), & t>0,\ x\in\mathbb{R}^3, \\ \nabla\cdot a = \nabla\cdot b = 0, & t>0,\ x\in\mathbb{R}^3, \\ (a,b)|_{t=0} = (a_0,b_0), & x\in\mathbb{R}^3. \end{cases} \tag{1}$$

In this context, the variables a, b, and p correspond to the velocity, magnetic, and scalar pressure fields, respectively. Additionally, we use $\rho_H > 0$ and $\nu > 0$ to represent the Hall coefficient and kinematic viscosity, respectively. The diffusive coefficient $\rho_D = \frac{1}{\rho\mu}$, where the constants $\rho > 0$ refer to the electrical conductivity and $\mu > 0$ refer to magnetic permeability. The given initial data a_0 signify the initial velocity and b_0 represent the initial magnetic field. The incompressible Hall-magnetohydrodynamic system finds applications in various fields, particularly in plasma physics: investigating the dynamics of plasmas in space, including the interaction between the solar wind and planetary magneto-spheres; astrophysics: studying the behavior of magneto-spheres of planets, such as Earth, where Hall effects become significant due to the presence of charged particles and magnetic fields; and engineering: developing propulsion systems for spacecraft utilizing magnetohydrodynamic principles, where Hall effects can influence the efficiency and control of propulsion.

Numerous studies have explored the global and local-in-time well-posedness and stability of equation (1) across various functional settings. Acheritogary et al. [2] obtained the global well-posedness of weak solutions for the system (1). Ferreira and Benvenutti [5] established the local-in-time well-posedness of strong solutions in the H^2 space. In [6], the authors have demonstrated the existence of locally smooth solutions for large data and globally smooth solutions for small data in the context of the incompressible viscous or inviscid Hall-MHD model. Recently, Raphael and Jin [8] delved into the existence and uniqueness issues of the three-dimensional incompressible Hall-magnetohydrodynamic equations within critical regularity spaces. In [18], Renhui Wan and Yong Zhou obtained two Fujita-Kato type results applicable to the 3D Hall-magnetohydrodynamic equations. In 2021, Nakasato [1] proved the existence of a local-in-time solution in

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the framework of Fourier-Besov spaces. Inspired by [1], the main purpose of this work is to show that the problem (1) is locally well-posed in Fourier-Besov-Morrey spaces (larger than Fourier-Besov spaces) by using the Fourier analysis, Littelwood-Paley theory and Banach fixed point theorem. It is noteworthy that when $\rho_H = 0$, the problem (1) simplifies to the well-known incompressible magnetohydrodynamic system,

$$\begin{cases} \partial_t a - \nu\Delta a + (a\cdot\nabla)a + \nabla p = (\nabla\times b)\times b, & t>0,\ x\in\mathbb{R}^3,\\ \partial_t b - \rho_D\Delta b = \nabla\times(a\times b), & t>0,\ x\in\mathbb{R}^3,\\ \nabla\cdot a = \nabla\cdot b = 0, & t>0,\ x\in\mathbb{R}^3,\\ (a,b)|_{t=0} = (a_0,b_0), & x\in\mathbb{R}^3. \end{cases} \tag{2}$$

Li and Zheng [11] established the well-posedness of the mild solution for the problem (2) in Fourier-Herz space, involving highly oscillating functions. In a different direction, the authors in [15] proved the global existence and a singular limit of the problem (2) in Fourier-Sobolev spaces. Examining the Cauchy problem (2), Miao and Yuan [13] obtained the global existence for small data and local existence for large data in the Besov space $\dot{\mathscr{B}}^{\frac{n}{p}-1}_{p,q}(\mathbb{R}^n)$ with $1\le p<\infty$ and $1\le q\le\infty$. They also established the concept of weak-strong uniqueness of solutions with initial data in $\dot{\mathscr{B}}^{\frac{n}{p}-1}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ for cases where $\frac{n}{2p}+\frac{2}{q}>1$. Moreover, we focus on a solution for the magnetic field that is close to a constant equilibrium $\mathscr{B}$ at spatial infinity, where $\mathscr{B}\in\mathbb{R}^3$ represents a constant magnetic field. To reformulate the problem (1) under the assumption that $\mathscr{B}\neq 0$, we introduce a new vector-valued function $B := b-\mathscr{B}$, leading to the following initial-value problem:

$$\begin{cases} \partial_t a - \nu\Delta a + \nabla\left(p+\frac{|B|^2}{2}\right) - (\nabla\times B)\times\mathscr{B} = h, & t>0,\ x\in\mathbb{R}^3,\\ \partial_t B - \rho_D\Delta B + \rho_H\nabla\times((\nabla\times B)\times\mathscr{B}) - \nabla\times(a\times\mathscr{B}) = \nabla\times g, & t>0,\ x\in\mathbb{R}^3,\\ \nabla\cdot a = \nabla\cdot B = 0, & t>0,\ x\in\mathbb{R}^3,\\ (a,B)|_{t=0} = (a_0,B_0), & x\in\mathbb{R}^3, \end{cases} \tag{3}$$

where h and g are the nonlinear terms, defined respectively by:

$$h := -(a\cdot\nabla)a + (B\cdot\nabla)B, \quad g := a\times B - \rho_H(\nabla\times B)\times B.$$

Numerous researchers have explored the global well-posedness result for the problem (3) (see references [7,10,17]). So as to determine the appropriate function class for a solution (a,p,b) to the problem (3), we examine the following linear system:

$$\begin{cases} \partial_t B - \rho_D\Delta B + \rho_H\nabla\times((\nabla\times B)\times\mathscr{B}) = f, & t>0,\ x\in\mathbb{R}^3,\\ B|_{t=0} = B_0, & x\in\mathbb{R}^3, \end{cases} \tag{4}$$

where $\rho_D,\rho_H>0$, $\mathscr{B}=(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3)^T\in\mathbb{R}^3$. Notice that $e^{tH}B_0$ is the solution of the linear problem (4) and we have,

$$e^{tH}B_0 \simeq e^{i|\mathscr{B}|t\Delta}e^{t\rho_D\Delta}B_0$$

$$\left|\widehat{e^{tH}B_0}\right| \simeq e^{-\sigma_D t|\xi|^2}\left|e^{-i|\mathscr{B}|t|\xi|^2}\widehat{B_0}\right| = \left|\widehat{e^{\rho_D t\Delta}B_0}\right|.$$

Let us present the scaling property of the system (1). If (a,b,p) solves (1) with initial data (a_0,b_0,p_0) then $(a_\gamma,b_\gamma,p_\gamma)$ also solves (1) with the initial data $(a_{0,\gamma},b_{0,\gamma},p_{0,\gamma})$, where

$$\begin{cases} a_\gamma(t,x) = \gamma a\left(\gamma^2 t,\gamma x\right),\\ p_\gamma(t,x) = \gamma^2 p\left(\gamma^2 t,\gamma x\right), & \gamma>0\\ b_\gamma(t,x) = \gamma b\left(\gamma^2 t,\gamma x\right). \end{cases}$$

The critical space for the system (1) is naturally defined as a result of this scaling invariant characteristic.

To facilitate clarity in our description, for $1\le p\le\infty$, we use $L^p = L^p(\mathbb{R}^n)$ to denote the Lebesgue space and $\widehat{L}^p(\mathbb{R}^n) = \widehat{L}^p$ to denote the Fourier-Lebesgue space defined by

$$\widehat{L}^p = \{f\in\mathscr{S}';\ \|f\|_{\widehat{L}^p}<\infty\}, \quad \text{where } \|f\|_{\widehat{L}^p} = \|\widehat{f}\|_{L^{p'}},$$

where p' is the conjugate of p satisfying $\frac{1}{p}+\frac{1}{p'}=1$. We denote by $\mathscr{P}(\mathbb{R}^n)$ the set of polynomials.

The work is organized as follows. In Section 2, we recall the fundamental setting, defining the functional spaces concerned. In Section 3, we present some technical lemmas and in Section 4, we state our main results on local well-posedness and we also prove it within critical Fourier-Besov-Morrey spaces.

In the following section, we will present the Littlewood-Paley theory and Fourier-Besov-Morrey spaces.

2 Preliminaries

Consider the Littlewood-Paley dyadic decomposition of unity denoted by $\{\theta_j\}_{j\in\mathbb{Z}}$, where θ is a positive radially symmetric function belonging to the space $\mathscr{S}'$ such that

$$\operatorname{supp}(\hat{\theta}) \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\},$$

and

$$\hat{\theta}_j(\xi) = \hat{\theta}\left(2^{-j}\xi\right), \quad \sum_{j\in\mathbb{Z}} \hat{\theta}_j(\xi) = 1.$$

First, we define the homogeneous Besov spaces.

Definition 1.*[4] Let $1 \leq p,q \leq +\infty$ and $s \in \mathbb{R}$, the homogeneous Besov space is defined as*

$$\dot{\mathscr{B}}^s_{p,q} = \left\{ f \in \mathscr{S}' : \hat{f} \in L^1_{loc}, \|f\|_{\dot{\mathscr{B}}^s_{p,q}(\mathbb{R}^n)} = \left\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\theta_j * f\|^q_{L^p} \right\}^{1/q} < +\infty \right\},$$

with appropriate modifications made when $q = \infty$.

Next, we define the homogeneous Fourier-Besov spaces.

Definition 2.*[4] Let $1 \leq p,q \leq +\infty$ and $s \in \mathbb{R}$, the homogeneous Fourier-Besov space is defined as*

$$\mathscr{F}\dot{\mathscr{B}}^s_{p,q} = \left\{ f \in \mathscr{S}' : \hat{f} \in L^1_{loc}, \|f\|_{\mathscr{F}\dot{\mathscr{B}}^s_{p,q}(\mathbb{R}^n)} = \left\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\hat{\theta}_j \hat{f}\|^q_{L^{p'}} \right\}^{1/q} < +\infty \right\},$$

with appropriate modifications made when $q = \infty$.

Now, we present the Morrey spaces $\mathscr{M}^\lambda_p(\mathbb{R}^n)$.

Definition 3.*([16]) Let $1 \leq p < \infty$, $0 \leq \lambda < n$, the Morrey space $\mathscr{M}^\lambda_p = \mathscr{M}^\lambda_p(\mathbb{R}^n)$ is given by $\mathscr{M}^\lambda_p(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) ; \|f\|_{\mathscr{M}^\lambda_p} < \infty \right\}$, where*

$$\|f\|_{\mathscr{M}^\lambda_p} = \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))}.$$

Remark.[9]

1) The space $\mathscr{M}^\lambda_p$ equipped with the norm $\|\cdot\|_{\mathscr{M}^\lambda_p}$ is a Banach space.

2) If $1 \leq p_1, p_2, p_3 < \infty$, $0 \leq \lambda_1, \lambda_2, \lambda_3 < n$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we obtain the Hölder inequality

$$\|fg\|_{\mathscr{M}^{\lambda_3}_{p_3}} \leq \|f\|_{\mathscr{M}^{\lambda_1}_{p_1}} \|g\|_{\mathscr{M}^{\lambda_2}_{p_2}}.$$

3) For $1 \leq p < \infty$ and $0 \leq \lambda < n$,

$$\|\varphi * f\|_{\mathscr{M}^\lambda_p} \leq \|\varphi\|_{L^1} \|f\|_{\mathscr{M}^\lambda_p}, \tag{5}$$

for all $\varphi \in L^1$ and $f \in \mathscr{M}^\lambda_p$.

Lemma 1.*[9] Let $1 \leq p_2 \leq p_1 < \infty, 0 \leq \lambda_1, \lambda_2 < n, \frac{n-\lambda_1}{p_1} \leq \frac{n-\lambda_2}{p_2}$ and let γ be a multi-index. If $supp(\widehat{f}) \subset \left\{|\xi| \leq A2^j\right\}$, then the following inequality holds for any constant $C > 0$ independent of f and j.*

$$\left\|(i\xi)^\gamma \widehat{f}\right\|_{\mathscr{M}^{\lambda_2}_{p_2}} \leq C 2^{j|\gamma| + j\left(\frac{n-\lambda_2}{p_2} - \frac{n-\lambda_1}{p_1}\right)} \|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p_1}}. \tag{6}$$

Now, we define the function spaces $\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}(\mathbb{R}^n)$.

Definition 4.[6] *Let $s \in \mathbb{R}, 0 \leq \lambda < n, 1 \leq p < +\infty$, and $1 \leq q \leq +\infty$. The space $\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,r}(\mathbb{R}^n)$ denotes the set of all $f \in \mathscr{S}'(\mathbb{R}\times\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}(\mathbb{R}^n)} = \left\{\sum_{j\in\mathbb{Z}} 2^{jqs}\left\|\hat{\theta}_j\hat{f}\right\|^q_{\mathscr{M}^{\lambda}_{p'}}\right\}^{1/q} < +\infty, \tag{7}$$

with appropriate modifications made when $q = \infty$.

Now, we define the Chemin-Lerner of Fourier-Besov-Morrey spaces.

Definition 5.[6] *Let $s \in \mathbb{R}, 1 \leq p < \infty, 1 \leq q, \rho \leq \infty, 0 \leq \lambda < n$ and $T \in (0,\infty]$. The Chemin-Lerner norm is defined on $f(t,x)$ by*

$$\|f(t,x)\|_{\mathscr{L}^{\rho}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}\right)} = \left\{\sum_{j\in\mathbb{Z}} 2^{jqs}\left\|\hat{\theta}_j\hat{f}\right\|^q_{L^{\rho}\left(0,T;\mathrm{M}^{\lambda}_{p'}\right)}\right\}^{1/q},$$

where $\mathscr{L}^{\rho}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}\right)$ is the set of distributions in $\mathscr{S}'(\mathbb{R}\times\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ with finite $\|\cdot\|_{\mathscr{L}^{\rho}(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q})}$ norm.

As a consequence of the Minkowski inequality, we can establish the following continuous embedding connecting the Chemin-Lerner spaces with the Bochner spaces and the Fourier-Besov-Morrey spaces,

$$\mathscr{L}^{r}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}\right) \subseteq L^{r}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}\right),\ q \leq r,$$
$$L^{r}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}\right) \subseteq \mathscr{L}^{r}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}\right),\ r \leq q.$$

In the following section, let us summarize key lemmas that will be used repeatedly.

3 A priory estimates

Let us proceed to illustrate the different types of product estimations within the Chemin-Lerner spaces of Fourier-Besov-Morrey spaces. The classical bilinear estimates are provided in the following lemma.

Lemma 2.*(Bilinear estimates) Let $s > 0$, $0 \leq \lambda,\ \lambda_1,\ \lambda_2 < n$, $1 \leq r_1,\ r_2 \leq \infty$, and $1 \leq p,\ p_1,\ p_2,\ q \leq \infty$, such that $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2} = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_1}$. Then there exists a positive constant C such that the following estimate holds:*

$$\|fg\|_{\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}} \leq C\left(\|f\|_{\hat{\mathscr{M}}^{\lambda_1}_{p_1}}\|g\|_{\mathscr{F}\dot{\mathscr{N}}^{s}_{p_2,\lambda_2,q}} + \|g\|_{\hat{\mathscr{M}}^{\lambda_1}_{r_1}}\|f\|_{\mathscr{F}\dot{\mathscr{N}}^{s}_{r_2,\lambda_2,q}}\right).$$

*Proof.*We put, $f_j = \theta_j * f$. Thanks to Bony's para-product formula, We break down the product of f and g in the following manner:

$$\begin{aligned} fg &= \sum_{k\in\mathbb{Z}} N_{k-1}fg_k + \sum_{k\in\mathbf{Z}} f_kN_{k-1}g + \sum_{k\in\mathbb{Z}} f_k\tilde{g}_k \\ &\equiv I_1 + I_2 + I_3, \end{aligned}$$

with $N_kf \equiv \sum_{l\leq k-1} f_l$. We first determine the estimation of I_1, we note that $\theta_j * \left(N_{j'-1}f\theta_{j'} * g\right) = 0\,(|j-j'| \geq 4)$, then we have

$$\begin{aligned} \left\|\widehat{I_1}\right\|_{\mathscr{M}^{\lambda}_{p'}} &\leq \sum_{|j-j'|\leq 3}\left\|\widehat{\theta}_j\mathscr{F}\left(N_{j'-1}fg_{j'}\right)\right\|_{\mathscr{M}^{\lambda}_{p'}} \\ &\leq C\sum_{|j-j'|\leq 3}\left\|\mathscr{F}\left(N_{j'-1}fg_{j'}\right)\right\|_{\mathscr{M}^{\lambda}_{p'}} \\ &\leq C\sum_{|j-j'|\leq 3}\left\|\mathscr{F}\left(N_{j'-1}f\right)\right\|_{\mathscr{M}^{\lambda_1}_{p'_1}}\left\|\widehat{\theta}j'\widehat{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}} \\ &\leq C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}}\sum_{|j-j'|\leq 3}\left\|\widehat{\theta}_{j'}\hat{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}}. \end{aligned} \tag{8}$$

Similarly, we obtain the estimation for I_2 in the following manner:

$$\|\widehat{I_2}\|_{\mathscr{M}^{\lambda}_{p'}} \leq C\|\widehat{g}\|_{\mathscr{M}^{\lambda_1}_{r'_1}} \sum_{|j-j'|\leq 3} \|\widehat{\theta} j' \widehat{f}\|_{\mathscr{M}^{\lambda_2}_{r'_2}}. \tag{9}$$

Whereas, we observe that $\theta_j * \left(f_{j'} \tilde{g}_{j'}\right) = 0\,(j' \leq j-4)$. Then we obtain

$$\begin{aligned}
\|\widehat{I_3}\|_{\mathscr{M}^{\lambda}_{p'}} &\leq \sum_{j'\geq j-3} \|\widehat{\theta}_j \mathscr{F}\left(f_{j'} \tilde{g}_{j'}\right)\|_{\mathscr{M}^{\lambda}_{p'}} \\
&\leq C \sum_{j'\geq j-3} \|\widehat{\theta}_{j'} \hat{f} * \widehat{\tilde{\theta}}_{j'} \widehat{g}\|_{\mathscr{M}^{\lambda}_{p'}} \\
&\leq C \sum_{j'\geq j-3} \left\|\widehat{\theta}_{j'} \widehat{f}\right\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \|\widehat{\tilde{\theta}}_{j'} \widehat{g}\|_{\mathscr{M}^{\lambda_2}_{p'_2}} \\
&\leq C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \sum_{j'\geq j-3} \left\|\widehat{\tilde{\theta}} j'^{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}}.
\end{aligned} \tag{10}$$

Combining (8)-(10), we have

$$\begin{aligned}
\left\|\widehat{\theta}_j \mathscr{F}(fg)\right\|_{\mathscr{M}^{\lambda}_{p'}} &\leq C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \left(\sum_{|j-j'\|\leq 3} \left\|\widehat{\theta}_{j'} \widehat{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}} + \sum_{j'\geq j-3} \left\|\widehat{\tilde{\theta}}_{j'} \widehat{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}}\right) \\
&+ C\|\widehat{g}\|_{\mathscr{M}^{\lambda_1}_{r'_1}} \sum_{|j-j'\|\leq 3} \left\|\widehat{\theta}_{j'} \widehat{f}\right\|_{\mathscr{M}^{\lambda_2}_{r'_2}}.
\end{aligned}$$

By Minkowski's inequality, we get

$$\begin{aligned}
\|fg\|_{\mathscr{F}\mathscr{N}^{s}_{p,\lambda,q}} &\leq C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left(\sum_{|j-j'|\leq 3} \left\|\widehat{\theta}_{j'} \widehat{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}}\right)^q\right)^{\frac{1}{q}} \\
&+ C\|\widehat{g}\|_{\mathscr{M}^{\lambda_1}_{r'_1}} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left(\sum_{|j-j'|\leq 3} \|\widehat{\theta}_{j'} \widehat{f}\|_{\mathscr{M}^{\lambda_2}_{r'_2}}\right)^q\right)^{\frac{1}{q}} \\
&+ C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left(\sum_{j-j'\leq 3} \left\|\widehat{\tilde{\theta}}_{j'} \widehat{g}\right\|_{\mathscr{M}^{\lambda_2}_{p'_2}}\right)^q\right)^{\frac{1}{q}} \\
&\leq C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \sum_{|l|\leq 3} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \|\,\widehat{\theta}_{j-l} \widehat{g}\|^q_{\mathscr{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
&\quad + C\|\widehat{g}\|_{\mathscr{M}^{\lambda_1}_{r'_1}} \sum_{|l|\leq 3} \left(\sum_{j\in\mathbf{Z}} 2^{sjq} \|\,\widehat{\theta}_{j-l} \widehat{f}\|^q_{\mathscr{M}^{\lambda_2}_{r'_2}}\right)^{\frac{1}{q}} \\
&\quad + C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \sum_{l\leq 3} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left\|\widehat{\tilde{\theta}}_{j-l} \widehat{g}\right\|^q_{\mathscr{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
&\quad \equiv J_1 + J_2 + J_3.
\end{aligned}$$

For J_1 and J_2, we obtain

$$\begin{aligned}
J_1 + J_2 =& C\|\widehat{f}\|_{\mathscr{M}^{\lambda_1}_{p'_1}} \sum_{|l|\leq 3} 2^{sl} \left(\sum_{j\in\mathbb{Z}} 2^{s(j-l)q} \|\,\widehat{\theta}_{j-l} \widehat{g}\|^q_{\mathscr{M}^{\lambda_2}_{p'_2}}\right)^{\frac{1}{q}} \\
&+ C\|\widehat{g}\|_{\mathscr{M}^{\lambda_1}_{r'_1}} \sum_{|l|\leq 3} 2^{sl} \left(\sum_{j\in\mathbb{Z}} 2^{s(j-l)q} \|\widehat{\theta}_{j-l} \widehat{f}\|_{\mathscr{M}^{\lambda_2}_{r'_2}}\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq C\left(\|f\|_{\mathscr{M}_{p_1}^{\lambda_1}}\|g\|_{\mathscr{F}\dot{\mathscr{N}}_{p_2,\lambda_2,q}^{s}}+\|g\|_{\mathscr{M}_{r_1}^{\lambda_1}}\|f\|_{\mathscr{F}\dot{\mathscr{N}}_{r_2,\lambda_2,q}^{s}}\right).$$

As for J_3, we get

$$h_3' = C\|\widehat{f}\|_{\mathscr{M}_{p_1'}^{\lambda_1}}\sum_{l\leq 3}\left(\sum_{j\in\mathbb{Z}}2^{s(j-l)q}2^{slq}\left\|\widehat{\theta}_{j-l}\widehat{g}\right\|^{q}_{\mathscr{M}_{p_2'}^{\lambda_2}}\right)^{\frac{1}{q}}$$
$$\leq C\|f\|_{\mathscr{M}_{p_1}^{\lambda_1}}\|g\|_{\mathscr{F}\dot{\mathscr{N}}_{p_2,\lambda_2,q}^{s}}.$$

Thus, we complete the proof of Lemma 2.

Moreover, it is essential to demonstrate the smoothing properties within the Chemin-Lerner spaces. First, we examine the inhomogeneous heat equation, which features a diffusive coefficient $(\nu > 0)$:

$$\begin{cases}\partial_t a - \nu\Delta a = f,\ t>0, x\in\mathbb{R}^3,\\ a|_{t=0} = a_0, \qquad\qquad x\in\mathbb{R}^3.\end{cases} \tag{11}$$

Lemma 3.(*Smoothing estimates for* (11)) *Let* $s\in\mathbb{R}$, $0\leq\lambda<3$ *and* $1\leq p,q\leq\infty$. *The inhomogeneous heat equation for* (11) *admits a unique solution u and there exists a positive constant* $C=C(r)>0$, *such that*

$$\nu^{\frac{1}{r}}\|a\|_{L^r\left(0,T;\mathscr{F}\dot{\mathscr{N}}_{p,\lambda,q}^{s+\frac{2}{r}}\right)}\leq C\left(\|a_0\|_{\mathscr{F}\dot{\mathscr{N}}_{p,\lambda,q}^{s}}+\nu^{-1+\frac{1}{r_1}}\|f\|_{\mathscr{L}^{r_1}\left(0,T;\mathscr{F}\dot{\mathscr{N}}_{p,\lambda,q}^{s-2+\frac{2}{r_1}}\right)}\right).$$

Proof. Suppose that u satisfies (11), then for any $j\in\mathbb{Z}$, we have

$$a_j(t) = e^{t\nu\Delta}\theta_j * a_0 + \int_0^t e^{(t-s)\nu\Delta}f_j(s)\mathrm{d}s. \tag{12}$$

In order to evaluate the initial component on the right-hand side of equation (12), we obtain by using $\operatorname{supp}\widehat{\theta}_j\subset\left\{\xi\in\mathbb{R}^n;\ 2^{j-1}\leq|\xi|\leq 2^{j+1}\right\}$ that

$$\begin{aligned}\left\|\widehat{\theta}_j e^{-t\nu|\xi|^2}\widehat{a}_0\right\|_{L^r\left(0,T;\mathscr{M}_{p'}^{\lambda}\right)} &\leq \left(\int_0^\infty e^{-\frac{\nu}{4}2^{2j}tr}z\right)^{\frac{1}{r}}\left\|\widehat{\theta}_j\widehat{a}_0\right\|_{\mathscr{M}_{p'}^{\lambda}}\\ &\leq C\left(\nu 2^{2j}\right)^{-\frac{1}{r}}\left\|\widehat{\theta}_j\widehat{a}_0\right\|_{\mathscr{M}_{p'}^{\lambda}}.\end{aligned} \tag{13}$$

Using Young's inequality, we obtain

$$\begin{aligned}\left\|\int_0^t e^{(t-s)\nu|\xi|^2}\widehat{\theta}_j\widehat{f}(s)\mathrm{d}s\right\|_{L^r\left(0,T;\mathscr{M}_{p'}^{\lambda}\right)} &\leq \left(\int_0^T\left(\int_0^t e^{-\frac{\nu}{4}2^{2j}(t-s)}\left\|\widehat{\theta}_j\widehat{f}\right\|_{\mathscr{M}_{p'}^{\lambda}}\mathrm{d}s\right)^r\mathrm{d}t\right)^{\frac{1}{r}}\\ &\leq \left\|e^{-\frac{\nu}{4}2^{2j}\cdot}\right\|_{L^{\theta}(0,\infty)}\left\|\widehat{\theta}_j\widehat{f}\right\|_{L^{r_1}\left(0,T;\mathscr{M}_{p'}^{\lambda}\right)}\\ &\leq C\left(\nu 2^{2j}\right)^{-1-\frac{1}{r}+\frac{1}{r_1}}\left\|\widehat{\theta}_j\widehat{f}\right\|_{L^{r_1}\left(0,T;\mathscr{M}_{p'}^{\lambda}\right)},\end{aligned} \tag{14}$$

where $\frac{1}{\theta}=1+\frac{1}{r}-\frac{1}{r_1}$. Combining (13) with (14) and multiplying by 2^{sj}, we get

$$\nu^{\frac{1}{r}}2^{\left(s+\frac{2}{r}\right)j}\|a_j\|_{L^r\left(0,T;\mathscr{M}_p^{\lambda}\right)}\leq C2^{sj}\|\theta_j * a_0\|_{\widehat{\mathscr{M}_p^{\lambda}}}+C\nu^{-1+\frac{1}{r_1}}2^{\left(s-2+\frac{2}{r_1}\right)j}\|f_j\|_{L^{r_1}\left(0,T;\widehat{\mathscr{M}_p^{\lambda}}\right)}.$$

Therefore, we take the $\ell^q(\mathbb{Z})$-norm and we get the result.

Furthermore, we show the following estimate for the system (4).

Lemma 4.*(Smoothing estimates for* (4)*) Let* $s \in \mathbb{R}, 1 \leq p,q \leq \infty$, $1 \leq r_1 \leq r \leq \infty$, $T \in \mathbb{R}+$ *and* $0 \leq \lambda < 3$. *Suppose that* $B_0 \in \mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}$ *and* $f \in \mathscr{L}^{r_1}(0,T;\mathscr{F}\dot{\mathscr{N}}^{s-2+\frac{2}{r_1}}_{p,\lambda,q})$. *Then, the problem* (4) *admits a unique solution B and there exists some constant* $C = C(r) > 0$ *such that*

$$\rho_D^{\frac{1}{r}} \|B\|_{L^r(0,T;\mathscr{F}\dot{\mathscr{N}}^{s+\frac{2}{r}}_{p,\lambda,q})} \leq C\left(\|B_0\|_{\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}} + \rho_D^{-1+\frac{1}{r_1}} \|f\|_{L^{r_1}\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{s-2+\frac{2}{r_1}}_{p,\lambda,q}\right)} \right).$$

Proof. Initially, we derive the integral forms of the solution to the linear equations described in (4) and the Fourier transform of (4) concerning $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, if $f \equiv 0$, yields

$$\partial_t \widehat{B} + \rho_D |\xi|^2 \widehat{B} + \rho_H \Omega_\xi \Omega_{\mathscr{B}} \Omega_\xi \widehat{B} = 0,$$

where $\Omega_\xi, \Omega_{\mathscr{B}}$ are defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}, \quad \Omega_{\mathscr{B}} = \begin{pmatrix} 0 & -\mathscr{B}_3 & \mathscr{B}_2 \\ \mathscr{B}_3 & 0 & -\mathscr{B}_1 \\ -\mathscr{B}_2 & \mathscr{B}_1 & 0 \end{pmatrix},$$

respectively. Put $\widehat{H}(\xi) := -\rho_D |\xi|^2 I - \rho_H \Omega_\xi \Omega_{\mathscr{B}} \Omega_\xi$, As can be observed, the characteristic polynomial associated with $\widehat{H}(\xi)$ is determined by

$$\left(\lambda + \rho_D |\xi|^2\right)\left(\lambda^2 + 2\rho_D |\xi|^2 + \rho_D^2 |\xi|^4 + \rho_H |\xi||\xi \cdot \mathscr{B}|^2\right). \tag{15}$$

The following expressions are present in the above equation (15)

$$\lambda_0(\xi) := -\rho_D |\xi|^2 \text{ and } \lambda_\pm(\xi) := -\rho_D |\xi|^2 \pm i\rho_H |\xi||\xi \cdot \mathscr{B}|.$$

We have for any $v \in \mathscr{S}\left(\mathbb{R}^3\right)$,

$$e^{t\hat{H}(\xi)} v = e^{t\lambda_0(\xi)} \widehat{P_0}(\xi) v + e^{t\lambda_+(\xi)} \widehat{P_+}(\xi) v + e^{t\lambda_-(\xi)} \widehat{P_-}(\xi) v,$$

where the Hermitian matrices $\widehat{P_i}(\xi)(i = 0, \pm)$ are defined as follows:

$$\widehat{P_0}(\xi) := \frac{1}{|\xi|^2} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix}$$

$$\widehat{P_+}(\xi) := \frac{1}{2|\xi|^2 |\xi \cdot \mathscr{B}|^2}$$

$$\begin{pmatrix} R_2^2 + R_3^2 & -R_1R_2 + i|\xi||\xi \cdot \mathscr{B}|R_3 & -R_1R_3 - i|\xi||\xi \cdot \mathscr{B}|R_2 \\ -R_1R_2 - i|\xi||\xi \cdot \mathscr{B}|R_3 & R_1^2 + R_3^2 & -R_2R_3 + i|\xi||\xi \cdot \mathscr{B}|R_1 \\ -R_1R_3 + i|\xi||\xi \cdot \mathscr{B}|R_2 & -R_2R_3 - i|\xi||\xi \cdot \mathscr{B}|R_1 & R_1^2 + R_2^2 \end{pmatrix},$$

$$\widehat{P_-}(\xi) := \widehat{P_+}(\xi)^T,$$

with $R_i = R_i(\xi) := \xi_i \xi \cdot \mathscr{B}(i = 1,2,3)$. Then, $e^{tH} B_0$ defined by $\mathscr{F}^{-1}\left[e^{t\hat{H}(\xi)} \widehat{B_0}\right]$ is the solution to the problem (4) with $f \equiv 0$. Using the Duhamel principle, we get the following integral equation,

$$B(t) = e^{tH} B_0 + \int_0^t e^{(t-s)H} f(s) \mathrm{d}s.$$

Using the definition of e^{tH}, we can prove that

$$\left\| \widehat{\theta} j e^{t\widehat{H}(\xi)} \widehat{v} \right\|_{\mathscr{M}^{\lambda}_{p'}} \leq C \left\| e^{-\rho_D |\xi|^2 t} \widehat{\theta}_j \widehat{v} \right\|_{\mathscr{M}^{\lambda}_{p'}}, \tag{16}$$

for all $j \in \mathbb{Z}$. Similarly to the proof of Lemma 3, and by using (16), we have

$$\begin{aligned}
&\left\|B_j\right\|_{L^r\left(0,T;\widehat{\mathscr{M}_p^\lambda}\right)} \\
&\leq \left\|e^{tH}\theta_j * B_0\right\|_{L^r\left(0,T;\widehat{\mathscr{M}_p^\lambda}\right)} + \left\|\int_0^t e^{(t-s)H} f_j(s)\mathrm{d}s\right\|_{L^r\left(0,T;\widehat{\mathscr{M}_p^\lambda}\right)} \\
&\leq C\left\|e^{-\rho_D|\xi|^2 t}\widehat{\theta} j\widehat{B}_0\right\|_{L^r\left(0,T;\mathscr{M}_{p'}^\lambda\right)} + C\left(\int_0^T\left(\int_0^t\left\|e^{-(t-s)\rho_D|\xi|^2}\widehat{\theta}_j\widehat{f}\right\|_{\mathscr{M}_{p'}^\lambda}\mathrm{d}s\right)^r \mathrm{d}t\right)^{\frac{1}{r}} \\
&\leq C\left(\rho_D 2^{2j}\right)^{-\frac{1}{r}}\left\|\widehat{\theta} j\widehat{B}_0\right\|_{\mathscr{M}_{p'}^\lambda} + C\left(\rho_D 2^{2j}\right)^{-1-\frac{1}{r}+\frac{1}{r_1}}\left\|\widehat{\theta} j\widehat{f}\right\|_{L^{r_1}\left(0,T;\mathscr{M}_{p'}^\lambda\right)}.
\end{aligned}$$

Then, the proof of Lemma 4 is completed.

4 Main result

We now proceed to state our main result which establishes the local existence.

Theorem 1.*(Local existence in critical Fourier-Besov-Morrey spaces) Let* $1 \leq p \leq \infty$, $0 \leq \lambda < 3$ *and*

$$(a_0, B_0) \in \mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\left(\mathbb{R}^3\right) \times \left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1} \cap \mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\left(\mathbb{R}^3\right)$$

where $\nabla \cdot a_0 = \nabla \cdot B_0 = 0$. *There exists a positive constant* $\varepsilon_0 \ll 1$ *such that if*

$$\|B_0\|_{\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}} \leq \varepsilon_0,$$

then there exists $T > 0$ *such that the problem* (3) *admits a unique solution* (a, B) *satisfying*

$$\begin{aligned}
a &\in C\left([0,T);\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right) \cap L^1\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right) \\
B &\in C\left([0,T);\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1} \cap \mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right) \cap L^1\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1} \cap \mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right).
\end{aligned}$$

For the proof, we introduce the complete metric spaces $\mathscr{S}_T$ and $\mathscr{Z}_T$, along with the pair of mappings $(\phi[(a,B)](t), \psi[(a,B)](t))$, defined as follows:

$$\begin{aligned}
\mathscr{S}_T &:= \left\{a \in L^2\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right); \|a\|_{\mathscr{S}_T} := \|a\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right) \cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \leq \varepsilon_0\right\}, \\
\mathscr{Z}_T &:= \left\{B \in L^\infty\left(0,T;\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right);\right. \\
&\qquad \left.\|B\|_{\mathscr{Z}_T} := \|B\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right) \cap L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1} \cap \mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right) \cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1} \cap \mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right)} \leq M\varepsilon_0\right\},
\end{aligned}$$

$$\begin{cases}
\phi[(a,B)](t) := e^{t\Delta}a_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}_\rho F(s)\mathrm{d}s, \\
\psi[(a,B)](t) := e^{tH}B_0 + \int_0^t e^{(t-s)H}\nabla \times k(s)\mathrm{d}s,
\end{cases} \tag{17}$$

with

$$\begin{aligned}
F &:= (\nabla \times B) \times \mathscr{B} - (a \cdot \nabla)a + (B \cdot \nabla)B, \\
k &:= a \times \mathscr{B} + a \times B - \rho_H(\nabla \times B) \times B,
\end{aligned}$$

where $M>0$ is an arbitrary number to be determined later and $\mathbb{P}_\rho$ is the Helmholtz projection given by $Id+(-\Delta)^{-1}\nabla\,\mathrm{div}$. We will divide the proof into three steeps. Initially, we prove that $(\phi[(a,B)](t),\psi[(a,B)](t))$ is the map from $\mathscr{S}_T\times\mathscr{Z}_T$ to itself.

Let $T>0$ be sufficiently small, then we have

$$\left\|e^{t\Delta}a_0\right\|_{\mathscr{S}_T}=\left\|e^{t\Delta}a_0\right\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\le\frac{\varepsilon_0}{2}. \tag{18}$$

We use the embedding $\mathscr{L}^2_T(\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,1})\hookrightarrow L^2_T(\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,1})$, Lemma 3 and (18), we obtain

$$\begin{aligned}\|\phi[(a,B)](t)\|_{\mathscr{S}_T}\le\frac{\varepsilon_0}{2}&+C\left\|\mathbb{P}_\rho(\nabla\times B)\times\mathscr{B}\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\&+C\left\|\mathbb{P}_\rho\nabla\cdot(a\otimes a)\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\&+C\left\|\mathbb{P}_\rho\nabla\cdot(B\otimes B)\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}.\end{aligned}$$

Since $\mathbb{P}_\rho$ is bounded from $\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}$ to itself, by using Lemma 3 and the fact that $\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\hookrightarrow\widehat{\mathscr{M}^{\lambda}_{\infty}}$, we can conclude that

$$\begin{aligned}\|\phi[(a,B)](t)\|_{\mathscr{S}_T}&\le\frac{\varepsilon_0}{2}+CT^{\frac{1}{2}}\|B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}+C\|a\|_{L^2\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}+C\|B\|^2_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\&\le\frac{\varepsilon_0}{2}+CMT^{\frac{1}{2}}\varepsilon_0+C\varepsilon_0^2+CM^2\varepsilon_0^2,\end{aligned} \tag{19}$$

for all $(a,B)\in\mathscr{S}_T\times\mathscr{Z}_T$. Now, we provide an estimate for $\psi[(a,B)](t)$. Similar to (18), we choose $T>0$ to be sufficiently small, we get

$$\|e^{tH}B_0\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\le\varepsilon_0. \tag{20}$$

Applying Lemma 4 to $\psi[(a,B)](t)$ and using (20), we can show that

$$\begin{aligned}&\|\psi[(a,B)](t)\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{1}}_{p,\lambda,1}\Big)}\\&\le\|e^{tH}B_0\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{l+\frac{3}{p}}_{p,\lambda,1}\right)}+\left\|\int_0^t e^{(t-s)H}\nabla\times k(s)\mathrm{d}s\right\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\\&\le\varepsilon_0+C\|\nabla\times(a\times(\mathscr{B}+B))\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|\nabla\times((\nabla\times B)\times B)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}.\end{aligned}$$

Applying Lemma 2, we obtain

$$\begin{aligned}\|\psi[(a,B)](t)\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}&\le\varepsilon_0+C\|a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}+C\|a\times B\|_{\left(L^1_T\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\&\quad+C\|(\nabla\times B)\times B\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\&\le\varepsilon_0+CT^{\frac{1}{2}}\|a\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}+C\|a\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\&\quad+\|\nabla B\|_{L^1_T\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,1}\Big)}\|B\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\&\le\varepsilon_0+CT^{\frac{1}{2}}\varepsilon_0+CM\varepsilon_0^2+CM^2\varepsilon_0^2.\end{aligned} \tag{21}$$

In a similar way, we show by using Lemmas 2 and 4 that

$$\begin{aligned}\|\psi[(a,B)](t)\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} &\leq \|B_0\|_{\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}} + C\|a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &\quad + C\|a\times B\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} + C\|(\nabla\times B)\times B\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &\leq C\varepsilon_0 + C\|B\|^2_{\mathscr{L}^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} + C\|a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\|B\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)} \\ &\quad + C\|B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\|a\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)} + \|B\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,1}\right)}\|B\|_{\mathscr{L}^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)} \\ &\leq \varepsilon_0 + CM\varepsilon_0^2 + CM^2\varepsilon_0^2.\end{aligned} \tag{22}$$

Thanks to (21) and (22), we conclude that

$$\|\psi[(a,B)](t)\|_{\mathscr{Z}_T} \leq (1+C)\varepsilon_0 + CT^{\frac{1}{2}}\varepsilon_0 + CM(1+M)\varepsilon_0^2. \tag{23}$$

Put $M = 2C+1$ and T, ε_0 as small enough, by using (19) and (23) we conclude that $(\phi[(a,B)], \psi[(a,B)])$ is map from $\mathscr{S}_T \times \mathscr{Z}_T$ to itself.

Secondly, we show that the map $(\phi[(a,B)](t), \psi[(a,B)](t))$ is a contraction mapping from $\mathscr{S}_T \times \mathscr{Z}_T$ to itself. First, we present the norm $\|\cdot\|_T$ of $\mathscr{S}_T \times \mathscr{Z}_T$ as follows:

$$\|(a_1-a_2, B_1-B_2)\|_T := K\|a_1-a_2\|_{\mathscr{S}_T} + \|B_1-B_2\|_{\mathscr{Z}_T} \quad \text{for} \quad a_1,a_2 \in \mathscr{S}_T, B_1,B_2 \in \mathscr{Z}_T,$$

where K is a positive constant that we will specify later. For simplicity, we denote $(\delta a, \delta B) := (a_1 - a_2, B_1 - B_2)$ and

$$(a_1\cdot\nabla)a_1 - (a_2\cdot\nabla)a_2 = \nabla\cdot(a_1\otimes\delta a) + \nabla\cdot(\delta a\otimes a_2).$$

Using Lemmas 2 and 3, we show that

$$\begin{aligned}&\|\phi[(a_1,B_1)](t) - \phi[(a_2,B_2)](t)\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &\leq C\left\|\mathbb{P}_\rho(\nabla\times\delta B)\times\mathscr{B}\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)} + C\left\|\mathbb{P}_\rho\nabla\cdot(a_1\otimes\delta a)\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &+ C\left\|\mathbb{P}_\rho\nabla\cdot(\delta a\otimes a_2)\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)} + C\left\|\mathbb{P}_\rho\nabla\cdot(B_1\otimes\delta B)\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &+ C\left\|\mathbb{P}_\rho\nabla\cdot(\delta B\otimes B_2)\right\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)} \leq CT^{\frac{1}{2}}\|\delta B\|_{\mathscr{Z}_T} \\ &+ C\left(\|a_1\|_{\mathscr{S}_T} + \|a_2\|_{\mathscr{S}_T}\right)\|\delta a\|_{\mathscr{S}_T} + C\left(\|B_1\|_{\mathscr{Z}_T} + \|B_2\|_{\mathscr{Z}_T}\right)\|\delta B\|_{\mathscr{Z}_T} \\ &\leq 2C\varepsilon_0\|\delta a\|_{\mathscr{S}_T} + C\left(2M\varepsilon_0 + T^{\frac{1}{2}}\right)\|\delta B\|_{\mathscr{Z}_T}.\end{aligned} \tag{24}$$

From Lemmas 2 and 4, we get the following estimate for ψ :

$$
\begin{aligned}
&\|\psi[(a_1,B_1)](t)-\psi[(a_2,B_2)](t)\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\Big)}\\
&\le C\|\nabla\times(\delta a\times\mathscr{B})\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|\nabla\times(\delta a\times B_1)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\nabla\times(a_2\times\delta B)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\nabla\times((\nabla\times\delta B)\times B_1)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\nabla\times((\nabla\times B_2)\times\delta B)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&\le C\left(T^{\frac{1}{2}}+M\varepsilon_0\right)\|\delta a\|_{\mathscr{S}_T}+C(\varepsilon_0+2M\varepsilon_0)\|\delta B\|_{\mathscr{Z}_T}.
\end{aligned}
\tag{25}
$$

Similarly, Lemmas 2 and 4, show that

$$
\begin{aligned}
&\|\psi[(a_1,B_1)](t)-\psi[(a_2,B_2)](t)\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^2_T\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\Big)\cap L^1_T\left(\hat{B}\mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&\le C|\mathscr{B}|\|\delta a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|\delta a\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|B_1\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\delta a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\|B_1\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|a_2\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|\delta B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}p,\lambda,1^{1+\frac{3}{p}}\right)}+C\|a_2\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\|\delta B\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\delta B\|L^2\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)\|B_1\|_{L^2\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,q}\right)}+C\|B_1\|_{L^\infty\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|\delta B\|_{L^\infty\left(\mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|\nabla\times((\nabla\times\delta B)\times B_1)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\nabla\times((\nabla\times B_2)\times\delta B)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&\le C\left(T^{\frac{1}{2}}+M\varepsilon_0\right)\|\delta a\|\mathscr{S}_T+C(\varepsilon_0+2M\varepsilon_0)\|\delta B\|_{\mathscr{Z}_T}.
\end{aligned}
\tag{26}
$$

Analogously, by using Lemmas 4 and 2, we get

$$
\begin{aligned}
&\|\psi[(a_1,B_1)](t)-\psi[(a_2,B_2)](t)\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{2+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&\le C\|\delta a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|\delta a\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|B_1\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|\delta a\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\|B_1\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|a_2\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|\delta B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|a_2\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{1}}_{p,\lambda,1}\right)}\|\delta B\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|\delta B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1}_{p,\lambda,1+\frac{3}{p}}\right)}\|B_1\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|B_1\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{p}_{p,\lambda,1}\right)}\|\delta B\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\\
&+C\|B_2\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1}_{p,\lambda,1+\frac{3}{p}}\right)}\|\delta B\|_{\mathscr{L}^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}+C\|\delta B\|_{L^\infty_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\|B_2\|_{\mathscr{L}^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \qquad (27)\\
&\le C_1\|\delta a\|_{\mathscr{S}_T}+CM\varepsilon_0\|\delta a\|_{\mathscr{S}_T}+C\varepsilon_0(1+2M)\|\delta B\|_{\mathscr{Z}_T}.
\end{aligned}
$$

Combining (26) with (27), we obtain

$$
\begin{aligned}
\|\psi[(a_1,B_1)](t)-\psi[(a_2,B_2)](t)\|_{\mathscr{Z}_T}\le&C_1\|\delta a\|\mathscr{S}_T+C\left(T^{\frac{1}{2}}+2M\varepsilon_0\right)\|\delta a\|_{\mathscr{S}_T}\\
&+2C\varepsilon_0(1+2M)\|\delta B\|_{\mathscr{Z}_T}.
\end{aligned} \qquad (28)
$$

Put $E:=4C_1+1$ and taking $E\times(25)+(28)$, we have

$$
\begin{aligned}
&\|\,|\,(\phi[(a_1,B_1)](t)-\phi[(a_2,B_2)](t),\psi[(a_1,B_1)](t)-\psi[(a_2,B_2)](t))\,|\,\|_T\\
&\le\frac{1}{2}\|\|(\delta a,\delta B)\|\|_T,
\end{aligned}
$$

where $\varepsilon_0<\min\left(\frac{1}{16CME},\frac{1}{2C(1+2M)},\frac{1}{16C},\frac{1}{32M}\right),T<\min\left(\frac{1}{16^2},\frac{1}{64C^2E^2}\right)$. Then, by Banach's contraction mapping theorem, we get the fixed point $(a,B)\in\mathscr{S}_T\times\mathscr{Z}_T$.

Finally, we prove the uniqueness of the solution to complete the proof of Theorem 1. Let (a_1,B_1) and (a_2,B_2) be two solutions of (3) for the same initial data, we have that

$$
\delta a(t)=\int_0^t e^{(t-s)\Delta}\mathbb{P}_\rho\delta f(s)\mathrm{d}s,\quad \delta B(t)=\int_0^t e^{(t-s)H}\nabla\times\delta h(s)\mathrm{d}s,
$$

where $(\delta a,\delta B):=(a_2-a_1,B_2-B_1)$ and

$$
\begin{aligned}
&\delta F=(\nabla\times\delta B)\times\mathscr{B}-(a_1\cdot\nabla)\,\delta a-(\delta a\cdot\nabla)a_2+(B_1\cdot\nabla)\,\delta B+(\delta B\cdot\nabla)B_2,\\
&\delta k=\delta a\times\mathscr{B}+\delta a\times B_1+a_2\times\delta B-\rho_H(\nabla\times\delta B)\times B_1-\rho_H\,(\nabla\times B_2)\times\delta B.
\end{aligned}
$$

By Lemmas 3 and 4 , we get for $I=(0,T)$,

$$
\begin{aligned}
&\|\delta a\|_{L^\infty\left(I;\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1\left(I;\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\le C\|\delta f\|_{L^1\left(I;\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)},\\
&\|(\delta B,\nabla\delta B)\|_{L^\infty\left(I;\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1\left(I;\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)}\le C\|(\delta k,\nabla\delta k)\|_{L^1\left(I,\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}.
\end{aligned}
$$

Notice that $\|f\|_{\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,1}}\le\|f\|^{1/2}_{\mathscr{F}\dot{\mathscr{N}}^{s-1}_{p,\lambda,1}}\|f\|^{1/2}_{\mathscr{F}\dot{\mathscr{N}}^{s+1}_{p,\lambda,1}}$, by using Lemma 2 and Young's inequality we can prove that for any $\varepsilon>0$,

$$
\begin{aligned}
\|\delta f\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)}\le&CT^{\frac{1}{2}}\|\delta B\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}+\varepsilon\|(\delta a,\delta B)\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{1+\frac{3}{5}}_{p,\lambda,1}\right)}\\
&+C\varepsilon^{-1}\int_0^T\|(a_1,a_2,B_1,B_2)\|^2_{\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}}\|(\delta a,\delta B)\|_{\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}}\mathrm{d}t.
\end{aligned} \qquad (29)
$$

Similarly, we obtain that for any $\varepsilon>0$,

$$
\begin{aligned}
\|\delta k\|_{L^1_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}\le&CT^{\frac{1}{2}}\|\delta a\|_{L^2_T\left(\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}\right)}+\varepsilon\|(\delta a,\delta B,\nabla\delta B)\|_{L^1_T}\\
&+C\varepsilon^{-1}\int_0^T\|(a_2,B_1,B_2,\nabla B_1)\|^2_{\mathscr{F}\dot{\mathscr{N}}^{\frac{3}{p}}_{p,\lambda,1}}\|(\delta a,\delta B,\nabla\delta B)\|_{\mathscr{F}\dot{\mathscr{N}}^{-1+\frac{3}{p}}_{p,\lambda,1}}\mathrm{d}t.
\end{aligned} \qquad (30)
$$

On the other hand, using $\|B_0\|_{\dot{\mathscr{FN}}^{\frac{3}{p}}_{p,\lambda,1}} \leq \varepsilon_0$, we get $\|B_0\|_{\mathscr{L}^\infty_T(\dot{\mathscr{FN}}^{\frac{3}{p}}_{p,\lambda,1})} \leq C\varepsilon_0$. Noting this smallness for B and using similar way we obtain that for any $\varepsilon > 0$,

$$\begin{aligned} \|\delta k\|_{L^1_T\left(\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} &\leq K\|\delta a\|_{L^1_T\left(\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} + C(\varepsilon+\varepsilon_0)\|(\delta a, \nabla\delta B)\|_{L^1_T\left(\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &+ C\varepsilon^{-1}\int_0^T \left(\|(a_2,\nabla a_2,\nabla B_1,\nabla B_2)\|_{\dot{\mathscr{FN}}^{\frac{3}{p}}_{p,\lambda,1}} + \|^2\nabla^2 B_1\|_{\dot{\mathscr{FN}}^{\frac{3}{p}}_{p,\lambda,1}}\right)\|(\delta a,\nabla\delta B)\|_{\dot{\mathscr{FN}}^{-1+\frac{3}{p}}_{p,\lambda,1}}\,\mathrm{d}t. \end{aligned} \tag{31}$$

Thanks to $(E+1)\times(29)+(30)+(31)$, we get

$$\begin{aligned} &\|\delta a\|_{L^\infty_T\left(\dot{\mathscr{FN}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} + \|(\delta B,\nabla\delta B)\|_{L^\infty_T\left(\dot{\mathscr{FN}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1_T\left(\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\right)} \\ &\leq C\int_0^T \left(\|(a_1,a_2,B_1,B_2,\nabla B_1,\nabla B_2)\|_{\dot{\mathscr{FN}}^{\frac{3}{p}}_{p,\lambda,1}} + \|(a_2,\nabla B_1)\|_{\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}}\right)\|(\delta a,\delta B,\nabla\delta B)\|_{\dot{\mathscr{FN}}^{-1+\frac{3}{p}}_{p,\lambda,1}}\,\mathrm{d}t. \end{aligned}$$

By Gronwall's inequality, we conclude that

$$\begin{aligned} &a_1 \equiv a_2 \text{ in } L^\infty\left(I;\dot{\mathscr{FN}}^{-1+\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1\left(I;\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\right), \\ &B_1 \equiv B_2 \text{ in } L^\infty\left(I;\dot{\mathscr{FN}}^{-1+\frac{3}{p}}_{p,\lambda,1}\cap\dot{\mathscr{FN}}^{\frac{3}{p}}_{p,\lambda,1}\right)\cap L^1\left(I;\dot{\mathscr{FN}}^{1+\frac{3}{p}}_{p,\lambda,1}\cap\dot{\mathscr{FN}}^{2+\frac{3}{p}}_{p,\lambda,1}\right). \end{aligned}$$

Therefore, we finish the proof of Theorem 1.

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