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On Reformulated Geometric-Harmonic Index of Graphs

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Abstract: In theoretical chemistry, the topological indices are mostly used to explore/develop QSAR and QSPR analyses of molecular graphs. The edge variant of the Geometric-Harmonic index is one type of topological indices it is described as $EGH(\Gamma) = \sum_{e\sim t\in E(\Gamma)} \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2}$, where $d_\Gamma(t)$ is the degree of a vertex t in Γ. In this paper, using several molecular structural parameters, we establish some new bounds on the the edge variant of Geometric-Harmonic index $EGH(\Gamma)$. Also, we connect these indices to a number of well-known molecular descriptors.

Keywords: Reformulated Zagreb indices; edge degree; vertex degree; platt number; order and size.

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1 Introduction

A topological index is a structural descriptor that may be derived from a molecular structure and is useful for numerically representing the size, shape, cyclicity, and branching of molecules. Relationships between a molecular compound's structure and its physicochemical properties or biological activity are frequently found using the topological indices of molecular structures. Distance-based topological indices, degree-based topological indices, eccentricity based topological indices, counting-related polynomials, and graph indices are some of the categories into which topological indices are divided. The significance of degree-based topological indices in chemical graph theory is particularly noteworthy in the field of chemistry. Let $\Gamma = (V,E)$ be a connected simple graph with order $n = |V|$ vertices and size $m = |E|$ edges. The degree of a vertex v is denoted as $d_\Gamma(v)$. Specifically, the maximum and minimum degree of Γ are denoted by $\Delta = \Delta(\Gamma)$ and $\delta = \delta(\Gamma)$, respectively. Also, an edge $e = xy \in E(\Gamma)$ is the number of edges incident to e and $d_\Gamma(e) = d_\Gamma(x) + d_\Gamma(y) - 2$ represents the degree of edge e. For convenience sake, $e \sim t$ represents edges $e, t \in E(\Gamma)$ are adjacent, we have included a list of degree-based topological indices that will be utilised in the next sections. In [15], the authors define the Geometric-Harmonic index as

$$GH(\Gamma) = \sum_{xy\in E(\Gamma)} \frac{(d_\Gamma(x)+d_\Gamma(y))\sqrt{d_\Gamma(x)\cdot d_\Gamma(y)}}{2}.$$

By using above graph index we can define edge variant of Geometric-Harmonic index as

$$EGH(\Gamma) = \sum_{e\sim t\in E(\Gamma)} \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2}. \tag{1}$$

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Lemma 1.[11,8] *The Platt number is the sum of degrees of all its edges of Γ and is denoted as*

$$\begin{aligned} Pl(\Gamma) &= \sum_{xy \in E(\Gamma)} d_\Gamma(xy) \\ &= \sum_{xy \in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y) - 2) \\ &= \sum_{x \in V(\Gamma)} d_\Gamma(x)(d_\Gamma(x) - 1). \end{aligned}$$

Lemma 2.[2] *For any connected graph Γ with $n > 2$,*

(i) $2m(\delta - 1) \leq Pl(\Gamma) \leq 2m(\Delta - 1)$

(ii) $m \leq Pl(\Gamma) \leq 2m(n-2)$.

In [4], the authors define the first Zagreb index as

$$M_1(\Gamma) = \sum_{x \in V(\Gamma)} d_\Gamma(x)^2 = \sum_{xy \in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y)).$$

Lemma 3.[1,3] *For any graph Γ with $n \geq 2$, $Pl(\Gamma) = M_1(\Gamma) - 2m$.*

Lemma 4.[9] *If $a_k \geq 0$, $b_k \geq 0$ for $k = 1,2,\ldots,n$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p \geq 1$, then*

$$\left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}} \geq \sum_{k=1}^{n} a_k b_k.$$

With equality holding if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1,2,\ldots,n$ where α and β are real non-negative constants with $\alpha^2 + \beta^2 > 0$.

2 Main Results

In this section, we obtain bounds on edge variant of Geometric-Harmonic index using the graph parameters namely minimum degree, maximum degree, order, size, first Zagreb index $M_1(\Gamma)$, second reformulated Zagreb index $EM_2(\Gamma)$ and first reformulated Zagreb index $EM_1(\Gamma)$, reformulated Hyper-Zagreb index $RHM(\Gamma)$, Randić edge index, reformulated second Gourava index, edge variant of Arithmetic-Geometric index and edge variant of Geometric-Arithmetic index.

2.1 Bounds in relation to minimum degree and maximum degree

Theorem 1. *If Γ is any graph of order n, $n \geq 2$, size m, maximum degree Δ and minimum degree δ, then*

$$4m(\delta - 1)^2 \leq EGH(\Gamma) \leq 4m(\Delta - 1)^2.$$

Additional equality is possible only in the case when Γ is regular.

Proof. By inequality (i) of Lemma 2 and the definition of $EGH(\Gamma)$, we have

$$4(\delta - 1)^2 \leq \frac{(d_\Gamma(e) + d_\Gamma(t))\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}}{2} \leq 4(\Delta - 1)^2 \tag{2}$$

The eqn. (2) holds true for $\frac{Pl(\Gamma)}{2}$ pairs of neighbouring edges, and the sum of those in equalities yields the result.

$$Pl(\Gamma) \cdot 2 \cdot (\delta - 1)^2 \leq \sum_{e \sim t} \frac{(d_\Gamma(e) + d_\Gamma(t))\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}}{2} \leq Pl(\Gamma) \cdot 2 \cdot (\Delta - 1)^2 \tag{3}$$

Again, by inequality (i) of Lemma 2 and eqn (3), we have

$$4m(\delta - 1)^2 \leq EGH(\Gamma) \leq 4m(\Delta - 1)^2.$$

Additional equality is possible only in the case when Γ is regular.

2.2 *Bounds in relation to order and size*

Theorem 2.*If Γ is any graph of order n, $n \geq 3$ and size m, then*

$$\frac{m}{2} \leq EGH(\Gamma) \leq 4m(n-2)^3.$$

Additional equality is possible only in the case when Γ is regular.

*Proof.*Using inequality (ii) of Lemma 2, we have

$$\frac{m}{2} \leq \frac{Pl(\Gamma)}{2} \leq m(n-2).$$

Therefore, $1 \leq \sqrt{d_\Gamma(e) \cdot d_\Gamma(f)} \leq 2(n-2)$

$$1 \leq \frac{(d_\Gamma(e)+d_\Gamma(f))\sqrt{d_\Gamma(e) \cdot d_\Gamma(f)}}{2} \leq 4(n-2)^2$$

The above inequality satiesfies for $\frac{Pl(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, we have

$$\frac{Pl(\Gamma)}{2} \leq \sum_{e \sim t} \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}}{2} \leq Pl(\Gamma) \cdot 2 \cdot (n-2)^2 \tag{4}$$

By using above expression, we have

$$\frac{m}{2} \leq EGH(\Gamma) \leq 4m(n-2)^3.$$

Additional equality is possible only in the case when Γ is regular.

2.3 *Bounds in relation to first Zagreb index $M_1(\Gamma)$*

Theorem 3.*If Γ is any graph of order n, $n \geq 3$, size m, maximum degree Δ and minimum degree δ, then*

(i) $\frac{M_1(\Gamma)-2m}{2} \leq EGH(\Gamma) \leq 2(M_1(\Gamma)-2m)(n-2)^2$

(ii) $2(\delta-1)^2(M_1(\Gamma)-2m) \leq EGH(\Gamma) \leq 2(\Delta-1)^2(M_1(\Gamma)-2m)$.

Additional equality is possible only in the case when Γ is regular.

*Proof.*The desired result is obtained by applying Lemma 3 on the inequality (4) to get (i) and on the inequality (3) to get (ii).

2.4 *Bounds in relation to second reformulated Zagreb index $EM_2(\Gamma)$ and first reformulated Zagreb index $EM_1(\Gamma)$*

The below is a definition of the first and second reformulated Zagreb indices [6]

$$EM_1(\Gamma) = \sum_{e \sim t \in E(\Gamma)} (d_\Gamma(e)+d_\Gamma(t)), EM_2(\Gamma) = \sum_{e \sim t \in E(\Gamma)} (d_\Gamma(e) \cdot d_\Gamma(t)). \tag{5}$$

Theorem 4.*If Γ is any graph of order n and size m, then*

$$EGH(\Gamma) \geq EM_2(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

*Proof.*By using eqn.(1)

$$EGH(\Gamma) = \sum_{e\sim t\in E(\Gamma)} \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2}$$
$$= \sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}\cdot\frac{(d_\Gamma(e)+d_\Gamma(t))}{2}$$
$$\frac{\sqrt{d_\Gamma(e)^2+d_\Gamma(t)^2}}{2} \geq \frac{(d_\Gamma(e)+d_\Gamma(t))}{2} \geq \sqrt{(d_\Gamma(e)\cdot d_\Gamma(t))} \geq \frac{2d_\Gamma(e)\cdot d_\Gamma(t)}{d_\Gamma(e)+d_\Gamma(t)} \tag{6}$$

It is clear that

$$\frac{(d_\Gamma(e)+d_\Gamma(t))}{2} \geq \sqrt{(d_\Gamma(e)\cdot d_\Gamma(t))}$$

Now,by multiplying both sides by $\sqrt{(d_\Gamma(e)\cdot d_\Gamma(t))}$

$$\text{we get, } \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{(d_\Gamma(e)\cdot d_\Gamma(t))}}{2} \geq (d_\Gamma(e)\cdot d_\Gamma(t))$$

The above inequality satiesfies for $\frac{Pl(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, we have

$$\frac{Pl(\Gamma)}{2}\cdot\frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{(d_\Gamma(e)\cdot d_\Gamma(t))}}{2} \geq \frac{Pl(\Gamma)}{2}\cdot(d_\Gamma(e)\cdot d_\Gamma(t))$$
$$EGH(\Gamma) \geq EM_2(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

Theorem 5.*If Γ is any graph of order n, $n \geq 2$ and size m, then*

$$2(\delta-1)EM_1(\Gamma) \leq EGH(\Gamma) \leq 2(\Delta-1)EM_1(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

Proof.

$$2(\delta-1) \leq \frac{\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2} \leq 2(\Delta-1)$$
$$2(\delta-1)(d_\Gamma(e)+d_\Gamma(t)) \leq (d_\Gamma(e)+d_\Gamma(t))\frac{\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2} \leq 2(\Delta-1)(d_\Gamma(e)+d_\Gamma(t))$$

this inequality satisfies for $\frac{Pl(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, to get

$$2(\delta-1)\sum_{e\sim t}(d_\Gamma(e)+d_\Gamma(t)) \leq \sum_{e\sim t}(d_\Gamma(e)+d_\Gamma(t))\frac{\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2} \leq 2(\Delta-1)\sum_{e\sim t}(d_\Gamma(e)+d_\Gamma(t))$$

Thus,

$$2(\delta-1)EM_1(\Gamma) \leq EGH(\Gamma) \leq 2(\Delta-1)EM_1(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

2.5 ***Bounds in relation to reformulated Hyper-Zagreb index*** $RHM(\Gamma)$

The below is a definition of the Hyper-Zagreb index [13]

$$HM(\Gamma) = \sum_{xy\in E(\Gamma)} (d_\Gamma(x)+d_\Gamma(y))^2.$$

Using above definition, the edge variant of the Hyper-Zagreb index defined as reformulated Hyper-Zagreb index

$$RHM(\Gamma) = \sum_{e\sim t\in E(\Gamma)} (d_\Gamma(e)+d_\Gamma(t))^2. \tag{7}$$

Theorem 6. *If Γ is any graph of order n and size m, then*

$$EGH(\Gamma) \leq \frac{\sqrt{RHM(\Gamma) \cdot EM_2(\Gamma)}}{2}.$$

Additional equality is possible only in the case when Γ is regular.

Proof. By setting $a_k = \frac{d_\Gamma(e)+d_\Gamma(t)}{2}$, $b_k = \sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}$, $p = q = 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ in Lemma 4, we get

$$\sum_{k=1}^{m} \frac{(d_\Gamma(e)+d_\Gamma(t))}{2} \sqrt{d_\Gamma(e) \cdot d_\Gamma(t)} \leq \left(\sum_{k=1}^{m} \left(\frac{d_\Gamma(e)+d_\Gamma(t)}{2} \right)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{m} d_\Gamma(e) \cdot d_\Gamma(t) \right)^{\frac{1}{2}}$$

Using the definition of $EGH(\Gamma)$, $RHM(\Gamma)$ and $EM_2(\Gamma)$, we get

$$EGH(\Gamma) \leq \frac{\sqrt{RHM(\Gamma) \cdot EM_2(\Gamma)}}{2}.$$

Additional equality is possible only in the case when Γ is regular.

2.6 ***Bounds in relation to Randić edge index***

The below is a definition of the edge variant General Randić index [14]

$$ER_\alpha(\Gamma) = \sum_{e \sim t \in E(\Gamma)} (d_\Gamma(e) \cdot d_\Gamma(t))^\alpha.$$

For $\alpha = \frac{1}{2}$

$$ER_{\frac{1}{2}}(\Gamma) = \sum_{e \sim t \in E(\Gamma)} (\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}). \tag{8}$$

Theorem 7. *If Γ is any graph of order n, $n \geq 2$ and size m, then*

$$2(\delta-1)ER_{\frac{1}{2}}(\Gamma) \leq EGH(\Gamma) \leq 2(\Delta-1)ER_{\frac{1}{2}}(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

Proof.

$$2(\delta-1) \leq \frac{(d_\Gamma(e)+d_\Gamma(t))}{2} \leq 2(\Delta-1)$$

$$2(\delta-1)\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)} \leq \frac{(d_\Gamma(e)+d_\Gamma(t))}{2} \sqrt{d_\Gamma(e) \cdot d_\Gamma(t)} \leq 2(\Delta-1)\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}$$

this inequality satiesfies for $\frac{Pl(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, to get

$$2(\delta-1)\sum_{e \sim t} \sqrt{d_\Gamma(e) \cdot d_\Gamma(t)} \leq \sum_{e \sim t} \frac{(d_\Gamma(e)+d_\Gamma(t))}{2} \sqrt{d_\Gamma(e) \cdot d_\Gamma(t)} \leq 2(\Delta-1)\sum_{e \sim t} \sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}$$

Using the definition of $ER_{\frac{1}{2}}$ and $EGH(\Gamma)$, we get

$$2(\delta-1)ER_{\frac{1}{2}}(\Gamma) \leq EGH(\Gamma) \leq 2(\Delta-1)ER_{\frac{1}{2}}(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

2.7 **Bounds in relation to reformulated second Gourava index**

The below is a definition of the second Gourava index [5]

$$GO_2(\Gamma) = \sum_{xy \in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y))(d_\Gamma(x) \cdot d_\Gamma(y)).$$

Using above definition, the edge variant of the second Gourava index defined as

$$EGO_2(\Gamma) = \sum_{e \sim t \in E(\Gamma)} (d_\Gamma(e) + d_\Gamma(t))(d_\Gamma(e) \cdot d_\Gamma(t))). \tag{9}$$

Theorem 8. *If Γ is any graph of order n and size m, then*

$$\frac{EGO_2(\Gamma)}{4(\Delta - 1)} \leq EGH(\Gamma) \leq \frac{EGO_2(\Gamma)}{4(\delta - 1)}.$$

Additional equality is possible only in the case when Γ is regular.

Proof. Since

$$= \frac{(d_\Gamma(e) + d_\Gamma(t))\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}}{2}$$
$$= \frac{(d_\Gamma(e) + d_\Gamma(t))(d_\Gamma(e) \cdot d_\Gamma(t))}{2\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}}$$

and by the inequality

$$\frac{1}{4(\Delta - 1)} \leq \frac{1}{2\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}} \leq \frac{1}{4(\delta - 1)}$$

we deduce

$$\frac{(d_\Gamma(e) + d_\Gamma(t))(d_\Gamma(e) \cdot d_\Gamma(t))}{4(\Delta - 1)} \leq \frac{(d_\Gamma(e) + d_\Gamma(t))(d_\Gamma(e) \cdot d_\Gamma(t))}{2\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}} \leq \frac{(d_\Gamma(e) + d_\Gamma(t))(d_\Gamma(e) \cdot d_\Gamma(t))}{4(\delta - 1)}$$

this inequality satiesfies for $\frac{Pl(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, to get

$$\frac{EGO_2(\Gamma)}{4(\Delta - 1)} \leq EGH(\Gamma) \leq \frac{EGO_2(\Gamma)}{4(\delta - 1)}.$$

Additional equality is possible only in the case when Γ is regular.

2.8 **Bounds in relation to edge variant of Arithmetic-Geometric index**

The below is a definition of the Arithmetic-Geometri index

$$AG(\Gamma) = \sum_{xy \in E(\Gamma)} \frac{d_\Gamma(x) + d_\Gamma(y)}{2\sqrt{d_\Gamma(x) \cdot d_\Gamma(y)}}.$$

Using above definition, the edge variant of the Arithmetic-Geometric index defined as

$$EAG(\Gamma) = \sum_{e \sim t \in E(\Gamma)} \frac{d_\Gamma(e) + d_\Gamma(t)}{2\sqrt{d_\Gamma(e) \cdot d_\Gamma(t)}}. \tag{10}$$

Theorem 9. *If Γ is any graph of order n and size m, then*

$$4(\delta-1)^2 EAG(\Gamma) \le EGH(\Gamma) \le 4(\Delta-1)^2 EAG(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

Proof. Since

$$= \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2}$$
$$= \frac{(d_\Gamma(e)+d_\Gamma(t))(d_\Gamma(e)\cdot d_\Gamma(t))}{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}$$

and by the inequality

$$(2\delta-2)^2 \le d_\Gamma(e)\cdot d_\Gamma(t) \le (2\Delta-2)^2$$

we deduce

$$4(\delta-1)^2\frac{(d_\Gamma(e)+d_\Gamma(t))}{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}} \le d_\Gamma(e)\cdot d_\Gamma(t) \le 4(\Delta-1)^2\frac{(d_\Gamma(e)+d_\Gamma(t))}{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}$$

this inequality satiesfies for $\frac{PI(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, to get

$$4(\delta-1)^2\sum_{e\sim t\in E(\Gamma)}\frac{(d_\Gamma(e)+d_\Gamma(t))}{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}} \le \sum_{e\sim t\in E(\Gamma)}\frac{(d_\Gamma(e)+d_\Gamma(t))(d_\Gamma(e)\cdot d_\Gamma(t))}{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}} \le 4(\Delta-1)^2\sum_{e\sim t\in E(\Gamma)}\frac{(d_\Gamma(e)+d_\Gamma(t))}{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}$$

$$4(\delta-1)^2 EAG(\Gamma) \le EGH(\Gamma) \le 4(\Delta-1)^2 EAG(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

2.9 ***Bounds in relation to edge variant of Geometric-Arithmetic index***

The below is a definition of edge variant of Geometric-Arithmetic index [10].

$$EGA(\Gamma) = \sum_{e\sim t\in E(\Gamma)}\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{d_\Gamma(e)+d_\Gamma(t)}.$$

Theorem 10. *If Γ is any graph of order n and size m, then*

$$4(\delta-1)^2 EGA(\Gamma) \le EGH(\Gamma) \le 4(\Delta-1)^2 EGA(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

Proof. Since

$$= \frac{(d_\Gamma(e)+d_\Gamma(t))\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{2}$$
$$= \frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)}\cdot\frac{((d_\Gamma(e)+d_\Gamma(t))^2}{4}$$

and by the inequality

$$4(\delta-1)^2 \le \frac{((d_\Gamma(e)+d_\Gamma(t))^2}{4} \le 4(\Delta-1)^2$$

we deduce

$$4(\delta-1)^2\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)} \leq \frac{(d_\Gamma(e)+d_\Gamma(t))^2}{4}\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)} \leq 4(\Delta-1)^2\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)}$$

this inequality satiesfies for $\frac{Pl(\Gamma)}{2}$ pairs of adjacent edges and sum of those inequalities, to get

$$4(\delta-1)^2\sum_{e\sim t\in E(\Gamma)}\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)} \leq \sum_{e\sim t\in E(\Gamma)}\frac{(d_\Gamma(e)+d_\Gamma(t))^2}{4}\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)} \leq 4(\Delta-1)^2\sum_{e\sim t\in E(\Gamma)}\frac{2\sqrt{d_\Gamma(e)\cdot d_\Gamma(t)}}{(d_\Gamma(e)+d_\Gamma(t)}$$

$$4(\delta-1)^2EGA(\Gamma) \leq EGH(\Gamma) \leq 4(\Delta-1)^2EGA(\Gamma).$$

Additional equality is possible only in the case when Γ is regular.

3 Conclusion

In this article, we found some bounds in terms of order, size, degree and well-known molecular descriptors. Additionally, when considering comparative advantages, applications, and mathematical perspectives, numerous questions arise. According to this study's findings, some recommendations include the following.

1. Determine the values of edge variant of Geometric-Harmonic index for each type of chemical graph and then contrast them with other topological indices based on edge degree. Additionally, investigate certain findings related to QSPR / QSAR.
2. Characterize the extremal properties of edge variant of Geometric-Harmonic index.
3. Find the edge variant of Geometric-Harmonic index of Transformation graphs.

Declarations

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