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Extropy and Entropy Estimation Based on Uniformly and Log-Normal Distributed Data

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Abstract: This paper proposes non-parametric estimates for the two information measures extropy and entropy when a progressively Type-I interval censored data is available. Different non-parametric approaches are used for deriving the estimates; namely Moments, Linear, Kernel and differential Approximation methods. Some properties of the proposed estimates are studied. The performance of the proposed estimates is studied under various censoring schemes via simulation studies considering the parent distributions of the data; namely Uniform and Log-Normal distributions. The results indicate that Moments Approximation (J_1 and H_1) and Linear Approximation (J_2 and H_2) estimates for the extropy and entropy have a smaller mean squared error than competing estimates. A real data set is presented and analysed.

Keywords: Entropy; Extropy; Mean Square Error; Non-parametric statistics; Monte Carlo simulation; Type-I Interval Censoring.

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1 Introduction

Shannon entropy (1948) of a random variable (r.v.) X whose probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$, is defined as:

$$H(X) = -\int f(x)log(f(x))\,dx. \tag{1}$$

The differential extropy of X is defined by Lad et al.(2015) as:

$$J(X) = -\frac{1}{2}\int f^2(x)dx. \tag{2}$$

The problem of deriving a sample estimate for Shannon entropy has been considered by several authors in the literature, see for example Noughabi and Noughabi (2013). There have also been studies on the characterizations of Shannon entropy based on ordered data by, see for example Tahmasebi and Eskandarzadeh (2017). Furthermore, the estimation problem of certain entropy measures for a particular distribution have been discussed in literature, see for example Alam and Nassar (2023).

Important properties of the extropy measures have been discussed in the literature, for example, Qiu (2017) and Qiu and Jia (2018a) studied properties of the residual extropy and the extropy of ordered statistics. Raqab and Qiu (2019) considered properties of the extropy measure under ranked set sampling. The problem of estimating the extropy based on complete sample data has been considered by some authors, including: Qiu and Jia (2018b) and Noughabi and Jarrahiferiz (2019).

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Recently Hazeb et al.(2021a and 2021b) introduced nonparametric estimates for extropy and entropy based on progressively Type-II censored data. Our main objective in this paper is to develop different methods for estimating the extropy and entropy measures based on progressively Type-I interval censored data.

Censoring schemes of statistical experiments arise naturally in survival and reliability, and medical studies. Cohen (1963) introduced progressive Type-I censoring as an extension of Type-I censoring. Aggarwala (2001) initially discussed progressive type-I interval censoring in literature and studied an exponential distribution using this censoring. Since then this censoring scheme has attracted attention among researchers. Progressive type-I interval censoring can be briefly described as follows: Suppose n identical items are placed simultaneously on life testing at time $t_0 = 0$, where inspection is at m pre-fixed censoring times $t_1 < t_2 < ..... < t_m$, where t_m is the scheduled time to terminate the experiment and m is pre-fixed number of stops. For $i = 1,2,...,m$, let k_i be the number of failures in the interval $(t_{i-1},t_i]$. Let S_i be the number of the surviving items at t_i and R_i be the number of removed items at t_i. In this censoring scheme, k_i and S_i are random numbers while R_i is the number of remaining items, which is also a random number. At the 1^{st} inspection time t_1, we observe k_1 failures, then R_1 surviving items are randomly withdrawn from the remaining $(n-k_1)$ items. One can see that after this step, the number of remaining items is $(n-k_1-R_1)$. Now, after time t_1 and at the 2^{nd} inspection time t_2, we observe k_2 failures where R_2 are randomly removed from $(n-k_1-k_2-R_1)$ items. Lastly, at the m^{th} inspection time (the last inspection time), we observe k_m failures and all remaining $(n-\sum_{i=1}^{m} k_i - \sum_{i=1}^{m-1} R_i)$ items are immediately removed from the experiment. The observed progressive Type-I interval censored data can be represented as: $\{(k_i,R_i,t_i), i = 1,2,\ldots,m\}$. The associated likelihood function of the parameter θ under the progressive type-I interval censoring is given by:

$$L(\theta) \propto \prod_{i=1}^{m} \left[F(t_i;\theta) - F(t_{i-1};\theta)\right]^{k_i} \left[1-F(t_i;\theta)\right]^{R_i}. \tag{3}$$

Note that R_i should not be greater than S_i, where the values of R_i for $i = 1,2,...,m$ are determined based on pre-specified removal proportions $q_1, q_2, ..., q_{k-1}$ and $q_m = 1$, such that $R_i = [S_i q_i]$, for $i = 1,2,\ldots,k-1$, where symbol $[b]$ is the greatest integer less than or equal to b. It can be easily seen that $n = \sum_{i=1}^{m}(R_i + k_i)$. If $R_i = 0$, for $i = 1,2,\ldots,m-1$, then Progressive type-I interval censoring reduces to the conventional type-I censoring.

Progressive type-I interval censoring approach has been considered by different authors in the literature including Ng and Wang (2009), Lio et al.(2011), Singh and Tripathi (2016), Du et al.(2018), Alotaibi et al.(2021) and Qubbaj et al.(2023).

The next theorem considers Type-I interval censoring using an underlying lifetime distribution, namely the uniform.

Theorem 1.*Let $U_{i:m:n} = F(t_i)$, $i = 1,2,...,m$ denote a progressively Type-I interval censoring sample obtained from the uniform $(0,1)$ distribution, assuming the sample size is n with progressive Type-I interval censored data $\{(k_i,R_i,t_i), i = 1,2,\ldots,m\}$. Let*

$$U_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} V_j,$$

where,

$$V_1 = \frac{1-U_{m:m:n}}{1-U_{m-1:m:n}}, V_2 = \frac{1-U_{m-1:m:n}}{1-U_{m-2:m:n}}, ..., V_m = 1-U_{1:m:n}, \tag{4}$$

are all independent identically distributed (iid) r.v.'s. Then

$$V_i \overset{d}{=} Beta\left(i + \sum_{j=m-i+2}^{m} k_j + \sum_{j=m-i+1}^{m} R_j, k_{m-i+1}+1\right), \qquad i = 1,2,...,m. \tag{5}$$

*Proof.*For simplicity, we denote $U_{i:m:n}$ by U_i.

Since U_i follows $U(0,1)$. Then the pdf and cdf of U_i are $f_{U_i}(u) = 1$ and $F_{U_i}(u) = u$, for $0 \le u \le 1$, respectively. So, using Eq. (3), the joint pdf of $U_1 \le U_2 \le ... \le U_m$ is obtained as follows:

$$f_{U_{1:m:n},U_{2:m:n},...,U_{m:m:n}}(u_1,u_2,..,u_m) \propto \prod_{i=1}^{m} (u_i - u_{i-1})^{k_i} (1-u_i)^{R_i}. \tag{6}$$

Since $U_i = 1 - \prod_{j=m-i+1}^{m} V_j$, one can see that:

$$\begin{aligned}
U_1 &= 1 - V_m \\
U_2 &= 1 - V_{m-1}V_m \\
U_3 &= 1 - V_{m-2}V_{m-1}V_m \\
&. \\
&. \\
&. \\
U_m &= 1 - V_1V_2V_3...V_m
\end{aligned} \tag{7}$$

Hence, assuming $U_0 = 0$, we have:

$$\begin{aligned}
U_1 - U_0 &= 1 - V_m \\
U_2 - U_1 &= V_m(1 - V_{m-1}) \\
U_3 - U_2 &= V_{m-1}V_m(1 - V_{m-2}) \\
&. \\
&. \\
&. \\
U_m - U_{m-1} &= V_2V_3...V_m(1 - V_1)
\end{aligned} \tag{8}$$

The Jacobian matrix of this transformation is given by the following lower triangular matrix:

$$J = \frac{\partial}{\partial V}U = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -V_m & -V_{m-1} \\ \vdots & \vdots & \ddots & \vdots & \\ -V_2V_3...V_m & -V_1V_3...V_m & \cdots & -V_1V_2...V_{m-2}V_m & -V_1V_2...V_{m-1} \end{pmatrix}$$

Therefore,

$$|J| = \left|\frac{\partial}{\partial V}U\right| = \prod_{j=2}^{m} V_j^{j-1}. \tag{9}$$

Now, in order to derive the joint pdf of $V_1, V_2, ..., V_m$, we simplify the terms of Eq. (6) separately.

Using Eq. (8), the first term in Eq. (6) is simplified to:

$$\prod_{i=1}^{m}(U_i - U_{i-1})^{k_i} = \prod_{i=1}^{m}(1 - V_{m-i+1})^{k_i}\prod_{i=1}^{m}V_{m-i+1}^{\sum_{j=i+1}^{m}k_j}. \tag{10}$$

Using Eq. (7), the second term in Eq. (6) is simplified to:

$$\prod_{i=1}^{m}(1 - U_i)^{R_i} = \prod_{i=1}^{m}V_{m-i+1}^{\sum_{j=i}^{m}R_j}. \tag{11}$$

Accordingly, using Eqs. (9), (10) and (11), the joint pdf of $V_1, ..., V_m$ is obtained as follows:

$$f_{V_1,V_2,...,V_m}(v_1, v_2, .., v_m) \propto \prod_{i=1}^{m}(1 - V_i)^{k_m+i-1}\prod_{i=2}^{m}V_i^{\sum_{j=m-i+2}^{m}k_j}\prod_{i=1}^{m}V_i^{\sum_{j=m-i+1}^{m}R_j}\prod_{i=2}^{m}V_i^{i-1}. \tag{12}$$

Since $\prod_{j=2}^{m}V_j^{j-1} = \prod_{j=1}^{m}V_j^{j-1}$ and $\prod_{i=2}^{m}V_i^{\sum_{j=m-i+2}^{m}k_j} = \prod_{i=1}^{m}V_i^{\sum_{j=m-i+2}^{m}k_j}$,
Eq. (12) can be simplified to:

$$f_{V_1,V_2,...,V_m}(v_1, v_2, .., v_m) \propto \prod_{i=1}^{m}(1 - V_i)^{k_m+i-1}V_i^{\sum_{j=m-i+2}^{m}k_j+\sum_{j=m-i+1}^{m}R_j+i-1} \tag{13}$$

where $0 < V_i < 1$.

By factorization theorem, we see that $V_1, V_2, ..., V_m$ are independent and

$$V_i \overset{d}{=} Beta\left(i + \sum_{j=m-i+2}^{m}k_j + \sum_{j=m-i+1}^{m}R_j, k_{m-i+1} + 1\right), \qquad i = 1, 2, ..., m.$$

Corollary 1.*As a result of Theorem 1, we find*

$$E(U_{i:m:n}) = 1 - \prod_{j=m-i+1}^{m} \gamma_j, \tag{14}$$

where,

$$\gamma_i = \frac{i + \sum_{j=m-i+2}^{m} k_j + \sum_{j=m-i+1}^{m} R_j}{1 + i + \sum_{j=m-i+1}^{m} k_j + R_j},$$

$$\textit{such that } \gamma_j = \gamma_1 \textit{ if } j \le 1 \textit{ and } \gamma_j = \gamma_m \textit{ if } j \ge m \textit{ provided that } \sum_{j=m+1}^{m} k_j = 0.$$

This paper is organized as follows: Non-parametric estimation for extropy and entropy measures based on progressive Type-I interval censoring are developed in Section 2. Simulation experiments as are performed in Section 3. Analyses of real life data are performed in Section 4. Finally, we end the the paper in Section 5 with a conclusion.

2 Non-parametric Extropy and Entropy Estimates

This section develops non-parametric estimates for the extropy and entropy measures based on progressively Type-I interval censored samples. It is of importance here to mention that for a random variable (r.v.) T, extropy and entropy measures $J(T)$ and $H(T)$ are expressed as:

$$J(T) = -\frac{1}{2}\int_0^1 \left(\frac{d}{dp}F^{-1}(p)\right)^{-1} dp \tag{15}$$

and

$$H(T) = \int_0^1 log\left(\frac{d}{dp}F^{-1}(p)\right) dp \tag{16}$$

In this section we will introduce four estimates for both extropy and entropy based on Kernel and Differential Approximation methods. Recently, we introduced four estimates for the extropy and entropy (Qubbaj et al. 2023) based on Moments Approximation Method ($\hat{J}_1$, $\hat{H}_1$) and Linear Approximation Method ($\hat{J}_2$, $\hat{H}_2$) under type-I interval censoring, shown below:

$$\hat{J}_1 = -\frac{1}{2m}\sum_{i=1}^{m} \frac{\prod_{j=m-(i-w)+1}^{m} \gamma_j - \prod_{j=m-(i+w)+1}^{m} \gamma_j}{T_{i+w:m:n} - T_{i-w:m:n}}, \tag{17}$$

$$\hat{H}_1 = \frac{1}{m}\sum_{i=1}^{m} log\left(\frac{T_{i+w:m:n} - T_{i-w:m:n}}{\prod_{j=m-(i-w)+1}^{m} \gamma_j - \prod_{j=m-(i+w)+1}^{m} \gamma_j}\right), \tag{18}$$

$$\hat{J}_2 = -\frac{1}{2m}\sum_{i=1}^{m} \frac{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})(\frac{\sum_{j=i-w}^{i+w}\prod_{k=m-j+1}^{m}\gamma_k}{2w+1} - \prod_{k=m-j+1}^{m}\gamma_k)}{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})^2}, \tag{19}$$

$$\hat{H}_2 = \frac{1}{m}\sum_{i=1}^{m} log\left[\frac{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})(\frac{\sum_{j=i-w}^{i+w}\prod_{k=m-j+1}^{m}\gamma_k}{2w+1} - \prod_{k=m-j+1}^{m}\gamma_k)}{\sum_{j=i-w}^{i+w}(\frac{\sum_{j=i-w}^{i+w}\prod_{k=m-j+1}^{m}\gamma_k}{2w+1} - \prod_{k=m-j+1}^{m}\gamma_k)^2}\right]. \tag{20}$$

where w is the window size.

2.1 Kernel-Based method

Here, the extropy J is represented as

$$J(T) = -\frac{1}{2}\int f^2(t)dt = -\frac{1}{2}E_f\left(f(t)\right). \quad (21)$$

Accordingly, a third estimate of $J(T)$ follows as

$$\hat{J}_3 = -\frac{1}{2m}\sum_{i=1}^{m}\hat{f}(t_{i:m:n}), \quad (22)$$

where $\hat{f}(t_{i:m:n})$ is estimated by the kernel density estimate,

$$\hat{f}(t_{i:m:n}) = \frac{1}{md}\sum_{j=1}^{m}K\left(\frac{t_{i:m:n}-t_{j:m:n}}{d}\right), \quad (23)$$

we used $K(x)$ the kernel function that is non-negative, smooth and symmetric function which satisfies the conditions:

$$\int K(x)dx = 1, \text{ and } \int xK(x)dx = 0,$$

where d is the bandwidth such that $d > 0$, also d is called the smoothing parametr or windowwidth by some authors, (cf. Dmitriew and Tarasenko (1973)).
This estimate is designed assuming the Kernel function is the standard normal density function; due its convenient mathematical properties the normal Kernel is frequently used

$$K(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$$

There are several choices for the bandwidth d. Here the bandwidth d is decided to be $1.06Sm^{-\frac{1}{5}}$, (cf. Silverman (1986)), where S is the sample standard deviation and m is the number of points.
Similarly, the entropy $H(T)$ can also be represented by the form $-E_f\left(log f(t)\right)$. Therefore, an estimate of H is

$$\hat{H}_3 = -\frac{1}{m}\sum_{i=1}^{m}log\left(\hat{f}(t_{i:m:n})\right). \quad (24)$$

Proposition 1.*Let $Y = aT + b$, $a > 0$. Then $\hat{J_3}^Y = \frac{1}{a}\hat{J_3}^T$ and $\hat{H_3}^Y = log(a) + \hat{H_3}^T$.*

*Proof.*Since
$f(Y_i) = \frac{1}{md_Y}\sum_{j=1}^{m}K(\frac{Y_i-Y_j}{d_Y}) = \frac{1}{mad_T}\sum_{j=1}^{m}K(\frac{aT_i-aT_j}{ad_T}) = \frac{1}{a}\hat{f}(T_i)$
And
$d_Y = 1.06SmY^{\frac{-1}{5}} = 1.06SamT^{\frac{-1}{5}} = cd_T$
Thus,
${H_3}^Y = -\frac{1}{m}\sum_{i=1}^{m}log(\hat{f}(Y_i)) = -\frac{1}{m}\sum_{i=1}^{m}log(\frac{1}{a}\hat{f}(T_i)) = loga + {H_3}^T$
Also we get,
${J_3}^Y = -\frac{1}{2m}\sum_{i=1}^{m}(\hat{f}(Y_i)) = \frac{-1}{2m}\sum_{i=1}^{m}(\frac{1}{a}\hat{f}(T_i)) = {J_3}^T/a$

Proposition 2.*$\hat{J}_3$ and $\hat{H}_3$ are consistent estimates for J and H, respectively, i.e.*
$\hat{J}_3 \xrightarrow{p} J$, and $\hat{H}_3 \xrightarrow{p} H$, as $m \to n$, $n \to \infty$, $w \to \infty$ and $w/m \to 0$.

*Proof.*We have

$$\hat{J}_3 = -\frac{1}{2m}\sum_{i=1}^{m}\hat{f}(T_{i:m:n}) = -\frac{1}{2}E(\hat{f}(T)).$$

By consistency of the Kernel pdf i.e.

$$\hat{f}(t) \to f(t),$$

Therefore;

$$\hat{J}_3 = -\frac{1}{2}E(\hat{f}(T)) \xrightarrow{p} -\frac{1}{2}E(f(T)) = J_3.$$

However, consistency of $\hat{H}_3$ can be proved similarly, thus

$$\hat{H}_3 = -E(\log(\hat{f}(T))) \xrightarrow{p} -E(\log(f(T))) = H_3.$$

2.2 Differential Approximation method

The fourth estimate can be derived based on the definition of the differentiation. In precise,

$$F(t_{i+w:m:n}) - F(t_{i-w:m:n}) \approx \frac{f(t_{i+w:m:n}) + f(t_{i-w:m:n})}{2}(t_{i+w:m:n} - t_{i-w:m:n}). \tag{25}$$

Substituting Eq.(25) into the following equation

$$J(T) \approx -\frac{1}{2m}\sum_{i=1}^{m}\frac{F(T_{i+w:m:n}) - F(T_{i-w:m:n})}{T_{i+w:m:n} - T_{i-w:m:n}}. \tag{26}$$

We get

$$\hat{J}_4 = -\frac{1}{4m}\sum_{i=1}^{m}\left(\hat{f}(t_{i+w:m:n}) + \hat{f}(t_{i-w:m:n})\right), \tag{27}$$

where w is the window size and $\hat{f}(t_{i:m:n})$ is estimated by the Kernel function proposed in the previous section. Another estimate for $H(T)$ is obtained by applying similar arguments

$$\hat{H}_4 = \frac{1}{m}\sum_{i=1}^{m} log\left(\frac{2}{\hat{f}(t_{i+w:m:n}) + \hat{f}(t_{i-w:m:n})}\right). \tag{28}$$

Proposition 3.*Let $Y = aT + b$, $a > 0$. Then $\hat{J}_4^{\,Y} = \frac{1}{a}\hat{J}_4^{\,T}$ and $\hat{H}_4^{\,Y} = log(a) + \hat{H}_4^{\,T}$*

Proof.: The proof of this proposition is similar to Proposition 2.1 and therefore it is omitted.

Proposition 4.*$\hat{J}_4$ and $\hat{H}_4$ are consistent estimates for J and H, respectively, i.e.*
$\hat{J}_4 \xrightarrow{p} J$, and $\hat{H}_4 \xrightarrow{p} H$, as $m \to n$, $n \to \infty$, $w \to \infty$ and $w/m \to 0$.

*Proof.*Since

$$\frac{1}{n}\sum_{i=1}^{n}(T_i) \to E(T),$$

We have,

$$J_4 \longrightarrow -\frac{1}{2}E\frac{\hat{f}(T_{i+w}) + \hat{f}(T_{i-w})}{2} = -\frac{1}{2}E\frac{F(T_{i+w}) - F(T_{i-w})}{T_{i+w} - T_{i-w}}$$

Here T_{i-w} and T_{i+w} belong to an interval in which $f(T)$ is positive and continuous, then there exist a value $T_i' \in (T_{i-w}, T_{i+w})$ such that

$$\left(\frac{F(T_{i+w}) - F(T_{i-w})}{T_{i+w} - T_{i-w}}\right) = \hat{f}(T_i')$$

Therefore,

$$J_4 \xrightarrow{p} J,$$

Now, by continuity of log function we have

$$\log(\hat{f}(t))) \longrightarrow \log(f(t)))$$

Then, by law of large numbers

$$H_4 = -E(\log(\hat{f}(T))) \xrightarrow{p} -E(\log(f(T))) = H.$$

and so the consistency of H_4 is proved.

3 Simulation study

In this section, we carry out a Monte Carlo simulation to analyse the behavior of our proposed estimates of extropy and entropy. In order to perform the simulation process, we consider different sample sizes, i.e. $n = 10, 20, 30$ and 50 with five different inspection times, i.e. $m = 5$. Next, we generate 1000 progressive type-I interval censoring data sets in each experimentation case. Also, we work with different withdrawal (removal) schemes as detailed in Table 1. We consider the uniform distribution $U(0,\theta)$ with $\theta = 1$ and the Log-Normal Distribution (μ,σ) with $\mu = 0, \sigma = 1$, which are commonly used.

Table 1: Progressive interval censoring schemes used in the Monte Carlo simulation study

Scheme No.	m	$(t_1,\ldots,t_m)$	$(q_1,\ldots,q_m)$
1	5	$0.1,0.3,0.5,0.7,0.9$	$0.25,0,0,0,1$
2	5	$0.1,0.3,0.5,0.7,0.9$	$0,0,0,0,1$
3	5	$0.1,0.3,0.5,0.7,0.9$	$0.25,0.25,0,0,1$
4	5	$0.1,0.3,0.5,0.7,0.9$	$0.1,0.1,0.2,0.2,1$

Table 2: MSEs of the Estimates of extropy $J(Y)$ and $H(Y)$ for $U(0,1)$, assuming $m = 5$

Scheme Number	n	J_1	J_2	J_3	J_4	H_1	H_2	H_3	H_4
1	10	0.0039	**0.0038**	0.0673	0.0575	**0.0145**	0.0158	0.0958	0.1025
	20	0.0038	**0.0031**	0.0228	0.0216	**0.0136**	0.0142	0.0373	0.0443
	30	0.0037	**0.0025**	0.0122	0.0119	**0.0123**	**0.0123**	0.0233	0.0263
	50	0.0030	**0.0014**	0.0047	0.0046	0.0063	**0.0056**	0.0109	0.0114
2	10	**0.0038**	**0.0038**	0.0343	0.0298	**0.0155**	0.0163	0.0589	0.0677
	20	0.0030	**0.0025**	0.0172	0.0165	**0.0101**	0.0104	0.0293	0.0347
	30	0.0028	**0.0018**	0.0099	0.0097	**0.0075**	**0.0075**	0.0184	0.0208
	50	0.0023	**0.0009**	0.0076	0.0075	0.0035	**0.0030**	0.0145	0.0156
3	10	**0.0045**	0.0048	0.0712	0.0607	**0.0178**	0.0201	0.1096	0.1137
	20	0.0042	**0.0035**	0.0195	0.0179	**0.0155**	0.0162	0.0364	0.0411
	30	0.0040	**0.0030**	0.0141	0.0137	0.0145	**0.0143**	0.0264	0.0299
	50	0.0036	**0.0021**	0.0074	0.0073	0.0107	**0.0096**	0.0142	0.0157
4	10	**0.0043**	0.0045	0.0567	0.0491	**0.0183**	0.0199	0.0799	0.0871
	20	**0.0036**	0.0039	0.0211	0.0197	**0.0175**	0.0184	0.0351	0.0417
	30	**0.0029**	0.0031	0.0165	0.0160	**0.0134**	0.0138	0.0263	0.0311
	50	**0.0024**	0.0025	0.0104	0.0103	0.0118	**0.0117**	0.0166	0.0190

Table 3: MSEs of the Estimates of $J(Y)$ and $H(Y)$ for Log-Normal Distribution $(0,1)$, assuming $m = 5$

Scheme Number	n	J_1	J_2	J_3	J_4	H_1	H_2	H_3	H_4
1	10	**0.0020**	0.0030	0.0198	0.0187	0.0465	**0.0436**	0.8745	0.4858
	20	**0.0018**	0.0028	0.0084	0.0080	0.0412	**0.0407**	0.8387	0.3528
	30	**0.0015**	0.0025	0.0064	0.0061	0.0366	**0.0341**	0.8287	0.3301
	50	**0.0014**	0.0024	0.0034	0.0033	0.0336	**0.0299**	0.8273	0.3013
2	10	**0.0020**	0.0029	0.0123	0.0114	**0.0385**	0.0410	0.8608	0.4420
	20	**0.0015**	0.0024	0.0054	0.0052	0.0377	**0.0375**	0.8364	0.3032
	30	**0.0013**	0.0024	0.0035	0.0034	0.0319	**0.0299**	0.8216	0.2866
	50	**0.0012**	0.0022	0.0027	0.0027	0.0290	**0.0240**	0.8112	0.2659

Scheme Number	n	J_1	J_2	J_3	J_4	H_1	H_2	H_3	H_4
3	10	**0.0020**	0.0030	0.0349	0.0296	**0.0344**	0.0347	0.9067	0.5024
	20	**0.0018**	0.0028	0.0124	0.0118	0.0333	**0.0303**	0.8592	0.3656
	30	**0.0015**	0.0024	0.0039	0.0038	0.0327	**0.0273**	0.8295	0.3338
	50	**0.0014**	0.0023	0.0028	0.0027	0.0313	**0.0255**	0.7652	0.2663
4	10	**0.0024**	0.0035	0.0190	0.0171	**0.0377**	0.0399	0.9721	0.4914
	20	**0.0022**	0.0032	0.0117	0.0114	0.0376	**0.0357**	0.9679	0.4130
	30	**0.0021**	0.0032	0.0071	0.0069	0.0306	**0.0285**	0.9452	0.3415
	50	**0.0020**	0.0031	0.0030	0.0028	0.0264	**0.0225**	0.8861	0.3125

For each generated data set, we compute the average estimate and the corresponding mean squared error (MSE) of the proposed extropy and entropy estimates over 1000 simulations. Simulation results are shown in Tables (2-3), where the bold type in these tables indicates the estimate achieving the minimal MSE.

Now, for Table (2), we observe that for scheme 1, estimate J_2 dominates other estimates for all sample sizes; while for scheme 2 we see that the MSEs of estimate J_2 are always smaller than those of other estimates except when $n = 10$, where J_1 and J_2 are equal. As for scheme 3, we see that MSEs of estimate J_2 are always smaller than those of other estimates except when $n = 10$. On the other hand, in scheme 4, we see that MSEs of estimate J_1 are always smaller than those of other estimates. Also from Table (2), we observe that for schemes 1 and 2, estimates of H_1 and H_2 perform satisfactorily and we also see that MSEs of estimate H_1 are always smaller than those of other estimates except for $n = 50$ and they are equal when $n = 30$. As for scheme 3, we see that MSEs of estimate H_2 are always smaller than those of other estimates except when $n = 10$ and $n = 20$. On the other hand, in scheme 4 we see that MSE's of estimate H_1 are always smaller than those of other estimates except for $n = 50$. Accordingly, the reader can observe that if the data comes from uniform distribution, then estimates J_1, J_2, H_1 and H_2 mostly perform better than other estimates for estimating extropy and entropy.

As shown in Table (3), estimate J_1 dominates other estimates under all censoring schemes for all sample sizes. Also from Table (3), we observe that for scheme 1, estimate H_2 dominates other estimates under all censoring schemes for all sample sizes. As for schemes 2, 3 and 4 in same table, MSEs of estimate H_2 are always smaller than those of other estimates except when $n = 10$ MSEs of estimate H_1 are the smallest. Accordingly, for estimating the extropy and the entropy, one might recommend estimates J_1, H_1 and H_2.

Generally, choosing the best extropy and entropy estimate depends on the sample size, censoring schemes and the type of distribution of data. And as expected, MSE decreases as the sample size n increases.

It is worth mentioning that, Moments Approximation estimates (J_1 and H_1) and Linear Approximation estimates (J_2 and H_2) for extropy and entropy perform better than the other estimates in most considered cases.

4 Real Data Analysis

In this section, we present an example to show the behaviour of the proposed extropy and entropy estimates in real case.
Example : The following data, represents smiling times of an eight-week old baby measured in seconds, which can be treated as independent observations of the random variable X:

```
0.7   1.3   2.1   2.6   3.3   3.4   3.7   4.5   4.9
5.8   5.8   5.9   6.3   6.7   6.9   7.3   7.6   7.8
8.9   8.9   9.4   9.4   9.8   10.0  10.4  10.7  10.9
11.1  11.6  11.8  11.9  12.5  12.8  13.4  13.8  13.9
14.5  14.8  15.9  16.3  16.8  17.1  17.8  17.9  17.9
18.6  18.8  19.0  19.2  19.6  20.0  21.6  21.7  22.8  22.8
```

This data was given in Illowsky and Dean(2018) and it was shown that it follows a $U(0,23)$ distribution. Using the integral transformation, we consider the transformed data $Y = X/23$ which follows a $U(0,1)$ distribution, the transformed data is as follows:

```
0.030  0.056  0.091  0.113  0.143  0.148  0.161  0.196   0.213
0.252  0.252  0.257  0.274  0.291  0.030  0.317  0.330   0.339
0.387  0.387  0.409  0.409  0.426  0.435  0.452  0.465   0.474
0.483  0.504  0.513  0.517  0.543  0.556  0.583  0.600   0.604
0.630  0.643  0.691  0.709  0.730  0.743  0.774  0.778   0.778
0.809  0.817  0.826  0.835  0.852  0.870  0.939  0.943   0.991  0.991
```

The MLE of θ based on the complete sample is equal to $\hat{\theta}$ which is considered to be 1. The extropy and entropy of X in this case are obtained to be

$$J(X) = -\frac{1}{2\theta}, \text{ and } H(X) = \log\ \theta,$$

respectively.

Upon using the MLE of θ, we can compute the MLEs of the entropy and extropy measures as $J(X) = -0.5$ and $H(X) = 0$, respectively. Now we shall study the behaviour of the proposed estimates based on the following progressive Type-I Interval censoring schemes in the following table and associated censoring samples, notice that $m = 3$ and $w = 1$ are considered fixed for all suggested censoring schemes:

Table 4: Progressive censoring schemes used in this real data example

Censoring scheme No.	(t_1,t_2,t_3)	(q_1,q_2,q_3)
1	$0.1, 0.5, 1$	$0.25, 0, 1$
2	$0.3, 0.5, 0.95$	$0, 0, 1$
3	$0.01, 0.4, 1.2$	$0, 0.25, 1$
4	$0.7, 0.8, 0.9$	$0.3, 0.3, 1$
5	$0.05, 0.3, 0.7$	$0.25, 0.25, 1$

Continuing with the exploration of progressive Type-I Interval censoring under this lifetime model, the following censored data is observed according to the applied censoring scheme on the insulation data. The generated censored data are summarized in the Tables (5-9).

Table 5: The observed censored data from scheme 1

Inspection times	Number of failures	Number of removals
$(0, 0.1]$	3	13
$(0.1, 0.5]$	18	0
$(0.5, 1]$	21	0

Table 6: The observed censored data from scheme 2

Inspection times	Number of failures	Number of removals
$(0, 0.3]$	15	0
$(0.3, 0.5]$	13	0
$(0.5, 0.95]$	25	2

Table 7: The observed censored data from scheme 3

Inspection times	Number of failures	Number of removals
$(0, 0.01]$	0	0
$(0.01, 0.4]$	20	9
$(0.4, 1.2]$	26	0

Table 8: The observed censored data from scheme 4

Inspection times	Number of failures	Number of removals
$(0, 0.15]$	6	5
$(0.15, 0.45]$	17	3
$(0.45, 0.95]$	23	1

Table 9: The observed censored data from scheme 5

Inspection times	Number of failures	Number of removals
$(0, 0.05]$	1	14
$(0.05, 0.3]$	10	8
$(0.3, 0.7]$	15	7

Tables (10) and (11) show that the results are in agreement with what have been concluded from the simulation studies. Precisely, the extropy and entropy estimates J_1, J_2, H_1 and H_2 provide closer estimation results to those are obtained using the complete sample based on the MLEs $J(X) = -0.5$ and $H(X) = 0$, respectively.

Table 10: Extropy Estimates for the example

Scheme Number	J_1	J_2	J_3	J_4
1	-0.5015	-0.5013	-0.6872	-0.6872
2	-0.5237	-0.5117	-0.6690	-0.6690
3	-0.4203	-0.4153	-0.6044	-0.6044
4	-0.5241	-0.5195	-0.6363	-0.6363
5	-0.6086	-0.5794	-0.9427	-0.9427

Table 11: Entropy Estimates for the example

Scheme Number	H_1	H_2	H_3	H_4
1	-0.0029	-0.0045	0.2462	-0.2626
2	-0.0454	-0.0281	0.2743	-0.2751
3	0.1763	0.1874	0.1814	-0.1817
4	-0.0466	-0.0420	0.1835	-0.1881
5	-0.1714	-0.1564	0.5679	-0.5999

5 Conclusions

In this paper, we have studied the estimation problem of the extropy and entropy measures based on progressive Type-I interval censoring. Non-parametric based methods involving moments approximation, linear approximation, Kernel-based and differential principal have been discussed. It is obvious that choosing the best estimates of the extropy and entropy measures depend on the parent model of the data, sample size and censoring scheme. As expected, MSE decreases as the sample size n increases. It has been noticed that, the Moments Approximation (J_1 and H_1) and Linear Approximation (J_2 and H_2) estimates for the extropy and entropy compete the other estimates in most considered cases.

Declarations

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