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Study of Bipolar Fuzzy Soft Hypervector Spaces

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Abstract: In this paper, three topics in bipolar fuzzy soft hypervector spaces are investigated. At first, four equivalent conditions for the definition of a bipolar fuzzy soft hypervector space are presented, from different points of view. Then some new bipolar fuzzy soft hypervector spaces are constructed using the notions of subhyperspace, level subset, generated subhyperspace, and previous bipolar fuzzy soft hypervector spaces. Finally, normal bipolar fuzzy soft hypervector spaces are studied as a special kind of bipolar fuzzy soft hypervector spaces. All concepts are supported by interesting examples.

Keywords: Hypervector space; Level subset; Bipolar fuzzy soft hypervector space; Subhyperspace; Normal bipolar fuzzy soft hypervector space.

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1 Introduction

There are several mathematical tools for modeling uncertainties, such as fuzzy sets, bipolar fuzzy sets, soft sets, fuzzy soft sets, and bipolar fuzzy soft sets. These theories have had some applications in algebraic structures. In the following, we briefly discuss them:

Fuzzy sets, introduced by Lotfi A. Zadeh [44] in 1965, are an extension of classical set theory. In classical set theory, an element either belongs to a set or does not. However, in the case of fuzzy sets, equipped by a function $\nu : A \to [0,1]$, elements can partially belong to a set, expressing degrees of membership rather than a strict binary classification. The concept of fuzzy sets has found numerous applications in various fields, including control systems, pattern recognition, decision making, artificial intelligence, image processing, medicine, and traffic control. These are just a few examples of the wide-ranging applications of fuzzy sets. Their ability to handle uncertainty and imprecision makes them valuable in situations where traditional binary logic and set theory fall short.

There are various generalizations of fuzzy sets, including intuitionistic fuzzy sets, type-2 fuzzy sets, and hesitant fuzzy sets. These generalizations aim to capture more complex types of uncertainty and vagueness in real-world decision-making and modeling scenarios. Each generalization offers a different way of representing and handling uncertainty, allowing for more nuanced and flexible modeling in various applications. Another extension of traditional fuzzy sets is the notion of bipolar fuzzy sets, introduced by Zhang [45], which the range of membership function is increased from interval $[0,1]$ to $[-1,0] \times [0,1]$. In fact, it has two membership degrees that represent the satisfaction level for a property and its counter-property. This allows for a more nuanced representation of uncertainty and ambiguity. Bipolar fuzzy sets have found applications in decision-making, pattern recognition, and various fields where the consideration of both positive and negative aspects is necessary. The concept is particularly useful in situations where an element's association with a set is not absolute and can be influenced by conflicting or contradictory information.

Also, the concept of "soft sets", developed by Molodtsov [34] in the late 1990s, is a mathematical framework that extends the traditional notion of a set by introducing the concept of "soft membership". In a soft set, instead of an element being only a member or non-member of a set, it can have a varying degree of membership, often represented by a real number between 0 and 1. This extension allows for more flexibility in modeling uncertainty or vagueness in a set membership. Soft sets have found applications in fields such as decision-making, pattern recognition, and risk analysis, where the notion of uncertainty is crucial. This idea was also studied in algebraic structures; for examples, soft groups, soft rings and soft vector spaces were investigated by Aktas [8], Acar [3] and Sezgin [41], respectively.

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In 2011, Cogman [16] defined fuzzy soft sets as a more precise tool for modeling. Fuzzy soft sets are an extension of both fuzzy sets and soft sets, combining the concepts of fuzzy logic and soft sets to provide a more comprehensive framework for dealing with uncertainty and vagueness. In a fuzzy soft set, similar to a soft set, elements can have degrees of membership, but in addition, the membership degrees can themselves be fuzzy. This means that the degree of membership of an element in a fuzzy soft set is not restricted to a precise real number, but can itself be a fuzzy value represented by a membership function. Fuzzy soft sets find applications in areas where both uncertainty regarding membership and degrees of vagueness are present, such as in decision-making under complex and imprecise conditions, pattern recognition in ambiguous data, and control systems where precise inputs or outputs are not guaranteed. The development and study of fuzzy soft sets have brought together theories from fuzzy sets, soft sets, and other related areas of mathematics and computer science, providing a more versatile framework for handling ambiguous and uncertain information. For example, Akram [6] studied this concept in Lie algebras.

After that, Abdollah [1] introduced bipolar fuzzy soft sets in 2014, as an extension of both bipolar fuzzy sets and fuzzy soft sets. These sets are designed to manage the notion of duality and to handle both positive and negative information. Essentially, bipolar fuzzy soft sets combine the concepts of bipolar fuzzy logic and soft sets to effectively deal with uncertainty and vagueness in a bipolar setting. In bipolar fuzzy soft sets, the membership degree of an element is not only represented in a fuzzy manner, but it also considers the positive and negative aspects of the membership. This allows for a more nuanced representation and analysis of information, which is particularly useful in decision-making processes where both positive and negative factors need to be considered. Bipolar fuzzy soft sets have applications in various fields, including decision analysis, pattern recognition, risk assessment, and sentiment analysis, where the consideration of positive and negative aspects of information is crucial. They provide a powerful framework for handling ambiguous and uncertain information while taking into account both affirmative and contradictory viewpoints. The study and application of bipolar fuzzy soft sets is an active area of research, and it continues to evolve with contributions from both mathematical theory and practical problem-solving domains. Furthermore, some researchers have applied this idea in different branches of pure mathematics: Akram [4,5,7] studied it in K-algebras and Lie algebras, Ali [9] used it in decision making, Abughazalah [2] applied the bipolar fuzzy sets in BCI-algebras, Riaz [38] discussed bipolar fuzzy soft topology, Mahmood [32] presented its complex extension, and Khan [31] studied bipolar fuzzy soft matrices.

On the other hand, Marty [33] introduced algebraic hyperstructures in 1934 through the generalization of operations into hyperoperations. While an operation assigns a unique element of the set X to any two elements of X , a hyperoperation assigns them a unique subset of X. This idea has been studied in various fields, and for more informaion, you can reference the books by Corsini [17], Davvaz [19] and Vougiouklis [43]. Particularly, Scafati-Tallini [42] introduced the notion of hypervector space in 1990. Hypervector spaces have been studied by Ameri [11], Sedghi [40] and the author [22,26,28].

The mentioned extensions of fuzzy sets have influenced algebraic hyperstructures ([18]). For example, fuzzy hypervector spaces were introduced by Ameri [10] in 2005, who investigated some properties of fuzzy hypervector spaces ([12,13,14,15]). The author continued this work and studied more results of fuzzy hypervector spaces ([20,21,23,24,30]). Ranjbar [37] also examined some properties of fuzzy soft hypervector spaces. Additionally, the author [25,29] investigated some results in soft hypervector spaces. Norouzi [36] introduced new directions on soft hypermodules and soft fuzzy hypermodules. Sarwar [39] and Muhiuddin [35] applied bipolar fuzzy soft sets in hypergraphs and hyper BCK-ideals, respectively.

In a recent publication, the author [27] explored the application of bipolar fuzzy soft sets in hypervector spaces, defining new operations and external hyperoperations on bipolar fuzzy soft sets over hypervector space V. We introduced the notion of bipolar fuzzy soft hypervector space, supported by non-trivial examples, and demonstrated that the constructed bipolar fuzzy soft sets are indeed bipolar fuzzy soft hypervector spaces. We also investigated the behavior of bipolar fuzzy soft hypervector spaces under linear transformations.

In this paper, the author follows the previous work [27] and presents four equivalent conditions for the defined notion in section 3. Furthermore, new structures are constructed in section 4 using subhyperspaces, (α,β)-level subsets, and generated bipolar fuzzy soft sets. The study of normal bipolar fuzzy soft hypervector spaces is briefly discussed in section 5. All new notions in the paper supported by interesting examples. The results obtained in this paper lay the necessary groundwork for further research. The paper also suggests areas for future exploration in Section 6.

2 Preliminaries

Here, some definitions and examples are presented from the published papers, for using in the article.

Definition 1.[45] *If $\mathscr{X}$ is a non-empty set, $\mathscr{B}^+ : \mathscr{X} \to [0,1]$ and $\mathscr{B}^- : \mathscr{X} \to [-1,0]$ show how much the member applies to the desired property and the implicit counter-property, respectively, then*

$$\mathscr{B} = \{(x, \mathscr{B}^+(x), \mathscr{B}^-(x)),\ x \in \mathscr{X}\},$$

is called a bipolar fuzzy set (shortly, bf set) in $\mathscr{X}$. By $BF^{\mathscr{U}}$ we denote the collection of all bf sets over $\mathscr{U}$.

Definition 2.[34] *If $\mathscr{B}$ is a set of parameters, $\mathscr{U}$ is the universe set and $\mathscr{G}:\mathscr{B}\to P(\mathscr{U})$ is a mapping, then $(\mathscr{G},\mathscr{B})$ is a soft set over $\mathscr{U}$.*

Definition 3.[1] *If $\mathscr{U}$ is the universe set and $\mathscr{G}:\mathscr{B}\to BF^{\mathscr{U}}$ assigns to any parameter $e\in\mathscr{B}$ a bf set $\mathscr{G}_e$, i.e.*

$$\forall e\in\mathscr{B};\ \mathscr{G}_e=\{(x,\mathscr{G}_e^+(x),\mathscr{G}_e^-(x)),\ x\in\mathscr{U}\},$$

then $(\mathscr{G},\mathscr{B})$ is a bipolar fuzzy soft set (shortly, bfs set) of $\mathscr{U}$.

Note that $(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{H},\mathscr{C})$, iff $\mathscr{B}\subseteq\mathscr{C}$ and $\mathscr{G}_e^+(x)\leq\mathscr{G}_e^+(x)$, $\mathscr{G}_e^-(x)\geq\mathscr{G}_e^-(x)$, $\forall e\in\mathscr{B},x\in\mathscr{U}$.

Definition 4.[42] *A hypervector space (shortly, hvs) is an algebra $(\mathscr{V},+,\circ,\mathscr{K})$ that $(\mathscr{V},+)$ is an Abelian group, $\mathscr{K}$ is a field and $\circ:\mathscr{K}\times\mathscr{V}\to P_*(\mathscr{V})$ ($P_*(\mathscr{V})=P(\mathscr{V})\setminus\emptyset$) is an external hyperoperation (shortly, eho), such that*

(H_1) $b\circ(y+z)\subseteq b\circ y+b\circ z$,
(H_2) $(b+c)\circ y\subseteq b\circ y+c\circ y$,
(H_3) $b\circ(c\circ y)=(bc)\circ y$,
(H_4) $b\circ(-y)=(-b)\circ y=-(b\circ y)$,
(H_5) $y\in 1\circ y$.

Note that in the right hand of (H_1), $b\circ y+b\circ z=\{r+s:r\in b\circ y,s\in b\circ z\}$. In a similar way, we have in (H_2). Moreover, $b\circ(c\circ y)=\bigcup\limits_{p\in c\circ y} b\circ p$.
If equality holds in (H_1), then $\mathscr{V}$ is strongly right distributive (shortly, srd). A strongly left distributive (shortly, sld) hvs is similarly defined.

If $\mathscr{V}_1,\mathscr{V}_2$ are hvs's over the field $\mathscr{K}$, such that $\mathscr{V}_1\subseteq\mathscr{V}_2$, then $\mathscr{V}_1$ is said to be a subhyperspace (shortly, shs) of $\mathscr{V}_2$, i.e. $y-z\in\mathscr{V}_1$ and $b\circ y\subseteq\mathscr{V}_1$, for all $b\in\mathscr{K}$, $y,z\in\mathscr{V}_1$.

Example 1.[13] $(\mathbb{R}^3,+,\circ,\mathbb{R})$ is a hvs, where $(\mathbb{R}^3,+)$ is the real vector space and $a\circ(x_0,y_0,z_0)=\{(ax_0,ay_0,z),\ z\in\mathbb{R}\}$.

Example 2.[27] Let $\mathscr{K}=\mathbb{Z}_2=\{0,1\}$ be the field of two numbers with the following operations:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Then $(\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ is a hvs, such that the operation "$+:\mathbb{Z}_4\times\mathbb{Z}_4\to\mathbb{Z}_4$" and the eho "$\circ:\mathbb{Z}_2\times\mathbb{Z}_4\to P_*(\mathbb{Z}_4)$" are defined as follow:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

∘	0	1	2	3
0	{0,2}	{0}	{0}	{0}
1	{0,2}	{1,2,3}	{0,2}	{1,2,3}

Definition 5.[27] *Let $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$ be a hvs and $(\mathscr{G},\mathscr{B})$ be a bfs set of $\mathscr{V}$. Then $(\mathscr{G},\mathscr{B})$ is a bipolar fuzzy soft hypervector space (shortly, bfs-hvs) of $\mathscr{V}$, iff*

1. $\mathscr{G}_e^+(y-z)\geq\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(z)$, $\mathscr{G}_e^-(y-z)\leq\mathscr{G}_e^-(y)\vee\mathscr{G}_e^-(z)$,
2. $\bigwedge\limits_{r\in b\circ y}\mathscr{G}_e^+(r)\geq\mathscr{G}_e^+(y)$, $\bigvee\limits_{r\in b\circ y}\mathscr{G}_e^-(r)\leq\mathscr{G}_e^-(y)$.

Example 3.[27] $(\mathscr{G},\{a,b\})$ is a bfs-hvs of $\mathscr{V}=(\mathbb{R}^3,+,\circ,\mathbb{R})$ defined in Example 1, where "$\mathscr{G}_a^+,\mathscr{G}_b^+:\mathbb{R}^3\to[0,1]$" and "$\mathscr{G}_a^-,\mathscr{G}_b^-:\mathbb{R}^3\to[-1,0]$" are given by the followings:

$$\mathscr{G}_a^+(t)=\begin{cases}0.7\ t\in X,\\ 0.3\ t\in Y\setminus X,\\ 0\quad o.w.\end{cases}\qquad \mathscr{G}_a^-(t)=\begin{cases}-0.8\ t\in X,\\ -0.4\ t\in Y\setminus X,\\ -0.2\ o.w.\end{cases}$$

$$\mathscr{G}_b^+(t)=\begin{cases}0.9\ t\in X,\\ 0.4\ t\in Y\setminus X,\\ 0.1\ o.w.\end{cases}\qquad \mathscr{G}_b^-(t)=\begin{cases}-0.6\ t\in X,\\ -0.5\ t\in Y\setminus X,\\ -0.1\ o.w.\end{cases}$$

where $X=\{0\}\times\{0\}\times\mathbb{R}$ and $Y=\mathbb{R}\times\{0\}\times\mathbb{R}$.

Example 4.[27] $(\mathscr{G},\{c,d,e\})$ is a bfs-hvs of $\mathscr{V}=(\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ defined in Example 2, where "$\mathscr{G}_c^+,\mathscr{G}_d^+,\mathscr{G}_e^+:\mathbb{Z}_4\to[0,1]$" and "$\mathscr{G}_c^-,\mathscr{G}_d^-,\mathscr{G}_e^-:\mathbb{Z}_4\to[-1,0]$" are given by the followings:

$$\mathscr{G}_c^+(t)=\begin{cases}0.5 & t=0,2,\\ 0.3 & o.w.\end{cases}\qquad \mathscr{G}_c^-(t)=\begin{cases}-0.4 & t=0,2,\\ -0.2 & o.w.\end{cases}$$

$$\mathscr{G}_d^+(t)=\begin{cases}0.7 & t=0,2,\\ 0.2 & o.w.\end{cases}\qquad \mathscr{G}_d^-(t)=\begin{cases}-0.6 & t=0,2,\\ -0.3 & o.w.\end{cases}$$

$$\mathscr{G}_e^+(t)=\begin{cases}0.8 & t=0,2,\\ 0.4 & o.w.\end{cases}\qquad \mathscr{G}_e^-(t)=\begin{cases}-0.7 & t=0,2,\\ -0.5 & o.w.\end{cases}$$

Definition 6.[27] *Let $(\mathscr{G},\mathscr{A})$ and $(\mathscr{H},\mathscr{B})$ be bfs sets of hvs $\mathscr{V}$ and $b\in\mathscr{K}$. Then the sum $(\mathscr{G},\mathscr{A})+(\mathscr{H},\mathscr{B})$ and the scalar product $b\circ(\mathscr{G},\mathscr{A})$ are defined as the bfs sets $(\mathscr{G}+\mathscr{H},\mathscr{A}\cap\mathscr{B})$ and $(a\circ\mathscr{G},\mathscr{A})$, respectively, where*

$$(\mathscr{G}+\mathscr{H})_e^+(x)=\sup_{x=y+z}(\mathscr{G}_e^+(y)\wedge\mathscr{H}_e^+(z)),$$

$$(\mathscr{G}+\mathscr{H})_e^-(x)=\inf_{x=y+z}(\mathscr{G}_e^-(y)\vee\mathscr{H}_e^-(z)),$$

$$(b\circ\mathscr{G})_e^+(x)=\begin{cases}\bigvee\limits_{y\in b\circ r}\mathscr{G}_e^+(r) & \exists r\in\mathscr{V},y\in b\circ r,\\ 0 & otherwise,\end{cases}$$

$$(b\circ\mathscr{G})_e^-(x)=\begin{cases}\bigwedge\limits_{y\in b\circ r}\mathscr{G}_e^-(r) & \exists r\in\mathscr{V},y\in b\circ r,\\ 0 & otherwise.\end{cases}$$

Lemma 1.[27] *If $(\mathscr{G},\mathscr{B})$ and $(\mathscr{H},\mathscr{C})$ are bfs sets of hvs $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$, then for all $e\in\mathscr{B}\cap\mathscr{C}$, $\acute{e}\in\mathscr{B}$, $y,z\in\mathscr{V}$, it follows that:*

1) $(\mathscr{G}+\mathscr{H})_e^+(y+z)\geq\mathscr{G}_e^+(y)\wedge\mathscr{H}_e^+(z)$, $(\mathscr{G}+\mathscr{H})_e^-(y+z)\leq\mathscr{G}_e^-(y)\vee\mathscr{H}_e^-(z)$.

2) $(\mathscr{G},\mathscr{B})\sqsubseteq 1\circ(\mathscr{G},\mathscr{B})$, $-(\mathscr{G},\mathscr{B})\sqsubseteq(-1)\circ(\mathscr{G},\mathscr{B})$, where $(-\mathscr{G})_{\acute{e}}^+(y)=\mathscr{G}_{\acute{e}}^+(-y)$, $(-\mathscr{G})_{\acute{e}}^-(y)=\mathscr{G}_{\acute{e}}^-(-y)$.

3 Equivalence Theorems

In this section we present some equivalent conditions to definition of a bfs-hvs (Definition 5). More precisely, these conditions are stated separately in theorems 1, 2, 3, 4, and are summarized in Corollary 1.

Theorem 1. *If $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$ is a hvs and $(\mathscr{G},\mathscr{B})$ is a bfs set of $\mathscr{V}$, then $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$ iff:*

1. *$\mathscr{G}+\mathscr{G}\sqsubseteq\mathscr{G}$,*
2. *$-\mathscr{G}\sqsubseteq\mathscr{G}$,*
3. *$b\circ\mathscr{G}\sqsubseteq\mathscr{G}$, $\forall b\in\mathscr{K}$.*

Proof. Let $(\mathscr{G},\mathscr{B})$ be a bfs-hvs, $x\in\mathscr{V}$ and $e\in\mathscr{A}$. Then
1) $(\mathscr{G}+\mathscr{G})_e^+(x)=\sup\limits_{x=y+z}\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(z)\leq\mathscr{G}_e^+(x)$ and $(\mathscr{G}+\mathscr{G})_e^-(x)=\inf\limits_{x=y+z}\mathscr{G}_e^-(y)\vee\mathscr{G}_e^-(z)\geq\mathscr{G}_e^-(x)$.
2) $(-\mathscr{G})_e^+(x)=\mathscr{G}_e^+(-x)\leq\mathscr{G}_e^+(-(-x))=\mathscr{G}_e^+(x)$ and $(-\mathscr{G})_e^-(x)=\mathscr{G}_e^-(-x)\geq\mathscr{G}_e^-(-(-x))=\mathscr{G}_e^-(x)$.
3) If there does not exist $r\in\mathscr{V}$ with $x\in b\circ r$, then $(b\circ\mathscr{G})_e^+(x)=0\leq\mathscr{G}_e^+(x)$ and $(b\circ\mathscr{G})_e^-(x)=0\geq\mathscr{G}_e^-(x)$. Also, if $r\in\mathscr{V}$ with $x\in b\circ r$, then $\mathscr{G}_e^+(r)\leq\inf\limits_{s\in b\circ r}\mathscr{G}_e^+(s)\leq\mathscr{G}_e^+(x)$ and $\mathscr{G}_e^-(r)\geq\sup\limits_{s\in b\circ r}\mathscr{G}_e^-(s)\geq\mathscr{G}_e^-(x)$. Thus $(b\circ\mathscr{G})_e^+(x)=\bigvee\limits_{x\in b\circ r}\mathscr{G}_e^+(t)\leq\mathscr{G}_e^+(x)$ and $(b\circ\mathscr{G})_e^-(x)=\bigwedge\limits_{x\in b\circ r}\mathscr{G}_e^-(r)\geq\mathscr{G}_e^-(x)$.

Conversely, if $e\in\mathscr{B}$, $y,z\in\mathscr{V}$ and $b\in\mathscr{K}$, then
1) By Lemma 1,

$$\begin{aligned}\mathscr{G}_e^+(y-z)&\geq(\mathscr{G}+\mathscr{G})_e^+(y-z)\\&\geq\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(-z)\\&\geq\mathscr{G}_e^+(y)\wedge(-\mathscr{G})_e^+(-z)\\&=\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(z),\end{aligned}$$

and

$$\begin{aligned}\mathscr{G}_e^-(y-z) &\le (\mathscr{G}+\mathscr{G})_e^-(y-z)\\ &\le \mathscr{G}_e^-(y)\vee\mathscr{G}_e^-(-z)\\ &\le \mathscr{G}_e^-(y)\vee(-\mathscr{G})_e^-(-z)\\ &= \mathscr{G}_e^-(y)\vee\mathscr{G}_e^-(z).\end{aligned}$$

2) For all $r\in b\circ y$, $\mathscr{G}_e^+(r)\ge(b\circ\mathscr{G})_e^+(r)=\sup_{r\in b\circ s}\mathscr{G}_e^+(s)\ge\mathscr{G}_e^+(y)$ and $\mathscr{G}_e^-(r)\le(b\circ\mathscr{G})_e^-(r)=\inf_{r\in b\circ s}\mathscr{G}_e^-(s)\le\mathscr{G}_e^-(y)$. Hence $\inf_{r\in b\circ y}\mathscr{G}_e^+(r)\ge\mathscr{G}_e^+(y)$ and $\inf_{r\in b\circ y}\mathscr{G}_e^-(r)\le\mathscr{G}_e^-(y)$.

Therefore, by Definition 5, $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$.

Lemma 2. *Suppose $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$ is an invertible hvs (i.e. $x\in b\circ y\Rightarrow y\in b^{-1}\circ x$, $\forall b\in\mathscr{K}\setminus\{0\}$). If $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$, then $(b\circ(\mathscr{G},\mathscr{B}))_e^+(y)\ge\mathscr{G}_e^+(y)$, $(b\circ(\mathscr{G},\mathscr{B}))_e^-(y)\le\mathscr{G}_e^+(y)$, for all $e\in\mathscr{B}$, $y\in\mathscr{V}$ and non-zero $b\in\mathscr{K}$.*

Proof. By Definition 4, $y\in 1\circ y=b\circ(b^{-1}\circ y)$, so $y\in b\circ r$ for some $r\in b^{-1}\circ y$. Then

$$(b\circ(\mathscr{G},\mathscr{B}))_e^+(y)=\sup_{y\in b\circ r}\mathscr{G}_e^+(r)=\sup_{r\in b^{-1}\circ y}\mathscr{G}_e^+(r)\ge\inf_{r\in b^{-1}\circ y}\mathscr{G}_e^+(r)\ge\mathscr{G}_e^+(y),$$

and

$$(b\circ(\mathscr{G},\mathscr{B}))_e^-(y)=\inf_{y\in b\circ r}\mathscr{G}_e^-(r)=\inf_{r\in b^{-1}\circ y}\mathscr{G}_e^-(r)\le\sup_{r\in b^{-1}\circ y}\mathscr{G}_e^-(r)\le\mathscr{G}_e^-(y).$$

Proposition 1. *Assume $\mathscr{V}$ is invertible and srd-hvs. If $(\mathscr{G},\mathscr{C})$ and $(\mathscr{H},\mathscr{C})$ are bfs-hvs of $\mathscr{V}$, then for all non-zero $b\in\mathscr{K}$,*

$$b\circ((\mathscr{G},\mathscr{C})+(\mathscr{H},\mathscr{C}))=b\circ(\mathscr{G},\mathscr{C})+b\circ(\mathscr{H},\mathscr{C}).$$

Proof. Let $e\in\mathscr{C}$, $x\in\mathscr{V}$, $\alpha_1=(b\circ((\mathscr{G},\mathscr{C})+(\mathscr{H},\mathscr{C})))_e^+(x)$, $\alpha_2=(b\circ((\mathscr{G},\mathscr{C})+(\mathscr{H},\mathscr{C})))_e^-(x)$, $\beta_1=(b\circ(\mathscr{G},\mathscr{C})+b\circ(\mathscr{G},\mathscr{C}))_e^+(x)$ and $\beta_2=(b\circ(\mathscr{G},\mathscr{C})+b\circ(\mathscr{H},\mathscr{C}))_e^-(x)$. More precisely,

$$\alpha_1=\begin{cases}\sup_{x\in b\circ t}(\mathscr{G}+\mathscr{H})_e^+(t) & \exists t\in\mathscr{V}; x\in b\circ t,\\ 0 & o.w.\end{cases}$$

$$\alpha_2=\begin{cases}\inf_{x\in b\circ t}(\mathscr{G}+\mathscr{H})_e^-(t) & \exists t\in\mathscr{V}; x\in b\circ t,\\ 0 & o.w.\end{cases}$$

$$\beta_1=\sup_{x=y+z}(b\circ(\mathscr{G},\mathscr{C}))_e^+(y)\wedge(b\circ(\mathscr{H},\mathscr{C}))_e^+(z),$$

$$\beta_2=\inf_{x=y+z}(b\circ(\mathscr{G},\mathscr{C}))_e^-(y)\vee(b\circ(\mathscr{H},\mathscr{C}))_e^-(z).$$

We must prove $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$.

If $\nexists t\in\mathscr{V}$, with $x\in b\circ t$, then $\alpha_1=\alpha_2=0$. In this case, for any $y,z\in\mathscr{V}$, with $x=y+z$, either there does not exist $r\in\mathscr{V}$, with $y\in b\circ r$, or there does not exist $s\in\mathscr{V}$, with $z\in b\circ s$ (because if $r,s\in\mathscr{V}$ such that $y\in b\circ r$ and $z\in b\circ s$, then $x=y+z\in b\circ r+b\circ s=b\circ(r+s)$, a contradiction). Thus $(b\circ(\mathscr{G},\mathscr{C}))_e^+(y)=(b\circ(\mathscr{G},\mathscr{C}))_e^-(y)=0$ or $(b\circ(\mathscr{H},\mathscr{C}))_e^+(z)=(b\circ(\mathscr{H},\mathscr{C}))_e^-(z)=0$. So $(b\circ(\mathscr{G},\mathscr{C}))_e^+(y)\wedge(b\circ(\mathscr{H},\mathscr{C}))_e^+(z)=0$ and $(b\circ(\mathscr{G},\mathscr{C}))_e^-(y)\vee(b\circ(\mathscr{H},\mathscr{C}))_e^-(z)=0$. Hence, $(b\circ(\mathscr{G},\mathscr{C})+b\circ(\mathscr{H},\mathscr{C}))_e^+(x)=0$ and $(b\circ(\mathscr{G},\mathscr{C})+b\circ(\mathscr{H},\mathscr{C}))_e^-(x)=0$. Therefore $\beta_1=0=\alpha_1$ and $\beta_2=0=\alpha_2$.

Also, if $t\in\mathscr{V}$, with $x\in b\circ t$, then for any $\varepsilon>0$, there exist $r,s\in\mathscr{V}$, $x\in b\circ r$, $x\in b\circ s$, with $(\mathscr{G}+\mathscr{H})_e^+(r)>(b\circ((\mathscr{G},\mathscr{C})+(\mathscr{H},\mathscr{C})))_e^+(x)-\varepsilon=\alpha_1-\varepsilon$, and $(\mathscr{G}+\mathscr{H})_e^-(s)<(b\circ((\mathscr{G},\mathscr{C})+(\mathscr{H},\mathscr{C})))_e^-(x)+\varepsilon=\alpha_2+\varepsilon$. Thus there are $y_1,y_2,z_1,z_2\in\mathscr{V}$, such that $y_1+z_1=r$, and $y_2+z_2=s$, such that $\mathscr{G}_e^+(y_1)\wedge\mathscr{H}_e^+(z_1)>\alpha_1-\varepsilon$ and $\mathscr{G}_e^-(y_2)\vee\mathscr{H}_e^-(z_2)<\alpha_2+\varepsilon$. Hence, $x\in b\circ r=b\circ(y_1+z_1)\subseteq b\circ y_1+b\circ z_1$, $x\in b\circ s=b\circ(y_2+z_2)\subseteq b\circ y_2+b\circ z_2$, and so $x=\acute{y}_1+\acute{z}_1$, for some $\acute{y}_1 \in b \circ y_1$, $\acute{z}_1 \in b \circ z_1$, and $x = \acute{y}_2 + \acute{z}_2$, for some $\acute{y}_2 \in b \circ y_2$, $\acute{z}_2 \in b \circ z_2$. By Definition 5 and Lemma 2,

$$\begin{aligned}
\beta_1 &= (b \circ (\mathscr{G}, \mathscr{C}) + b \circ (\mathscr{H}, \mathscr{C}))_e^+ (x) \\
&\geq (b \circ (\mathscr{G}, \mathscr{C}))_e^+ (\acute{y}_1) \wedge (b \circ (\mathscr{H}, \mathscr{C}))_e^+ (\acute{z}_1) \\
&\geq \mathscr{G}_e^+(\acute{y}_1) \wedge \mathscr{H}_e^+(\acute{z}_1) \\
&\geq \left(\bigwedge_{t_1 \in b \circ y_1} \mathscr{G}_e^+(t_1) \right) \wedge \left(\bigwedge_{l_1 \in a \circ z_1} \mathscr{H}_e^+(l_1) \right) \\
&\geq \mathscr{G}_e^+(y_1) \wedge \mathscr{H}_e^+(z_1) \\
&> \alpha_1 - \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\beta_2 &= (b \circ (\mathscr{G}, \mathscr{C}) + b \circ (\mathscr{H}, \mathscr{C}))_e^- (x) \\
&\leq (b \circ (\mathscr{G}, \mathscr{C}))_e^- (\acute{y}_2) \vee (b \circ (\mathscr{H}, \mathscr{C}))_e^- (\acute{z}_2) \\
&\leq \mathscr{G}_e^-(\acute{y}_2) \vee \mathscr{H}_e^-(\acute{z}_2) \\
&\leq \left(\bigvee_{t_2 \in b \circ y_2} \mathscr{G}_e^-(t_2) \right) \vee \left(\bigvee_{l_2 \in a \circ z_2} \mathscr{H}_e^-(l_2) \right) \\
&\leq \mathscr{G}_e^-(y_2) \vee \mathscr{G}_e^-(z_2) \\
&< \alpha_2 + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\beta_1 \geq \alpha_1$ and $\beta_2 \leq \alpha_2$.

On the other hand, for all $\varepsilon > 0$, there exist $y_1, y_2, z_1, z_2 \in \mathscr{V}$, such that $x = y_1 + z_1 = y_2 + z_2$, $(b \circ (\mathscr{G}, \mathscr{C}))_e^+ (y_1) \wedge (b \circ (\mathscr{H}, \mathscr{C}))_e^+ (z_1) > \beta_1 - \varepsilon$ and $(b \circ (\mathscr{G}, \mathscr{C}))_e^- (y_2) \vee (b \circ (\mathscr{H}, \mathscr{C}))_e^- (z_2) < \beta_2 + \varepsilon$. Taking $\beta_1 > \varepsilon$ and $\beta_2 < \varepsilon$ (if $\beta_1 = 0$, then $\alpha_1 = 0$ and if $\beta_2 = 0$, then $\alpha_2 = 0$, and there is nothing to prove), it follows that: $(b \circ (\mathscr{G}, \mathscr{C}))_e^+ (y_1) > 0$, $(b \circ (\mathscr{H}, \mathscr{C}))_e^+ (z_1) > 0$, $(b \circ (\mathscr{G}, \mathscr{C}))_e^- (y_2) < 0$ and $(b \circ (\mathscr{H}, \mathscr{C}))_e^- (z_2) < 0$. Thus there are $\acute{y}_1, \acute{z}_1, \acute{y}_2, \acute{z}_2 \in \mathscr{V}$, such that $y_1 \in b \circ \acute{y}_1$, $z_1 \in b \circ \acute{z}_1$, $y_2 \in b \circ \acute{y}_2$ and $z_2 \in b \circ \acute{z}_2$, with $\mathscr{G}_e^+(\acute{y}_1) \wedge \mathscr{H}_e^+(\acute{z}_1) > \beta_1 - \varepsilon$ and $\mathscr{G}_e^-(\acute{y}_2) \vee \mathscr{H}_e^-(\acute{z}_2) < \beta_2 + \varepsilon$. But $x = y_1 + z_1 \in b \circ \acute{y}_1 + b \circ \acute{z}_1 = b \circ (\acute{y}_1 + \acute{z}_1)$, $x = y_2 + z_2 \in b \circ \acute{y}_2 + b \circ \acute{z}_2 = b \circ (\acute{y}_2 + \acute{z}_2)$, and so by Lemma 1, $\alpha_1 = \sup_{x \in b \circ t} (\mathscr{G} + \mathscr{H})_e^+(t) \geq (\mathscr{G} + \mathscr{H})_e^+(\acute{y}_1 + \acute{z}_1) \geq \mathscr{G}_e^+(\acute{y}_1) \wedge \mathscr{H}_e^+(\acute{z}_1) > \beta_1 - \varepsilon$, and $\alpha_2 = \inf_{x \in b \circ t} (\mathscr{G} + \mathscr{H})_e^-(t) \leq (\mathscr{G} + \mathscr{H})_e^-(\acute{y}_2 + \acute{z}_2) \leq \mathscr{G}_e^-(\acute{y}_2) \vee \mathscr{H}_e^-(\acute{z}_2) < \beta_2 + \varepsilon$. Hence $\alpha_1 \geq \beta_1$ and $\alpha_2 \leq \beta_2$, since $\varepsilon > 0$ was arbitrary.

Therefore, the proof is completed.

Definition 7. *If $(\mathscr{G}, \mathscr{B})$ is a bfs set of hvs $\mathscr{V}$, $\alpha \in (0,1]$, $\beta \in [-1,0)$, then the soft set*

$$(\mathscr{G}, \mathscr{B})_{\alpha,\beta} = \{(\mathscr{G}_e)_{\alpha,\beta}\,;\, e \in \mathscr{B}\},$$

is called (α, β)-level soft subset of $\mathscr{V}$, where

$$(\mathscr{G}_e)_{\alpha,\beta} = \{v \in \mathscr{V};\, \mathscr{G}_e^+(v) \geq \alpha,\, \mathscr{G}_e^-(v) \leq \beta\},$$

is an (α, β)-level subset of the bfs set $\mathscr{G}_e = (\mathscr{G}_e^+, \mathscr{G}_e^-)$.

Example 5. Let $(\mathscr{G}, \mathscr{B})$ be the bfs set of the hvs $\mathscr{V} = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 3. Then

$$(\mathscr{G}_a)_{0.4,-0.3} = \{x \in \mathbb{R}^3;\, \mathscr{G}_a^+(x) \geq 0.4,\, \mathscr{G}_a^-(x) \leq -0.3\} = X \cap Y = X,$$

$$(\mathscr{G}_b)_{0.4,-0.3} = \{x \in \mathbb{R}^3;\, \mathscr{G}_b^+(x) \geq 0.4,\, \mathscr{G}_b^-(x) \leq -0.3\} = Y \cap Y = Y.$$

Thus

$$\begin{aligned}
(\mathscr{G}, \mathscr{B})_{0.4,-0.3} &= \{(\mathscr{G}_a)_{0.4,-0.3}, (\mathscr{G}_b)_{0.4,-0.3}\} \\
&= \{\{0\} \times \{0\} \times \mathbb{R}, \mathbb{R} \times \{0\} \times \mathbb{R}\}.
\end{aligned}$$

Similarly,

$$(\mathscr{G}, \mathscr{B})_{0.8,-0.5} = \{\emptyset, \{0\} \times \{0\} \times \mathbb{R}\}.$$

*Example 6.*Let $\mathscr{V} = (\mathbb{R}^3, +, \circ, \mathbb{R})$ be the hvs defined in Example 1. Define a bfs set $(\mathscr{G}, \mathscr{B})$ of $\mathscr{V}$, where $\mathscr{B} = \{a, b\}$, "$\mathscr{G}_a^+, \mathscr{G}_b^+ : \mathbb{R}^3 \to [0,1]$" and "$\mathscr{G}_a^-, \mathscr{G}_b^- : \mathbb{R}^3 \to [-1,0]$" are given by the followings:

$$\mathscr{G}_a^+(x,y,z) = \begin{cases} 0.3 \; x \geq 0, yz \geq 0, \\ 0.7 \; x \geq 0, yz < 0, \\ 0.5 \; x < 0, \end{cases}$$

$$\mathscr{G}_a^-(x,y,z) = \begin{cases} -0.1 \; y \geq 0, xz \geq 0, \\ -0.4 \; y \geq 0, xz < 0, \\ -0.6 \; y < 0, \end{cases}$$

$$\mathscr{G}_b^+(x,y,z) = \begin{cases} 0.6 \; x \geq 0, y \geq 0, z \geq 0, \\ 0.7 \; x \geq 0, y \geq 0, z < 0, \\ 0.3 \; otherwise, \end{cases}$$

$$\mathscr{G}_b^-(x,y,z) = \begin{cases} -0.7 \; z \geq 0, \\ -0.3 \; z < 0. \end{cases}$$

Then

$$(\mathscr{G}, \mathscr{B})_{0.6,-0.5} = \left\{ (G_a)_{0.6,-0.5}, (G_b)_{0.6,-0.5} \right\},$$

where,

$$(\mathscr{G}_a)_{0.6,-0.5} = \{(x,y,z) \in \mathbb{R}^3;\ x \geq 0, y < 0, z > 0\},$$

and

$$(\mathscr{G}_b)_{0.6,-0.5} = \{(x,y,z) \in \mathbb{R}^3;\ x \geq 0, y \geq 0, z \geq 0\}.$$

*Example 7.*Let $(\mathscr{G}, \mathscr{C})$ be the bfs set of the hvs $\mathscr{V} = (\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ in Example 4. Then

$$\begin{aligned} (\mathscr{G}_c)_{0.5,-0.5} &= \{x \in \mathbb{Z}_4;\ \mathscr{G}_c^+(x) \geq 0.5,\ \mathscr{G}_c^-(x) \leq -0.5\} \\ &= \{0,2\} \cap \emptyset \\ &= \emptyset, \end{aligned}$$

$$\begin{aligned} (\mathscr{G}_d)_{0.5,-0.5} &= \{x \in \mathbb{Z}_4;\ \mathscr{G}_d^+(x) \geq 0.5,\ \mathscr{G}_d^-(x) \leq -0.5\} \\ &= \{0,2\} \cap \{0,2\} \\ &= \{0,2\}, \end{aligned}$$

and

$$\begin{aligned} (\mathscr{G}_e)_{0.5,-0.5} &= \{x \in \mathbb{Z}_4;\ \mathscr{G}_e^+(x) \geq 0.5,\ \mathscr{G}_e^-(x) \leq -0.5\} \\ &= \{0,2\} \cap \{0,1,2,3\} \\ &= \{0,2\}. \end{aligned}$$

Thus

$$\begin{aligned} (\mathscr{G}, \mathscr{C})_{0.5,-0.5} &= \{(\mathscr{G}_c)_{0.5,-0.5}, (\mathscr{G}_d)_{0.5,-0.5}, (\mathscr{G}_e)_{0.5,-0.5}\} \\ &= \{\emptyset, \{0,2\}, \{0,2\}\}. \end{aligned}$$

*Example 8.*Let $\mathscr{V} = (\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ be the hvs defined in Example 2. Consider a bfs set $(\mathscr{G}, \mathscr{A})$ of $\mathscr{V}$, where $\mathscr{A} = \{c, d, e\}$ and "$\mathscr{G}_c^+, \mathscr{G}_d^+, \mathscr{G}_e^+ : \mathbb{Z}_4 \to [0,1]$" and "$\mathscr{G}_c^-, \mathscr{G}_d^-, \mathscr{G}_e^- : \mathbb{Z}_4 \to [-1,0]$" are given by the followings:

x	0	1	2	3
$\mathscr{G}_c^+(x)$	0.4	0.3	0.2	0.7

x	0	1	2	3
$\mathscr{G}_c^-(x)$	-0.1	-0.3	-0.6	-0.6

x	0	1	2	3
$\mathscr{G}_d^+(x)$	1	0.5	0.4	0.1

x	0	1	2	3
$\mathscr{G}_d^-(x)$	-0.5	-0.7	-0.2	-0.3

x	0	1	2	3
$\mathscr{G}_e^+(x)$	0.4	0	0.2	0.8

x	0	1	2	3
$\mathscr{G}_e^-(x)$	0	-0.1	-0.7	-0.5

Then

$$\begin{aligned}(\mathscr{G}_c)_{0.3,-0.4} &= \{x \in \mathbb{Z}_4;\ \mathscr{G}_c^+(x) \geq 0.3, \mathscr{G}_c^-(x) \leq -0.4\}\\ &= \{0,1,3\} \cap \{1,2,3\}\\ &= \{1,3\},\end{aligned}$$

$$\begin{aligned}(\mathscr{G}_d)_{0.3,-0.4} &= \{x \in \mathbb{Z}_4;\ \mathscr{G}_d^+(x) \geq 0.3, \mathscr{G}_d^-(x) \leq -0.4\}\\ &= \{0,1,2\} \cap \{0,1\}\\ &= \{0,1\},\end{aligned}$$

and

$$\begin{aligned}(\mathscr{G}_e)_{0.3,-0.4} &= \{x \in \mathbb{Z}_4;\ \mathscr{G}_e^+(x) \geq 0.3, \mathscr{G}_e^-(x) \leq -0.4\}\\ &= \{0,3\} \cap \{2,3\}\\ &= \{3\}.\end{aligned}$$

Hence,

$$\begin{aligned}(\mathscr{G},\mathscr{A})_{0.3,-0.4} &= \left\{(\mathscr{G}_c)_{0.3,-0.4}, (\mathscr{G}_d)_{0.3,-0.4}, (\mathscr{G}_e)_{0.3,-0.4}\right\}\\ &= \{\{1,3\},\{0,1\},\{3\}\}.\end{aligned}$$

Theorem 2. *Consider $\mathscr{V} = (\mathscr{V},+,\circ,\mathscr{K})$ as a hypervector space and $(\mathscr{G},\mathscr{B})$ as a bfs set of $\mathscr{V}$. Then $(\mathscr{G},\mathscr{B})$ is classified as a bfs-hvs of $\mathscr{V}$ iff for all values of α within the range of $(0,1]$ and β within the range of $[-1,0)$, the (α,β)-level soft subset $(\mathscr{G},\mathscr{B})_{\alpha,\beta}$ qualifies as a soft hvs of $\mathscr{V}$, i.e. for all $\alpha \in (0,1]$, $\beta \in [-1,0)$ and $e \in \mathscr{B}$, $(\mathscr{G}_e)_{\alpha,\beta}$ is a shs of $\mathscr{V}$.*

Proof. Suppose $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$, $\alpha \in (0,1]$, $\beta \in [-1,0)$, $e \in \mathscr{B}$, $x,y \in (\mathscr{G}_e)_{\alpha,\beta}$ and $b \in \mathscr{K}$. Then $\mathscr{G}_e^+(x-y) \geq \mathscr{G}_e^+(x) \wedge \mathscr{G}_e^+(y) \geq \alpha$, $\mathscr{G}_e^-(x-y) \leq \mathscr{G}_e^-(x) \vee \mathscr{G}_e^-(y) \leq \beta$, and so $x-y \in (\mathscr{G}_e)_{\alpha,\beta}$. Also, for all $z \in b \circ x$,

$$\mathscr{G}_e^+(z) \geq \inf_{t \in b \circ x} \mathscr{G}_e^+(t) \geq \mathscr{G}_e^+(x) \geq \alpha,$$

and

$$\mathscr{G}_e^-(z) \leq \sup_{t \in b \circ x} \mathscr{G}_e^-(t) \leq \mathscr{G}_e^-(x) \leq \beta.$$

Thus $z \in (\mathscr{G}_e)_{\alpha,\beta}$. Hence $b \circ x \subseteq (\mathscr{G}_e)_{\alpha,\beta}$. Therefore, $(\mathscr{G}_e)_{\alpha,\beta}$ is a shs of $\mathscr{V}$.
Conversely, suppose $(\mathscr{G}_e)_{\alpha,\beta}$ is a shs of $\mathscr{V}$, for all $\alpha \in (0,1]$, $\beta \in [-1,0)$ and $e \in \mathscr{A}$. Let $x,y \in \mathscr{V}$ and $b \in \mathscr{K}$. Choose $e \in \mathscr{B}$ and set $\alpha = \mathscr{G}_e^+(x) \wedge \mathscr{G}_e^+(y)$ and $\beta = \mathscr{G}_e^-(x) \vee \mathscr{G}_e^-(y)$. Then $\mathscr{G}_e^+(x), \mathscr{G}_e^+(y) \geq \alpha$, $\mathscr{G}_e^-(x), \mathscr{G}_e^-(y) \leq \beta$, and so $x,y \in (\mathscr{G}_e)_{\alpha,\beta}$. Thus $x-y \in (\mathscr{G}_e)_{\alpha,\beta}$. Hence $\mathscr{G}_e^+(x-y) \geq \alpha = \mathscr{G}_e^+(x) \wedge \mathscr{G}_e^+(y)$ and $\mathscr{G}_e^-(x-y) \leq \beta = \mathscr{G}_e^-(x) \vee \mathscr{G}_e^-(y)$.
Now choose $e \in \mathscr{B}$ and put $\alpha = \mathscr{G}_e^+(x)$ and $\beta = \mathscr{G}_e^-(x)$. Then $x \in (\mathscr{G}_e)_{\alpha,\beta}$ and so $b \circ x \subseteq (\mathscr{G}_e)_{\alpha,\beta}$. Thus

$$\inf_{t \in b \circ x} \mathscr{G}_e^+(t) \geq \inf_{s \in (\mathscr{G}_e)_{\alpha,\beta}} \mathscr{G}_e^+(s) \geq \alpha,$$

and

$$\sup_{t \in b \circ x} \mathscr{G}_e^-(t) \leq \sup_{s \in (\mathscr{G}_e)_{\alpha,\beta}} \mathscr{G}_e^-(s) \leq \beta.$$

Hence, $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$.

Theorem 3. *Suppose $(\mathscr{G},\mathscr{B})$ is a bfs set of sld-hvs $\mathscr{V}$. Then $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$, iff*

$$\inf_{t \in a \circ x + b \circ y} \mathscr{G}_e^+(t) \geq \mathscr{G}_e^+(x) \wedge \mathscr{G}_e^+(y),$$

and

$$\sup_{t \in a \circ x + b \circ y} \mathscr{G}_e^-(t) \leq \mathscr{G}_e^-(x) \vee \mathscr{G}_e^-(y),$$

for all $e \in \mathscr{B}$, $x,y \in \mathscr{V}$, $a,b \in \mathscr{K}$.

*Proof.*If $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$, $e\in\mathscr{B}$, $x,y\in\mathscr{V}$ and $a,b\in\mathscr{K}$, then

$$\begin{aligned}\inf_{t\in a\circ x+b\circ y}\mathscr{G}_e^+(t) &= \inf_{t=t_1+t_2,t_1\in a\circ x,t_2\in b\circ y}\mathscr{G}_e^+(t)\\ &= \inf_{t_1\in a\circ x,t_2\in b\circ y}\mathscr{G}_e^+(t_1+t_2)\\ &\geq \left(\inf_{t_1\in a\circ x}\mathscr{G}_e^+(t_1)\right)\wedge\left(\inf_{t_2\in b\circ y}\mathscr{G}_e^+(t_2)\right)\\ &\geq \mathscr{G}_e^+(x)\wedge\mathscr{G}_e^+(y),\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in a\circ x+b\circ y}\mathscr{G}_e^-(t) &= \sup_{t=t_1+t_2,t_1\in a\circ x,t_2\in b\circ y}\mathscr{G}_e^-(t)\\ &= \sup_{t_1\in a\circ x,t_2\in b\circ y}\mathscr{G}_e^-(t_1+t_2)\\ &\leq \left(\sup_{t_1\in a\circ x}\mathscr{G}_e^-(t_1)\right)\vee\left(\sup_{t_2\in b\circ y}\mathscr{G}_e^-(t_2)\right)\\ &\leq \mathscr{G}_e^-(x)\vee\mathscr{G}_e^-(y).\end{aligned}$$

Conversely, if $e\in\mathscr{B}$, $x,y\in\mathscr{V}$ and $a\in\mathscr{K}$, then by Definition 4,

$$\mathscr{G}_e^+(x-y)\geq\inf_{t\in 1\circ x-1\circ y}\mathscr{G}_e^+(t)\geq\mathscr{G}_e^+(x)\wedge\mathscr{G}_e^+(y),$$

and

$$G_e^-(x-y)\leq\sup_{t\in 1\circ x-1\circ y}\mathscr{G}_e^-(t)\leq\mathscr{G}_e^-(x)\vee\mathscr{G}_e^-(y).$$

Also, $0\in 0\circ x$, for all $x\in\mathscr{V}$, since $\mathscr{V}$ is sld. Thus

$$\inf_{t\in a\circ x}\mathscr{G}_e^+(t)\geq\inf_{t\in 0\circ x+a\circ x}\mathscr{G}_e^+(t)\geq\mathscr{G}_e^+(x)\wedge\mathscr{G}_e^+(x)=\mathscr{G}_e^+(x),$$

and

$$\sup_{t\in a\circ x}\mathscr{G}_e^-(t)\leq\sup_{t\in 0\circ x+a\circ x}\mathscr{G}_e^-(t)\leq\mathscr{G}_e^-(x)\vee\mathscr{G}_e^-(x)=\mathscr{G}_e^-(x).$$

Therefore, by Definition 5, $(\mathscr{G},\mathscr{B})$ is a bfs-hvs.

Theorem 4.*Suppose $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$ is a sld-hvs and $(\mathscr{G},\mathscr{B})$ is a bfs set of $\mathscr{V}$. Then $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$ iff*

$$\forall b,c\in\mathscr{K};\ b\circ(\mathscr{G},\mathscr{B})+c\circ(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B}).$$

*Proof.*Let $(\mathscr{G},\mathscr{B})$ be a bfs-hvs of $\mathscr{V}$, $b,c\in\mathscr{K}$, $e\in\mathscr{B}$ and $x\in\mathscr{V}$. Then by Theorem 1, it follows that:

$$\begin{aligned}(b\circ\mathscr{G}+c\circ\mathscr{G})_e^+(x) &= \sup_{x=y+z}((b\circ\mathscr{G})_e^+(y)\wedge(c\circ\mathscr{G})_e^+(z))\\ &\leq \sup_{x=y+z}(\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(z))\\ &= (\mathscr{G}+\mathscr{G})_e^+(x)\\ &\leq \mathscr{G}_e^+(x),\end{aligned}$$

$$\begin{aligned}(b\circ\mathscr{G}+c\circ\mathscr{G})_e^-(x) &= \inf_{x=y+z}((b\circ\mathscr{G})_e^-(y)\vee(c\circ\mathscr{G})_e^-(z))\\ &\geq \inf_{x=y+z}(\mathscr{G}_e^-(y)\vee\mathscr{G}_e^-(z))\\ &= (\mathscr{G}+\mathscr{G})_e^-(x)\\ &\geq \mathscr{G}_e^-(x).\end{aligned}$$

Thus $b\circ(\mathscr{G},\mathscr{B})+c\circ(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$.

Conversely, set $b=c=1$. Then by Lemma 1,

$$(\mathscr{G},\mathscr{B})+(\mathscr{G},\mathscr{B}) \sqsubseteq 1\circ(\mathscr{G},\mathscr{B})+1\circ(\mathscr{G},\mathscr{B}) \sqsubseteq (\mathscr{G},\mathscr{B}).$$

Now, put $b=1$ and $c=-1$. Then by Lemma 1, for all $e\in\mathscr{B}$ and $x\in\mathscr{V}$, it follows that:

$$\begin{aligned}
\mathscr{G}_e^+(0) &\geq (1\circ(\mathscr{G},\mathscr{B})+(-1)\circ(\mathscr{G},\mathscr{B}))_e^+(0)\\
&= \sup_{0=y+z}\left((1\circ(\mathscr{G},\mathscr{B}))_e^+(y)\wedge((-1)\circ(\mathscr{G},\mathscr{B}))_e^+(z)\right)\\
&\geq (1\circ(\mathscr{G},\mathscr{B}))_e^+(x)\wedge((-1)\circ(\mathscr{G},\mathscr{B}))_e^+(-x)\\
&\geq \mathscr{G}_e^+(x)\wedge(-\mathscr{G})_e^+(-x)\\
&= \mathscr{G}_e^+(x)\wedge\mathscr{G}_e^+(x)\\
&= \mathscr{G}_e^+(x),
\end{aligned}$$

$$\begin{aligned}
\mathscr{G}_e^-(0) &\leq (1\circ(\mathscr{G},\mathscr{B})+(-1)\circ(\mathscr{G},\mathscr{B}))_e^-(0)\\
&= \inf_{0=y+z}\left((1\circ(\mathscr{G},\mathscr{B}))_e^-(y)\vee((-1)\circ(\mathscr{G},\mathscr{B}))_e^-(z)\right)\\
&\leq (1\circ(\mathscr{G},\mathscr{B}))_e^-(x)\vee((-1)\circ(\mathscr{G},\mathscr{B}))_e^-(-x)\\
&\leq \mathscr{G}_e^-(x)\vee(-\mathscr{G})_e^-(-x)\\
&= \mathscr{G}_e^-(x)\vee\mathscr{G}_e^-(x)\\
&= \mathscr{G}_e^-(x).
\end{aligned}$$

So

$$(0\circ\mathscr{G})_e^+(0) = \sup_{0\in 0\circ t}\mathscr{G}_e^+(t) = \mathscr{G}_e^+(0) = \sup_{s\in\mathscr{V}}\mathscr{G}_e^+(s),$$

$$(0\circ\mathscr{G})_e^-(0) = \sup_{0\in 0\circ t}\mathscr{G}_e^-(t) = \mathscr{G}_e^-(0) = \inf_{s\in\mathscr{V}}\mathscr{G}_e^-(s).$$

Now, suppose $b=0$ and $c=-1$. Then $0\circ(\mathscr{G},\mathscr{B})+(-1)\circ(\mathscr{G},\mathscr{B}) \sqsubseteq (\mathscr{G},\mathscr{B})$, and so for all $e\in\mathscr{B}$, $x\in\mathscr{V}$, by Lemma 1, we have:

$$\begin{aligned}
\mathscr{G}_e^+(x) &\geq (0\circ\mathscr{G}+(-1)\circ\mathscr{G})_e^+(x)\\
&= \sup_{x=y+z}\left((0\circ\mathscr{G})_e^+(y)\wedge((-1)\circ\mathscr{G})_e^+(z)\right)\\
&\geq (0\circ\mathscr{G})_e^+(0)\wedge((-1)\circ\mathscr{G})_e^+(x)\\
&= \mathscr{G}_e^+(0)\wedge((-1)\circ\mathscr{G})_e^+(x)\\
&= ((-1)\circ\mathscr{G})_e^+(x) \quad \left(\sup_{t\in\mathscr{V}}((-1)\circ\mathscr{G})_e^+(t) = \sup_{t\in\mathscr{V}}\mathscr{G}_e^+(t) = \mathscr{G}_e^+(0)\right)\\
&\geq (-\mathscr{G})_e^+(x),
\end{aligned}$$

$$\begin{aligned}
\mathscr{G}_e^-(x) &\leq (0\circ\mathscr{G}+(-1)\circ\mathscr{G})_e^-(x)\\
&= \inf_{x=y+z}\left((0\circ\mathscr{G})_e^-(y)\vee((-1)\circ\mathscr{G})_e^-(z)\right)\\
&\leq (0\circ\mathscr{G})_e^-(0)\vee((-1)\circ\mathscr{G})_e^-(x)\\
&= G_e^-(0)\vee((-1)\circ\mathscr{G})_e^-(x)\\
&= ((-1)\circ\mathscr{G})_e^-(x) \quad \left(\inf_{t\in\mathscr{V}}((-1)\circ\mathscr{G})_e^-(t) = \inf_{t\in\mathscr{V}}\mathscr{G}_e^-(t) = \mathscr{G}_e^-(0)\right)\\
&\leq (-\mathscr{G})_e^-(x).
\end{aligned}$$

Hence, $-(\mathscr{G},\mathscr{B}) \sqsubseteq (\mathscr{G},\mathscr{B})$.

Now, by taking $c=0$, $b\circ(\mathscr{G},\mathscr{B})+0\circ(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$. But for all $e\in\mathscr{B}$ and $x\in\mathscr{V}$,

$$\begin{aligned}((b\circ\mathscr{G})_e^+ + (0\circ\mathscr{G})_e^+)(x) &= \sup_{x=y+z}((b\circ\mathscr{G})_e^+(y)\wedge(0\circ\mathscr{G})_e^+(z))\\ &\geq (b\circ\mathscr{G})_e^+(x)\wedge(0\circ\mathscr{G})_e^+(0)\\ &= (b\circ\mathscr{G})_e^+(x)\wedge\mathscr{G}_e^+(0)\\ &= (b\circ\mathscr{G})_e^+(x), \quad \left(\sup_{t\in\mathscr{V}}(b\circ\mathscr{G})_e^+(t)=\sup_{t\in\mathscr{V}}\mathscr{G}_e^+(t)=\mathscr{G}_e^+(0)\right)\end{aligned}$$

$$\begin{aligned}((b\circ\mathscr{G})_e^- + (0\circ\mathscr{G})_e^-)(x) &= \inf_{x=y+z}((b\circ\mathscr{G})_e^-(y)\vee(0\circ\mathscr{G})_e^-(z))\\ &\leq (b\circ\mathscr{G})_e^-(x)\vee(0\circ\mathscr{G})_e^-(0)\\ &= (b\circ\mathscr{G})_e^-(x)\vee\mathscr{G}_e^-(0)\\ &= (b\circ\mathscr{G})_e^-(x). \quad \left(\inf_{t\in\mathscr{V}}(b\circ\mathscr{G})_e^-(t)=\inf_{t\in\mathscr{V}}\mathscr{G}_e^-(t)=\mathscr{G}_e^-(0)\right)\end{aligned}$$

Thus $b\circ(\mathscr{G},\mathscr{B})\sqsubseteq b\circ(\mathscr{G},\mathscr{B})+0\circ(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$, for all $b\in\mathscr{K}$.
Therefore, by Theorem 1, $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$.

Corollary 1. *The following terms, about bfs set $(\mathscr{G},\mathscr{B})$ of $\mathscr{V}$ are equivalent:*

1. $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$;
2. $(\mathscr{G},\mathscr{B})+(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$, $-(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$ and $b\circ(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$, $\forall b\in\mathscr{K}$.
3. For all $e\in\mathscr{B}$, $\alpha\in(0,1]$ and $\beta\in[-1,0)$, $(\mathscr{G}_e)_{\alpha,\beta}$ is a shs of $\mathscr{V}$;

Moreover, if $\mathscr{V}$ is sld, the above terms are equivalent to the followings:

(4) $\inf_{r\in b\circ y+c\circ z}\mathscr{G}_e^+(r)\geq\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(z)$, $\sup_{r\in b\circ y+c\circ z}\mathscr{G}_e^-(r)\leq\mathscr{G}_e^-(y)\vee\mathscr{G}_e^-(z)$, $\forall e\in\mathscr{B}$, $y,z\in\mathscr{V}$, $b,c\in\mathscr{K}$.
(5) $b\circ(\mathscr{G},\mathscr{B})+c\circ(\mathscr{G},\mathscr{B})\sqsubseteq(\mathscr{G},\mathscr{B})$, for all $b,c\in\mathscr{K}$.

Proof. (1)$\Longleftrightarrow$(2): Theorem 1;
(1)$\Longleftrightarrow$(3): Theorem 2;
(1)$\Longleftrightarrow$(4): Theorem 3;
(1)$\Longleftrightarrow$(5): Theorem 4.

4 New Constructed Bipolar Fuzzy Soft Hypervector Spaces

In this section, the paper introduces different methods for constructing new bipolar fuzzy soft hypervector spaces. Specifically:

1. From every subhyperspace of $\mathscr{V}$, Proposition 2 details the construction of two new bipolar fuzzy soft hypervector spaces.

2. Using the notion of (α,β)-level subsets (defined in Definition 7), Theorem 5 presents a method for obtaining a new bipolar fuzzy soft hypervector space from every bfs-hvs $(\mathscr{F},\mathscr{A})$ of $\mathscr{V}$ and for every $e\in\mathscr{B}$, $\alpha\in(0,1]$, $\beta\in[-1,0)$.

3. The paper also explores a method for constructing the bfs-hvs generated by a bfs set $(\mathscr{F},\mathscr{A})$ of $\mathscr{V}$ in Theorem 6.

These constructions provide valuable insights into the generation of new bipolar fuzzy soft hypervector spaces from existing structures.

Proposition 2. *The following terms about a subset $\mathscr{X}$ of hvs $\mathscr{V}$ are equivalent:*

1. $\mathscr{X}$ is a shs of $\mathscr{V}$.
2. $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$, where $\mathscr{G}_e^+$, $\mathscr{G}_e^-$ are $\chi_{\mathscr{X}}$, the characteristic function of $\mathscr{X}$ and the zero function, respectively, for every $e\in\mathscr{B}$.
3. $(\mathscr{H},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$, where

$$\forall e\in\mathscr{B},\ \mathscr{H}_e^+(x)=0,\ \mathscr{H}_e^-(x)=\begin{cases}-1 & x\in\mathscr{X},\\ 0 & x\notin\mathscr{X}.\end{cases}$$

Proof.(1)$\Longleftrightarrow$(2): Let $\mathscr{X}$ be a shs of $\mathscr{V}$. Then for all $e \in \mathscr{B}$, $\alpha \in (0,1]$, $\beta \in [-1,0)$, $(\mathscr{G}_e)_{\alpha,\beta}$ is a shs of $\mathscr{V}$. Thus by Theorem 2, $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$. Conversely, $(\mathscr{G}_e)_{1,\beta} = \mathscr{X}$ is a shs of $\mathscr{V}$, for arbitrary $e \in \mathscr{B}$ and $\beta \in [-1,0)$.
(1)$\Longleftrightarrow$(3): It is similar to (1)$\Leftrightarrow$(2).

Theorem 5.*If $(\mathscr{F},\mathscr{A})$ is a bfs-hvs of $\mathscr{V}$, then for any $\acute{e} \in \mathscr{A}$, $\alpha \in (0,1]$ and $\beta \in [-1,0)$, the bfs set $(\mathscr{G},\mathscr{A})$ of $\mathscr{V}$ given by*

$$\mathscr{G}_e^+(x) = \begin{cases} \mathscr{F}_e^+(x) & x \notin (\mathscr{F}_{\acute{e}})_{\alpha,\beta} \\ 1 & x \in (\mathscr{F}_{\acute{e}})_{\alpha,\beta} \end{cases} \qquad \text{and} \qquad \mathscr{G}_e^-(x) = \begin{cases} \mathscr{F}_e^-(x) & x \notin (\mathscr{F}_{\acute{e}})_{\alpha,\beta} \\ -1 & x \in (\mathscr{F}_{\acute{e}})_{\alpha,\beta} \end{cases}$$

is a bfs-hvs of $\mathscr{V}$.

*Proof.*Let's take a look at the requirements outlined in Definition 5 and evaluate if they are met. Suppose $\mathscr{W} = (\mathscr{F}_{\acute{e}})_{\alpha,\beta}$, $e \in \mathscr{A}$, $y,z \in \mathscr{V}$ and $a \in \mathscr{K}$.
1) If $y,z \in \mathscr{W}$, then $y+z \in \mathscr{W}$ and so $\mathscr{G}_e^+(y+z) = 1 = \mathscr{G}_e^+(y) \wedge \mathscr{G}_e^+(z)$ and $\mathscr{G}_e^-(y+z) = -1 = \mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z)$.
If $y \notin \mathscr{W}$, $z \notin \mathscr{W}$ and $y+z \in \mathscr{W}$, then $\mathscr{G}_e^+(y+z) = 1 \geq \mathscr{G}_e^+(y) \wedge \mathscr{G}_e^+(z)$ and $\mathscr{G}_e^-(y+z) = -1 \leq \mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z)$.
If $y \notin \mathscr{W}$, $z \notin \mathscr{W}$ and $y+z \notin \mathscr{W}$, then

$$\mathscr{G}_e^+(y+z) = \mathscr{F}_e^+(y+z) \geq \mathscr{F}_e^+(y) \wedge \mathscr{F}_e^+(z) = \mathscr{G}_e^+(y) \wedge \mathscr{G}_e^+(z),$$

and

$$G_e^-(y+z) = \mathscr{F}_e^-(y+z) \leq \mathscr{F}_e^-(y) \vee \mathscr{F}_e^-(z) = \mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z).$$

If $y \in \mathscr{W}$ and $z \notin \mathscr{W}$, then $y+z \notin \mathscr{W}$ (if $y+z = w \in \mathscr{W}$, then $z = w-y \in \mathscr{W}$, a contradiction), implies that

$$\begin{aligned} \mathscr{G}_e^+(y+z) &= \mathscr{F}_e^+(y+z) \\ &\geq \mathscr{F}_e^+(y) \wedge \mathscr{F}_e^+(z) \\ &\geq \mathscr{F}_e^+(z) \\ &= \mathscr{G}_e^+(z) \\ &= 1 \wedge \mathscr{G}_e^+(z) \\ &= \mathscr{G}_e^+(y) \wedge \mathscr{G}_e^+(z), \end{aligned}$$

and

$$\begin{aligned} \mathscr{G}_e^-(y+z) &= \mathscr{F}_e^-(y+z) \\ &\leq \mathscr{F}_e^-(y) \vee \mathscr{F}_e^-(z) \\ &\leq \mathscr{F}_e^-(z) \\ &= \mathscr{G}_e^-(z) \\ &= -1 \wedge \mathscr{G}_e^-(z) \\ &= \mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z). \end{aligned}$$

If $y \notin \mathscr{W}$ and $z \in \mathscr{W}$, the result is similarly obtained.
Thus in any case, $\mathscr{G}_e^+(y+z) \geq \mathscr{G}_e^+(y) \wedge \mathscr{G}_e^+(z)$, and $\mathscr{G}_e^-(y+z) \leq \mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z)$. Moreover, by Theorem 2, $\mathscr{W}$ is a shs of $\mathscr{V}$ and so

$$\mathscr{G}_e^+(-y) = \begin{cases} \mathscr{F}_e^+(-y) & -y \notin \mathscr{W} \\ 1 & o.w. \end{cases} = \begin{cases} \mathscr{F}_e^+(y) & y \notin \mathscr{W} \\ 1 & o.w. \end{cases} = \mathscr{G}_e^+(y),$$

and

$$\mathscr{G}_e^-(-y) = \begin{cases} \mathscr{F}_e^-(-y) & -y \notin \mathscr{W} \\ -1 & o.w. \end{cases} = \begin{cases} \mathscr{F}_e^-(y) & y \notin \mathscr{W} \\ -1 & o.w. \end{cases} = \mathscr{G}_e^-(y).$$

2) If $a \circ y \subseteq \mathscr{W}$, then $\inf_{t \in a \circ y} \mathscr{G}_e^+(t) = 1 \geq \mathscr{G}_e^+(y)$ and $\sup_{t \in a \circ y} \mathscr{G}_e^-(t) = -1 \leq \mathscr{G}_e^-(y)$. If $a \circ y \nsubseteq \mathscr{W}$, then $y \notin \mathscr{W}$ (if $y \in \mathscr{W}$, then $a \circ y \subseteq \mathscr{W}$) and so

$$\begin{aligned} \inf_{t \in a \circ y} \mathscr{G}_e^+(t) &= \left(\inf_{t \in a \circ y \cap \mathscr{W}} \mathscr{G}_e^+(t) \right) \wedge \left(\inf_{t \in a \circ y \setminus \mathscr{W}} \mathscr{G}_e^+(t) \right) \\ &= \inf_{t \in a \circ y \setminus \mathscr{W}} \mathscr{G}_e^+(t) \\ &= \inf_{t \in a \circ y \setminus \mathscr{W}} \mathscr{F}_e^+(t) \\ &\geq \mathscr{F}_e^+(y) \\ &= \mathscr{G}_e^+(y), \end{aligned}$$

and

$$
\begin{aligned}
\sup_{t\in a\circ y}\mathscr{G}_e^-(t) &= \left(\sup_{t\in a\circ y\cap\mathscr{W}}\mathscr{G}_e^-(t)\right)\vee\left(\sup_{t\in a\circ y\setminus\mathscr{W}}\mathscr{G}_e^-(t)\right)\\
&= \sup_{t\in a\circ y\setminus\mathscr{W}}\mathscr{G}_e^-(t)\\
&= \sup_{t\in a\circ y\setminus\mathscr{W}}\mathscr{F}_e^-(t)\\
&\le \mathscr{F}_e^-(y)\\
&= \mathscr{G}_e^-(y).
\end{aligned}
$$

Therefore, $(\mathscr{G},\mathscr{A})$ is a bfs-hvs of $\mathscr{V}$.

If $\mathscr{S}$ is a non-empty subset of a hypervector space $\mathscr{V}$, the linear span of $\mathscr{S}$ refers to the smallest shs of $\mathscr{V}$ that includes all the elements in $\mathscr{S}$. In other words, it is the intersection of all shs's $\mathscr{W}$ of $\mathscr{V}$ that contain $\mathscr{S}$. The linear span of $\mathscr{S}$ can be expressed as

$$\langle\mathscr{S}\rangle=\left\{t\in\sum_{i=1}^{n}a_i\circ s_i, a_i\in\mathscr{K}, s_i\in\mathscr{S}, n\in\mathbb{N}\right\}.$$

Theorem 6. *Suppose $(\mathscr{F},\mathscr{A})$ is a bfs set of hvs $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$ such that $|Im(\mathscr{F}_e^+)|,|Im(\mathscr{F}_e^-)|<\infty$, for all $e\in\mathscr{A}$. Define subhyperspaces $\mathscr{W}_i$ of $\mathscr{V}$, $1\le i\le l$, by the followings:*

$$\mathscr{W}_0=\langle\mathscr{U}_0\rangle,\ \ \mathscr{W}_i=\langle\mathscr{W}_{i-1},\mathscr{U}_i\rangle,$$

where

$$\mathscr{U}_0=\left\{x\in\mathscr{V},\ \mathscr{F}_e^+(x)=\sup_{t\in\mathscr{V}}\mathscr{F}_e^+(t),\mathscr{F}_e^-(x)=\inf_{t\in\mathscr{V}}\mathscr{F}_e^-(t),\forall e\in\mathscr{A}\right\},$$

$$\mathscr{U}_i=\left\{x\in\mathscr{V},\ \mathscr{F}_e^+(x)=\sup_{t\in\mathscr{V}\setminus\mathscr{W}_{i-1}}\mathscr{F}_e^+(t),\mathscr{F}_e^-(x)=\inf_{t\in\mathscr{V}\setminus\mathscr{W}_{i-1}}\mathscr{F}_e^-(t),\forall e\in\mathscr{A}\right\},$$

$l\le|Im(\mathscr{F}_e^+)|,|Im(\mathscr{F}_e^-)|$ and $\mathscr{W}_l=\mathscr{V}$. Then the bfs set $(\mathscr{G},\mathscr{A})$ of $\mathscr{V}$ defined by

$$\left[\begin{array}{l}\mathscr{G}:\mathscr{A}\longrightarrow BF^{\mathscr{V}}\\ \qquad e\mapsto\mathscr{G}_e:\mathscr{V}\to[0,1]\times[-1,0]\\ \qquad\qquad x\mapsto(\mathscr{G}_e^+(x),\mathscr{G}_e^-(x)),\end{array}\right.$$

is the smallest bfs-hvs of $\mathscr{V}$ that contains $(\mathscr{F},\mathscr{A})$, where

$$\mathscr{G}_e^+(x)=\begin{cases}\bigvee_{t\in\mathscr{V}}\mathscr{F}_e^+(t) & x\in\hat{\mathscr{W}}_0=\mathscr{W}_0,\\ \bigvee_{t\in\mathscr{V}\setminus\mathscr{W}_{i-1}}\mathscr{F}_e^+(t) & x\in\hat{\mathscr{W}}_i=\mathscr{W}_i\setminus\mathscr{W}_{i-1},\end{cases}$$

and

$$\mathscr{G}_e^-(x)=\begin{cases}\bigwedge_{t\in\mathscr{V}}\mathscr{F}_e^-(t) & x\in\hat{\mathscr{W}}_0=\mathscr{W}_0,\\ \bigwedge_{t\in\mathscr{V}\setminus\mathscr{W}_{i-1}}\mathscr{F}_e^-(t) & x\in\hat{\mathscr{W}}_i=\mathscr{W}_i\setminus\mathscr{W}_{i-1}.\end{cases}$$

Proof. For all $\alpha\in(0,1]$, $\beta\in[-1,0)$ and $e\in\mathscr{A}$, $(\mathscr{G}_e)_{\alpha,\beta}$ is a subhyperspace of $\mathscr{V}$, so by Theorem 2, $(\mathscr{G},\mathscr{A})$ is a bfs-hvs of $\mathscr{V}$.
Now let $(\mathscr{H},\mathscr{A})$ be a bfs-hvs of $\mathscr{V}$ containing $(\mathscr{F},\mathscr{A})$. We prove that $(\mathscr{F},\mathscr{A})\sqsubseteq(\mathscr{G},\mathscr{A})\sqsubseteq(\mathscr{H},\mathscr{A})$, i.e.

$$\forall e\in\mathscr{A},\forall x\in\mathscr{V},\ \mathscr{G}_e^+(x)\le\mathscr{H}_e^+(x),\ \mathscr{G}_e^-(x)\ge\mathscr{H}_e^-(x).$$

If $e \in \mathscr{A}$, $x \in \mathscr{W}_0$, then there exist $a_{01}, a_{02}, \ldots, a_{0n} \in \mathscr{K}$ and $t_{01}, t_{02}, \ldots, t_{0n} \in \mathscr{U}_0$ such that $x \in a_{01} \circ t_{01} + a_{02} \circ t_{02} + \cdots + a_{0n} \circ t_{0n}$. Thus by Theorem 3,

$$\begin{aligned}
\mathscr{H}_e^+(x) &\geq \bigwedge_{t \in a_{01} \circ t_{01} + \cdots + a_{0n} \circ t_{0n}} \mathscr{H}_e^+(t) \\
&\geq \bigwedge_{t \in a_{01} \circ t_{01} + \cdots + a_{0n} \circ t_{0n}} \mathscr{F}_e^+(t) \\
&\geq \mathscr{F}_e^+(t_{01}) \wedge \cdots \wedge \mathscr{F}_e^+(t_{0n}) \\
&= \bigvee_{t \in \mathscr{V}} \mathscr{F}_e^+(t) \\
&\geq \mathscr{F}_e^+(x),
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{H}_e^-(x) &\leq \bigvee_{t \in a_{01} \circ t_{01} + \cdots + a_{0n} \circ t_{0n}} \mathscr{H}_e^-(t) \\
&\leq \bigvee_{t \in a_{01} \circ t_{01} + \cdots + a_{0n} \circ t_{0n}} \mathscr{F}_e^-(t) \\
&\leq \mathscr{F}_e^-(t_{01}) \vee \cdots \vee \mathscr{F}_e^-(t_{0n}) \\
&= \bigwedge_{t \in \mathscr{V}} \mathscr{F}_e^-(t) \\
&\leq \mathscr{F}_e^-(x).
\end{aligned}$$

Hence $\mathscr{F}_e^+(x) \leq \mathscr{G}_e^+(x) \leq \mathscr{H}_e^+(x)$ and $\mathscr{F}_e^-(x) \geq \mathscr{G}_e^-(x) \geq \mathscr{H}_e^-(x)$, for all $x \in \mathscr{W}_0$, $e \in \mathscr{A}$.
If $e \in \mathscr{A}$ and $x \in \hat{\mathscr{W}}_i$, $i = 1, \ldots, l$, then there exist $a_{i1}, a_{i2}, \ldots, a_{in} \in \mathscr{K}$ and $t_{i1}, t_{i2}, \ldots, t_{in} \in \mathscr{U}_i$ such that $x \in a_{i1} \circ t_{i1} + a_{i2} \circ t_{i2} + \cdots + a_{in} \circ t_{in}$, and so

$$\begin{aligned}
\mathscr{H}_e^+(x) &\geq \bigwedge_{t \in a_{i1} \circ t_{i1} + \cdots + a_{in} \circ t_{in}} \mathscr{H}_e^+(t) \\
&\geq \bigwedge_{t \in a_{i1} \circ t_{i1} + \cdots + a_{in} \circ t_{in}} \mathscr{F}_e^+(t) \\
&\geq \mathscr{F}_e^+(t_{i1}) \wedge \cdots \wedge \mathscr{F}_e^+(t_{in}) \\
&= \bigvee_{t \in \mathscr{V} \setminus \mathscr{W}_{i-1}} \mathscr{F}_e^+(t) \\
&\geq \bigvee_{t \in \hat{\mathscr{W}}_i} \mathscr{F}_e^+(t) \\
&\geq \mathscr{F}_e^+(x),
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{H}_e^-(x) &\leq \bigvee_{t \in a_{i1} \circ t_{i1} + \cdots + a_{in} \circ t_{in}} \mathscr{H}_e^-(t) \\
&\leq \bigvee_{t \in a_{i1} \circ t_{i1} + \cdots + a_{in} \circ t_{in}} \mathscr{F}_e^-(t) \\
&\leq \mathscr{F}_e^-(t_{i1}) \vee \cdots \vee \mathscr{F}_e^-(t_{in}) \\
&= \bigwedge_{t \in \mathscr{V} \setminus \mathscr{W}_{i-1}} \mathscr{F}_e^-(t) \\
&\leq \bigwedge_{t \in \hat{\mathscr{W}}_i} \mathscr{F}_e^-(t) \\
&\leq \mathscr{F}_e^-(x).
\end{aligned}$$

Then $\mathscr{F}_e^+(x) \leq \mathscr{G}_e^+(x) \leq \mathscr{H}_e^+(x)$ and $\mathscr{F}_e^-(x) \geq \mathscr{G}_e^-(x) \geq \mathscr{H}_e^-(x)$, for all $x \in \hat{\mathscr{W}}_i$, $e \in \mathscr{A}$.
Therefore, $(\mathscr{G}, \mathscr{A})$ is the smallest bfs-hvs of $\mathscr{V}$ containing $(\mathscr{F}, \mathscr{A})$.

$(\mathscr{G}, \mathscr{A})$ defined in Theorem 6, is said to be the bipolar fuzzy soft hypervector space generated by $(\mathscr{F}, \mathscr{A})$.

5 Normal Bipolar Fuzzy Soft Hypervector Spaces

Here a particular type of bfs-hvs's is shortly studied, supported by some examples. It is shown that we can obtain normal bfs-hvs's from bfs-hvs's, also from subhyperspaces of $\mathscr{V}$.

Definition 8. *A bfs-hvs $(\mathscr{G},\mathscr{B})$ of $\mathscr{V}$ is considered normal if there exists x in $\mathscr{V}$ with $\mathscr{G}_e^+(x)=1$ and $\mathscr{G}_e^-(x)=-1$, for all $e\in\mathscr{B}$.*

The fact is clearly evident, if $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$, then $\mathscr{G}_e^+(x)\leq\mathscr{G}_e^+(0)$ and $\mathscr{G}_e^-(x)\geq\mathscr{G}_e^-(0)$, $\forall e\in\mathscr{B}$, $x\in\mathscr{V}$. Thus if $(\mathscr{G},\mathscr{B})$ is a normal bfs-hvs of $\mathscr{V}$,$\mathscr{G}_e^+(0)=1$ and $\mathscr{G}_e^-(0)=-1$, for all $e\in\mathscr{B}$. Hence $(\mathscr{G},\mathscr{B})$ is a normal bfs-hvs of $\mathscr{V}$, iff $\mathscr{G}_e^+(0)=1$ and $\mathscr{G}_e^-(0)=-1$, for all $e\in\mathscr{B}$.

Example 9. Let $\mathscr{V}=(\mathbb{R}^3,+,\circ,\mathbb{R})$ be the hvs in Example 1, and $\mathscr{A}=\{a,b\}$ be the parameters. Define "$\mathscr{F}_a^+,\mathscr{F}_b^+:\mathbb{R}^3\to[0,1]$" and "$\mathscr{F}_a^-,\mathscr{F}_b^-:\mathbb{R}^3\to[-1,0]$" by the followings:

$$\mathscr{F}_a^+(t)=\begin{cases}1 & t\in X,\\ 0.4 & t\in Y\setminus X,\\ 0 & otherwise,\end{cases}$$

$$\mathscr{F}_a^-(t)=\begin{cases}-1 & t\in X,\\ -0.6 & t\in Y\setminus X,\\ -0.2 & otherwise,\end{cases}$$

$$\mathscr{F}_b^+(t)=\begin{cases}1 & t\in X,\\ 0.5 & t\in Y\setminus X,\\ 0.3 & otherwise,\end{cases}$$

$$\mathscr{F}_b^-(t)=\begin{cases}-1 & t\in X,\\ -0.9 & t\in Y\setminus X,\\ -0.1 & otherwise.\end{cases}$$

Then $(\mathscr{F},\mathscr{A})$ is a normal bfs-hvs of $\mathbb{R}^3$.

Example 10. Let $\mathscr{V}=(\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ be the hvs in Example 2, and $\mathscr{A}=\{c,d,e\}$ be the parameters. Define "$\mathscr{F}_c^+,\mathscr{F}_d^+,\mathscr{F}_e^+:\mathbb{Z}_4\to[0,1]$" and "$\mathscr{F}_c^-,\mathscr{F}_d^-,\mathscr{F}_e^-:\mathbb{Z}_4\to[-1,0]$" by:

$$\mathscr{F}_c^+(t)=\begin{cases}1 & t=0,2,\\ 0.4 & t=1,3,\end{cases}\qquad \mathscr{F}_c^-(t)=\begin{cases}-1 & t=0,2,\\ -0.3 & t=1,3,\end{cases}$$

$$\mathscr{F}_d^+(t)=\begin{cases}1 & t=0,2,\\ 0.6 & t=1,3,\end{cases}\qquad \mathscr{F}_d^-(t)=\begin{cases}-1 & t=0,2,\\ -0.2 & t=1,3,\end{cases}$$

$$\mathscr{F}_e^+(t)=\begin{cases}1 & t=0,2,\\ 0.1 & t=1,3,\end{cases}\qquad \mathscr{F}_e^-(t)=\begin{cases}-1 & t=0,2,\\ -0.8 & t=1,3.\end{cases}$$

Then $(\mathscr{F},\mathscr{A})$ is a normal bfs-hvs of $\mathbb{Z}_4$.

Proposition 3. *If $(\mathscr{G},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}=(\mathscr{V},+,\circ,\mathscr{K})$, then the bfs set $(\tilde{\mathscr{G}},\mathscr{B})$ of $\mathscr{V}$ is a normal bfs-hvs of $\mathscr{V}$, containing $(\mathscr{G},\mathscr{B})$, where*

$$\forall e\in\mathscr{B},\ \tilde{\mathscr{G}}_e^+(x)=\mathscr{G}_e^+(x)+1-\mathscr{G}_e^+(0),\ \tilde{\mathscr{G}}_e^-(x)=\mathscr{G}_e^-(x)-1+\mathscr{G}_e^-(0).$$

Proof. Let $e\in\mathscr{B}$, $y,z\in\mathscr{V}$ and $b\in\mathscr{K}$. Then

$$\begin{aligned}\tilde{\mathscr{G}}_e^+(y-z)&=\mathscr{G}_e^+(y-z)+1-\mathscr{G}_e^+(0)\\&\geq(\mathscr{G}_e^+(y)\wedge\mathscr{G}_e^+(z))+1-\mathscr{G}_e^+(0)\\&=(\mathscr{G}_e^+(y)+1-\mathscr{G}_e^+(0))\wedge(\mathscr{G}_e^+(z)+1-\mathscr{G}_e^+(0))\\&=\tilde{\mathscr{G}}_e^+(y)\wedge\tilde{\mathscr{G}}_e^+(z),\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathscr{G}}_e^-(y-z) &= \mathscr{G}_e^-(y-z) - 1 + \mathscr{G}_e^-(0) \\
&\leq (\mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z)) - 1 + \mathscr{G}_e^-(0) \\
&= (\mathscr{G}_e^-(y) - 1 + \mathscr{G}_e^-(0)) \vee (\mathscr{G}_e^-(z) - 1 + \mathscr{G}_e^-(0)) \\
&= \tilde{\mathscr{G}}_e^-(y) \vee \tilde{\mathscr{G}}_e^-(z).
\end{aligned}$$

Also,

$$\bigwedge_{t\in b\circ y} \tilde{\mathscr{G}}_e^+(t) = \bigwedge_{t\in b\circ y} (\mathscr{G}_e^+(t) + 1 - \mathscr{G}_e^+(0)) \geq \mathscr{G}_e^+(y) + 1 - \mathscr{G}_e^+(0) = \tilde{\mathscr{G}}_e^+(y),$$

and

$$\bigvee_{t\in b\circ y} \tilde{\mathscr{G}}_e^-(t) = \bigvee_{t\in b\circ y} (\mathscr{G}_e^-(t) - 1 + \mathscr{G}_e^-(0)) \leq \mathscr{G}_e^-(y) - 1 + \mathscr{G}_e^-(0) = \tilde{\mathscr{G}}_e^-(y).$$

Hence, by Definition 5, $(\tilde{\mathscr{G}}, \mathscr{B})$ is a bfs-hvs of $\mathscr{V}$. Moreover, $\tilde{\mathscr{G}}_e^+(0) = 1$ and $\tilde{\mathscr{G}}_e^-(0) = -1$, for all $e \in \mathscr{B}$. Next, for all $e \in \mathscr{B}$ and $x \in \mathscr{V}$ we have, $\tilde{\mathscr{G}}_e^+(y) = \mathscr{G}_e^+(y) + 1 - \mathscr{G}_e^+(0) \geq \mathscr{G}_e^+(y)$ and $\tilde{\mathscr{G}}_e^-(y) = \mathscr{G}_e^-(y) - 1 + \mathscr{G}_e^-(0) \leq \mathscr{G}_e^-(y)$. Thus $(\mathscr{G}, \mathscr{B}) \sqsubseteq (\tilde{\mathscr{G}}, \mathscr{B})$. The proof is now completed.

Clearly, a bfs-hvs $(\mathscr{G}, \mathscr{B})$ of $\mathscr{V}$ is normal, iff $(\tilde{\mathscr{G}}, \mathscr{B}) = (\mathscr{G}, \mathscr{B})$.

Proposition 4.*If $\mathscr{X}$ is a shs of $\mathscr{V}$, then the bfs set $(\mathscr{G}, \mathscr{B})$ of $\mathscr{V}$ given by the following is a normal bfs-hvs of $\mathscr{V}$.*

$$\forall e \in \mathscr{B},\ \mathscr{G}_e^+(x) = \begin{cases} 1 & x \in \mathscr{X}, \\ 0 & o.w. \end{cases} \quad \textit{and} \quad \mathscr{G}_e^-(x) = \begin{cases} -1 & x \in \mathscr{X}, \\ 0 & o.w. \end{cases}$$

*Proof.*Clearly, for all $e \in \mathscr{B}$, $x \in \mathscr{X}$, $\mathscr{G}_e^+(x) = 1$ and $\mathscr{G}_e^-(x) = -1$.

Theorem 7.*If $(\mathscr{G}, \mathscr{B})$ is a non-zero bfs-hvs of V, then $(\breve{\mathscr{G}}, B)$ given by the following is a normal bfs-hvs of V, containing $(\mathscr{G}, \mathscr{B})$.*

$$\forall e \in B, x \in V,\ \breve{\mathscr{G}}_e^+(x) = \frac{\mathscr{G}_e^+(x)}{\mathscr{G}_e^+(0)},\ \breve{\mathscr{G}}_e^-(x) = -\frac{\mathscr{G}_e^-(x)}{\mathscr{G}_e^-(0)}.$$

*Proof.*Let $e \in \mathscr{B}$, $y, z \in \mathscr{V}$ and $b \in \mathscr{K}$. Then

$$\begin{aligned}
\breve{\mathscr{G}}_e^+(y-z) &= \frac{\mathscr{G}_e^+(y-z)}{\mathscr{G}_e^+(0)} \\
&\geq \frac{\mathscr{G}_e^+(y) \wedge \mathscr{G}_e^+(z)}{\mathscr{G}_e^+(0)} \\
&= \frac{\mathscr{G}_e^+(y)}{\mathscr{G}_e^+(0)} \wedge \frac{\mathscr{G}_e^+(z)}{\mathscr{G}_e^+(0)} \\
&= \breve{\mathscr{G}}_e^+(y) \wedge \breve{\mathscr{G}}_e^+(z),
\end{aligned}$$

and

$$\begin{aligned}
\breve{\mathscr{G}}_e^-(y-z) &= -\frac{\mathscr{G}_e^-(y-z)}{\mathscr{G}_e^-(0)} \\
&\leq -\frac{\mathscr{G}_e^-(y) \vee \mathscr{G}_e^-(z)}{\mathscr{G}_e^-(0)} \\
&= \left(-\frac{\mathscr{G}_e^-(y)}{\mathscr{G}_e^-(0)}\right) \vee \left(-\frac{\mathscr{G}_e^-(z)}{\mathscr{G}_e^-(0)}\right) \\
&= \breve{\mathscr{G}}_e^-(y) \vee \breve{\mathscr{G}}_e^-(z).
\end{aligned}$$

Also,

$$\begin{aligned}
\inf_{t\in b\circ y} \breve{\mathscr{G}}_e^+(t) &= \inf_{t\in b\circ y} \frac{\mathscr{G}_e^+(t)}{\mathscr{G}_e^+(0)} \\
&= \frac{\inf\limits_{t\in b\circ y} \breve{\mathscr{G}}_e^+(t)}{\mathscr{G}_e^+(0)} \\
&\geq \frac{\mathscr{G}_e^+(y)}{\mathscr{G}_e^+(0)} \\
&= \breve{\mathscr{G}}_e^+(y),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in b\circ y} \breve{\mathscr{G}}_e^-(t) &= \sup_{t\in b\circ y} \left(-\frac{\mathscr{G}_e^-(t)}{\mathscr{G}_e^-(0)}\right) \\
&= -\frac{\sup\limits_{t\in b\circ y} \breve{\mathscr{G}}_e^-(t)}{\mathscr{G}_e^-(0)} \\
&\leq -\frac{\mathscr{G}_e^-(y)}{\mathscr{G}_e^-(0)} \\
&= \breve{\mathscr{G}}_e^-(y).
\end{aligned}$$

Thus by Definition 5, $(\breve{\mathscr{G}},\mathscr{B})$ is a bfs-hvs of $\mathscr{V}$. Moreover, $\breve{\mathscr{G}}_e^+(0)=1$, $\breve{\mathscr{G}}_e^-(0)=-1$, and for all $y\in\mathscr{V}$, $\mathscr{G}_e^+(y)\leq\breve{\mathscr{G}}_e^+(y)$, $\mathscr{G}_e^-(y)\geq\breve{\mathscr{G}}_e^-(y)$. Hence the proof is now completed.

6 Conclusion

Bipolar fuzzy soft hypervector spaces combine several areas of study:

Fuzzy Set Theory: This branch of mathematics deals with uncertainty and approximation, where elements can belong to a set to a certain degree. Bipolar fuzzy sets extend this idea by allowing elements to have a pair of membership degrees that represent the degree of satisfaction to the property related to a fuzzy set and its counter-property.

Soft Set Theory: Soft sets are a general mathematical tool for dealing with uncertainty, vagueness, and ambiguity in decision-making processes and data analysis. It allows for the gradual transition from total rejection to total acceptance of an element.

Hypervector Spaces: Hypervector spaces are a generalization of vector spaces, with various types, based on the number of operations and external hyperoperations.

From the algebraic point of view, by combining these three concepts, a bipolar fuzzy soft hypervector space likely aims to provide a framework for handling complex, uncertain, and ambiguous data in a way that's suitable for a wide range of applications, such as decision-making systems, pattern recognition, and data analysis.

It's great to see the continuous development of the concept of bipolar fuzzy soft hypervector spaces, building upon the ideas introduced by the author in reference [27], such as equivalent conditions of fuzzy bipolar soft hypervector space, bipolar fuzzy soft hypervector spaces generated by a bipolar fuzzy soft set and normal bipolar fuzzy soft hypervector spaces. It shows a strong continuity in the research and a commitment to exploring and understanding the various aspects of this concept.

With this information, here are some potential directions for future work in this area:

Theoretical Foundations: Further development of the theoretical foundations of bipolar fuzzy soft hypervector spaces, including the exploration of their properties, axiomatic foundations, and relationships with other mathematical structures.

Algebraic Structures: Investigation into the algebraic structures associated with bipolar fuzzy soft hypervector spaces, such as the development of specific algebraic operations and their properties within this context.

Applications in Decision Making: Exploring applications of bipolar fuzzy soft hypervector spaces in decision making, pattern recognition, image processing, and data analysis, and studying their advantages over existing models in these domains.

Empirical Studies: Conducting empirical studies and experiments to validate the effectiveness of these spaces in practical applications and comparing their performance with other existing models.

Generalization and Specialization: Generalization of the concept to incorporate more complex information structures and specialization to address specific domains or applications.

Hybrid Models: Exploration of hybrid models that combine bipolar fuzzy soft hypervector spaces with other related mathematical models to harness the complementary strengths of different approaches.

Multi-disciplinary Research: Collaboration with researchers from diverse fields, including mathematics, computer science, engineering, and social sciences, to explore the broad applicability and impact of bipolar fuzzy soft hypervector spaces.

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