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On The Possibility of Mikusiński's Operational Calculus For The Fractional Laplacian

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Abstract: We try to adapt the framework of Mikusiński's operational calculus in order to apply it to differential equations involving the fractional Laplacian. We find a fundamental theorem of calculus which connects the fractional Laplacian with its inverse in general, even when no decay conditions are imposed. Ultimately, we discover that no suitable function space exists satisfying all the required conditions, and so the method of Mikusiński, in the sense that it has previously been used for fractional operators, cannot be applied to the fractional Laplacian.

Keywords: fractional Laplacian; Mikusiński's operational calculus; fundamental theorem of fractional calculus.

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1 Introduction

Operational calculus, in a non-rigorous form involving the algebraic manipulation of integro-differential operators to solve equations correctly, dates back to Heaviside [31]. The rigorous formulation of an operational calculus method for solving integro-differential equations was done by the Polish mathematician Jan Mikusiński in the 1950s, and this method is now called Mikusiński's operational calculus. It has been used to solve ordinary differential equations [28], partial differential equations [18], integral equations [17], and fractional differential equations using assorted fractional-order operators: Riemann–Liouville [19], Caputo [26], Erdélyi–Kober [24], Hilfer [20], with respect to functions [13], Prabhakar [15], and Sonine kernels [25].

In pure mathematical analysis, one of the most important fractional-order operators is considered to be the fractional Laplacian, due to its global nature and easy extension to arbitrary dimensions, as well as its natural relationship with the Fourier transform [21,23]. This operator has been studied by many authors and used in fractional partial differential equations [23] which have been solved by methods such as Caffarelli–Silvestre extension [7], weak solutions [12,36], perturbation methods [2], eigenvalue methods [11], etc. Note that this is unrelated to the fractional p-Laplacian, which is a different operator [34].

In the 1960s, first in a doctoral thesis [29] and then in a published paper [30], Norris considered the issue of defining a topology on the Mikusiński field. In the process of his work, he used a function space consisting of continuous functions $f:\mathbb{R}\to\mathbb{C}$ whose support is bounded below, i.e. such that there exists $X\in\mathbb{R}$ with $f(x)=0$ for all $t<X$. This is the closest attempt, in the literature so far, to obtain a Mikusiński-style operational calculus for functions defined on the whole real line. However, given the condition imposed on this function space, Norris's work essentially deals with functions defined on a half-line $[X,\infty)$. This is not the same as the half-line $[0,\infty)$ of Mikusiński, but recent work has shown [14] that conjugation relations allow Mikusiński's operational calculus to be easily extended to any half-line $[X,\infty)$ in the same way as the original $[0,\infty)$.

So far, it seems that the abstract algebraic structure of Mikusiński's operational calculus has not been applied to global fractional-order operators such as the fractional Laplacian, acting on the whole line $(-\infty,\infty)$. We seek to investigate whether these two topics can be combined, applying Mikusiński's algebraic methodology to differential

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equations involving the fractional Laplacian. Our approach will be to examine the structure of Mikusiński's method and the conditions that must be satisfied in order for it to be usable, and then either to construct the necessary mathematical framework or to prove that it cannot exist.

The structure of our manuscript is as follows. In Section 2, we provide the theoretical basics for both Mikusiński's operational calculus (in Subsection 2.1) and the fractional Laplacian operator (in Subsection 2.2). The main results are contained in Section 3, where we first prove a suitable inversion relation for the fractional Laplacian (in Subsection 3.1) and then consider what function space requirements will be necessary for applying the usual Mikusiński's operational calculus (in Subsection 3.2), before finally proving that such requirements are impossible to fulfil (in Subsection 3.3). The work concludes in Section 4 with some discussion of the impact and significance of our work.

2 Preliminaries

2.1 Mikusiński's Operational Calculus

In general, the methodology of Mikusiński's operational calculus is along the following lines.

1. Choose a suitable function space C which forms a commutative ring or rng (ring without identity), under pointwise addition and some type of convolution operator for multiplication.
2. Choose a suitable integration operator $\mathscr{I}$ which can be identified with multiplication by some element in C, namely $\mathscr{I}(f) = f * h$ for all $f \in C$ and for some fixed $h \in C$.
3. Find a suitable inversion relation between $\mathscr{I}$ and a suitable differentiation operator $\mathscr{D}$, a sort of second fundamental theorem of calculus connecting these two operators.
4. Construct the field of fractions M from the rng C, and use the inversion relation from step 3 to interpret the differentiation operator $\mathscr{D}$ algebraically in terms of the inverse $h^{-1} \in M$ of the element $h \in C \subset M$.

After following these steps, where in step 4 embedding C into a field M requires that the ring or rng C has no zero divisors, we have an abstract field M in which algebraic manipulations can be performed, and an algebraic interpretation via h^{-1} of the differentiation operator $\mathscr{D}$. This enables differential equations involving $\mathscr{D}$ to be interpreted and solved purely algebraically. Note that M is not a function space: its elements are, in general, algebraic abstractions which cannot necessarily be interpreted as functions. The elements of M are a sort of "generalised functions" (called "operators" by Mikusiński [28]), although not the same sort as those used in distribution theory. For example, the identity element in M can be thought of as a sort of Dirac delta, since it has the property that its convolution with any function $f \in H$ is again f. In cases where C has zero divisors, such as the space of continuous functions on a finite interval [10, §1.1.1], Mikusiński's operational calculus may also be performed but the construction and properties are necessarily slightly different, since a ring or rng with zero divisors cannot be embedded into a field.

We now briefly describe how the above methodology has been applied in some particular settings.

In the classical theory of Mikusiński for ordinary differential equations [28], the space C is taken to be simply the space $C[0,\infty)$ of continuous functions on the closed half-line. By a theorem of Titchmarsh, this forms a commutative rng without zero divisors under the operations of pointwise addition and the following convolution:

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau. \tag{1}$$

The "integration element" h is then the constant function $\ell = 1$, since $(\ell * f)(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$ for all $f \in C[0,\infty)$. The algebraic inverse element $s = \ell^{-1}$ is then related to differentiation by the relation

$$s * f = f' + f(0), \tag{2}$$

which follows immediately from the fundamental theorem of calculus. Thus, ordinary differential equations can be solved by using (2) to transform them to algebraic equations in the field M and solving those equations algebraically.

In the theory developed by Luchko and his collaborators for fractional differential equations of Riemann–Liouville, Caputo, and Hilfer types, the space C is taken to be the space C_{-1} defined by Dimovski [9] as follows:

$$C_{-1} = \Big\{ \text{functions } f(t),\ t > 0, \text{ s.t. } \exists\, p > -1 \text{ and } f_1 \in C[0,\infty) \text{ with } f(t) = t^p f_1(t) \Big\}.$$

This forms a commutative rng without zero divisors under the operations of pointwise addition and the convolution defined in (1). The "integration element" $h \in C_{-1}$ corresponding to the Riemann–Liouville fractional integral is the

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function $h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\alpha > 0$ is the fractional order. The algebraic inverse element $S_\alpha = h_\alpha^{-1}$ is then related to fractional differentiation by relations analogous to (2), essentially fractional fundamental theorems of calculus. Thus, many fractional differential equations can be solved by transforming them to algebraic equations in the field M and solving those equations algebraically.

The ideas of Mikusiński have also been applied in other, more general, settings: for example, by Luchko for operators with Sonine kernels [25] and for 1st-level fractional derivatives [1], and by the second author for fractional derivatives with respect to functions [13] and for Prabhakar fractional derivatives [15].

2.2 Fractional Laplacian

The fractional Laplacian is a global operator which is frequently used in the study of partial differential equations due to its natural relationship with the Fourier transform. It has an advantage over other fractional operators (Riemann–Liouville, Caputo, etc.) in that it can be easily extended to arbitrary dimensions. There are multiple ways to define the fractional Laplacian operator in the entire n-dimensional space $\mathbb{R}^n$, for any $n \in \mathbb{N}$, and when dealing with an open subset $\Omega \neq \mathbb{R}^n$, different versions of the fractional Laplacian can be defined, taking into account information from both the boundary and the exterior of the domain.

An excellent review of Kwaśnicki [21] details ten different definitions of the fractional Laplacian operator, which are all established to be equivalent. To gain a deeper understanding of research on fractional Laplacian operators, the reader is referred to [22,5,6] for further study. These publications provide fundamental estimates and some regularity results for solutions to problems governed by $(-\Delta)^s$. Additionally, the reference [8] presents a characterisation of the realisation of the fractional Laplacian operator in $\mathbb{L}^2(\Omega)$ using different exterior conditions, employing the theory of semigroups.

However, in this research, we restrict ourselves to considering the fractional Laplacian in one dimension, for functions defined everywhere in $\mathbb{R}$. This will be enough to draw conclusions about the applicability of Mikusiński's operational calculus to fractional Laplacian operators, and it simplifies the work by not requiring the consideration of higher dimensions.

Definition 1. *Let $0 < \alpha < 2$ and let $\Delta = \frac{\partial^2}{\partial x^2}$ denote the classical Laplacian operator in one dimension. The fractional Laplacian of a function $u : \mathbb{R} \longrightarrow \mathbb{R}$ with sufficient regularity is given as:*

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{1,\frac{\alpha}{2}} P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+\alpha}} \, \mathrm{d}y, \tag{3}$$

where P.V. denotes the Cauchy principal value and $C_{1,\frac{\alpha}{2}}$ is a normalisation constant given by

$$C_{1,\frac{\alpha}{2}} = \alpha \frac{2^{\alpha-1}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}.$$

Equivalently, letting $\alpha = 2s$, we can rewrite (3) as follows:

$$(-\Delta)^s u(x) = C_{1,s} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+2s}} \, \mathrm{d}y, \tag{4}$$

where $0 < s < 1$ and $C_{1,s}$ is a normalization constant given by:

$$C_{1,s} = \frac{4^s}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+2s}{2}\right)}{|\Gamma(-s)|}.$$

The motivation behind this definition, and behind the odd choice of normalisation constant, lies in the Fourier transform. It is well known that the Fourier transform of the Laplacian is given by

$$\mathscr{F}[\Delta u](k) = -k^2 \mathscr{F}[u](k) = -|k|^2 \mathscr{F}[u](k),$$

and the latter formula also extends to higher dimensions. The fractional Laplacian is notated by writing a fractional power of $(-\Delta)$, and this makes sense when we discover that its Fourier transform is given by

$$\mathscr{F}[(-\Delta)^s u](k) = |k|^{2s} \mathscr{F}[u].$$

In this sense, the fractional Laplacian (3)–(4) is the only natural way to define a fractional power of the Laplacian operator. Considering Fourier transforms and multipliers is of course the genesis of the theory of pseudo-differential operators [33], of which the fractional Laplacian is a special case.

Definition 2. *Let $0 < s < \frac{1}{2}$ and $\Delta = \frac{\partial^2}{\partial x^2}$ as before. The inverse fractional Laplacian $(-\Delta)^{-s}$ of a function $u : \mathbb{R} \longrightarrow \mathbb{R}$ with sufficient regularity is given as:*

$$(-\Delta)^{-s}u(x) = \frac{\Gamma\left(\frac{1-2s}{2}\right)}{4^s\sqrt{\pi}\Gamma(s)} \int_{\mathbb{R}} \frac{u(y)}{|x-y|^{1-2s}} dy. \tag{5}$$

The meaning of "inverse Laplacian" can be justified by the fact [37] that the operators in the above two definitions are inverse to each other on certain function spaces, assuming conditions such as decay at infinity for the function u. However, in the work below, we will find a different kind of inversion relation which is valid under weaker assumptions on u.

3 Main Results

Recalling the outline of Mikusiński's operational calculus given in Subsection 2.1, we see that there are several choices that must be fixed in order to construct an operational calculus for a particular operator. We must choose a function space, a suitable convolution operation, an integration operator, and a differentiation operator, in such a way that the function space is closed under convolution, the integration operator is given by convolution with some fixed function, and the differentiation and integration operators obey a fundamental theorem of calculus together.

Here, the differentiation operator should be the fractional Laplacian (4), so it makes sense for the integration operator to be the inverse fractional Laplacian (5), as long as we can construct a suitable fundamental theorem of calculus connecting these two operators.

Since we are working on the line $\mathbb{R}$ rather than the half-line $[0,\infty)$, the convolution operation (1) is no longer the right one to use: we use instead the Fourier-type convolution, defined by

$$(f \star g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y. \tag{6}$$

With this type of convolution, we can interpret the inverse fractional Laplacian (5) as follows:

$$(-\Delta)^{-s}u(x) = u(x) \star \left(\frac{K}{|x|^{1-2s}}\right) = \frac{\Gamma\left(\frac{1-2s}{2}\right)}{4^s\sqrt{\pi}\Gamma(s)} \int_{\mathbb{R}} \frac{u(y)}{|x-y|^{1-2s}}\,\mathrm{d}y,$$

where $0 < s < \frac{1}{2}$ and $K = \frac{\Gamma\left(\frac{1-2s}{2}\right)}{4^s\sqrt{\pi}\Gamma(s)}$ and u has sufficient regularity.

Now we have operators of convolution, integration, and differentiation, where integration is given by convolution with a fixed function. It remains to establish a fundamental theorem of calculus and to find a suitable function space.

3.1 Inversion Relation

The inversion relation, or fundamental theorem of calculus property, connecting the fractional Laplacian (4) and the inverse fractional Laplacian (5), is given in general by the following result.

Theorem 1. *Let $u \in C^1(\mathbb{R})$ and $0 < s < \frac{1}{2}$. Then,*

$$(-\Delta)^{-s}(-\Delta)^{s}u(x) = u(x) - \frac{u(-\infty)+u(\infty)}{2},$$

for all $x \in \mathbb{R}$, where $u(\pm\infty)$ denotes $\lim_{x\to\pm\infty} u(x)$ respectively.

Proof. Directly from the definitions (4)–(5), we have:

$$\begin{aligned}(-\Delta)^{-s}(-\Delta)^{s}u(x) &= \frac{s\Gamma\left(\frac{1-2s}{2}\right)\Gamma\left(\frac{1+2s}{2}\right)}{\pi\Gamma(s)\Gamma(1-s)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{u(y)-u(z)}{|x-y|^{1-2s}|y-z|^{1+2s}}\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{s}{\pi}\tan(s\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{z}^{y} u'(w)|x-y|^{2s-1}|y-z|^{-2s-1}\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}z,\end{aligned} \tag{7}$$

where the fundamental theorem of calculus has been used to obtain the third (innermost) integral, and the constant outside the integrals has been manipulated using the reflection formula for the gamma function:

$$\frac{\Gamma\left(\frac{1-2s}{2}\right)\Gamma\left(\frac{1+2s}{2}\right)}{\Gamma(s)\Gamma(1-s)}=\frac{\sin(\pi s)}{\sin\left(\frac{\pi}{2}(1-2s)\right)}=\frac{\sin\pi s}{\cos\pi s}.$$

The outer two integrals in (7) are over $(y,z)\in\mathbb{R}^2$, which we split into four subsets as follows:

$$\begin{aligned}
z<y<x &\implies |x-y|=x-y, \quad |y-z|=y-z;\\
y<z<x, y<x<z &\implies |x-y|=x-y, \quad |y-z|=z-y;\\
z<x<y, x<z<y &\implies |x-y|=y-x, \quad |y-z|=y-z;\\
x<y<z &\implies |x-y|=y-x, \quad |y-z|=z-y.
\end{aligned}$$

These four subsets give rise to four separate triple integrals, which we now evaluate one by one. Note that we ignore the outer factor of $\frac{s}{\pi}\tan(s\pi)$ in (7), which will be reintroduced later.

First case: $z<y<x$. Here, we have $z<w<y<x$. We put the z-integral inside and the w-integral outside:

$$\begin{aligned}
\int\!\!\int\!\!\int &= \int_{-\infty}^{x} u'(w)\int_{w}^{x}(x-y)^{2s-1}\int_{-\infty}^{w}(y-z)^{-2s-1}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}w\\
&= \int_{-\infty}^{x}\frac{1}{2s}u'(w)\int_{w}^{x}(x-y)^{2s-1}(y-w)^{-2s}\,\mathrm{d}y\,\mathrm{d}w\\
&= \int_{-\infty}^{x}\frac{1}{2s}u'(w)\int_{0}^{1}(1-v)^{2s-1}v^{-2s}(x-w)^{2s-1-2s+1}\,\mathrm{d}v\,\mathrm{d}w\\
&= \int_{-\infty}^{x}\frac{1}{2s}u'(w)B(2s,1-2s)\,\mathrm{d}w\\
&= \frac{\Gamma(2s)\Gamma(1-2s)}{2s}\int_{-\infty}^{x}u'(w)\,\mathrm{d}w\\
&= \frac{\pi}{2s\sin 2\pi s}\left(u(x)-\lim_{w\to-\infty}u(w)\right),
\end{aligned}\tag{8}$$

where in the third line we used the substitution $v=\frac{y-w}{x-w}$.

Second case: $y<z<x$ or $y<x<z$. Here, we have both $y<x$ and also $y<w<z$, meaning it makes more sense to consider $u(y)-u(z)=\int_y^z -u'(w)\,\mathrm{d}w$ rather than $\int_z^y u'(w)\,\mathrm{d}w$ as written in (7). Thus, in this case,

$$\begin{aligned}
\int\!\!\int\!\!\int &= \int_{-\infty}^{\infty}-u'(w)\int_{-\infty}^{\min(w,x)}(x-y)^{2s-1}\int_{w}^{\infty}(z-y)^{-2s-1}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}w\\
&= \int_{-\infty}^{\infty}\frac{-1}{2s}u'(w)\int_{-\infty}^{\min(w,x)}(x-y)^{2s-1}(w-y)^{-2s}\,\mathrm{d}y\,\mathrm{d}w.
\end{aligned}\tag{9}$$

Third case: $z<x<y$ or $x<z<y$. Here, we have both $x<y$ and also $z<w<y$, meaning that we start from (7) in an unaltered form:

$$\begin{aligned}
\int\!\!\int\!\!\int &= \int_{-\infty}^{\infty}u'(w)\int_{\max(w,x)}^{\infty}(y-x)^{2s-1}\int_{-\infty}^{w}(y-z)^{-2s-1}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}w\\
&= \int_{-\infty}^{\infty}\frac{1}{2s}u'(w)\int_{\max(w,x)}^{\infty}(y-x)^{2s-1}(y-w)^{-2s}\,\mathrm{d}y\,\mathrm{d}w\\
&= \int_{-\infty}^{\infty}\frac{1}{2s}u'(w)\int_{-\infty}^{\min(w,x)}(w-y')^{2s-1}(x-y')^{-2s}\,\mathrm{d}y'\,\mathrm{d}w,
\end{aligned}\tag{10}$$

where in the last step we have substituted $y'=x+w-y$ in order to make this expression look more similar to (9).

Second and third cases together: $y<z<x$ or $y<x<z$ or $z<x<y$ or $x<z<y$. Combining the expressions (9) and (10) together, we obtain an expression that converges:

$$\begin{aligned}\iiint &= \int_{-\infty}^{\infty}\frac{1}{2s}u'(w)\int_{-\infty}^{\min(w,x)}\left((w-y)^{2s-1}(x-y)^{-2s}-(x-y)^{2s-1}(w-y)^{-2s}\right)\mathrm{d}y\,\mathrm{d}w\\ &= \int_{-\infty}^{x}\frac{1}{2s}u'(w)\int_{-\infty}^{w}\left((w-y)^{2s-1}(x-y)^{-2s}-(x-y)^{2s-1}(w-y)^{-2s}\right)\mathrm{d}y\,\mathrm{d}w\\ &\quad+\int_{x}^{\infty}\frac{1}{2s}u'(w)\int_{-\infty}^{x}\left((w-y)^{2s-1}(x-y)^{-2s}-(x-y)^{2s-1}(w-y)^{-2s}\right)\mathrm{d}y\,\mathrm{d}w,\end{aligned}$$

where we have split the double integral into the two cases $w<x$ and $w>x$ in order to resolve the $\min(w,x)$ more simply. Then, substituting $v=\frac{x-y}{x-w}$ in the first one and $v=\frac{w-y}{w-x}$ in the second one, we get:

$$\begin{aligned}\iiint &= \int_{-\infty}^{x}\frac{1}{2s}u'(w)\int_{\infty}^{1}\left[(v-1)^{2s-1}v^{-2s}-v^{2s-1}(v-1)^{-2s}\right](x-w)^{2s-1-2s}(-(x-w))\,\mathrm{d}v\,\mathrm{d}w\\ &\quad+\int_{x}^{\infty}\frac{1}{2s}u'(w)\int_{\infty}^{1}\left[(v-1)^{-2s}v^{2s-1}-v^{-2s}(v-1)^{2s-1}\right](w-x)^{2s-1-2s}(-(w-x))\,\mathrm{d}v\,\mathrm{d}w\\ &= \int_{-\infty}^{x}\frac{1}{2s}u'(w)\int_{1}^{\infty}\left[(v-1)^{2s-1}v^{-2s}-v^{2s-1}(v-1)^{-2s}\right]\mathrm{d}v\,\mathrm{d}w\\ &\quad+\int_{x}^{\infty}\frac{1}{2s}u'(w)\int_{1}^{\infty}\left[(v-1)^{-2s}v^{2s-1}-v^{-2s}(v-1)^{2s-1}\right]\mathrm{d}v\,\mathrm{d}w.\end{aligned}\tag{11}$$

At this point, it is enough to find one of the inner integrals (which are both the same upon replacing s by $\frac{1}{2}-s$). Writing $\mathbb{I}=\int_1^\infty\left[(v-1)^{-2s}v^{2s-1}-v^{-2s}(v-1)^{2s-1}\right]\mathrm{d}v$, and substituting $t=1-\frac{1}{v}$, we have:

$$\begin{aligned}\mathbb{I} &= \int_{1}^{\infty}\frac{\left(1-\frac{1}{v}\right)^{-2s}-\left(1-\frac{1}{v}\right)^{2s-1}}{v}\,\mathrm{d}v\\ &= \int_{0}^{1}\frac{t^{-2s}-t^{2s-1}}{\frac{1}{1-t}}\cdot\frac{\mathrm{d}t}{(1-t)^2}\\ &= \int_{0}^{1}\frac{t^{-2s}-t^{1-2s}}{1-t}\,\mathrm{d}t\\ &= \int_{0}^{1}\frac{t^{-2s}-1+1-t^{1-2s}}{1-t}\,\mathrm{d}t\\ &= \int_{0}^{1}\frac{1-t^{1-2s}}{1-t}\,\mathrm{d}t-\int_{0}^{1}\frac{1-t^{-2s}}{1-t}\,\mathrm{d}t\\ &= \Psi(2s)+\gamma-[\Psi(1-2s)+\gamma]=\Psi(2s)-\Psi(1-2s),\end{aligned}$$

where Ψ is the digamma function and we have used the well-known fact that $\Psi(z)+\gamma=\int_0^1\frac{1-x^{z-1}}{1-x}\,\mathrm{d}x$.

Substituting the above evaluation of $\mathbb{I}$ into (11), we have:

$$\begin{aligned}\iiint &= \int_{-\infty}^{x}\frac{1}{2s}u'(w)\left[\Psi(1-2s)-\Psi(2s)\right]\mathrm{d}w+\int_{x}^{\infty}\frac{1}{2s}u'(w)\left[\Psi(2s)-\Psi(1-2s)\right]\mathrm{d}w\\ &= \frac{\Psi(1-2s)-\Psi(2s)}{2s}\left[u(x)-\lim_{w\to-\infty}u(w)\right]+\frac{\Psi(2s)-\Psi(1-2s)}{2s}\left[\lim_{w\to\infty}u(w)-u(x)\right]\\ &= \frac{\Psi(1-2s)-\Psi(2s)}{2s}\left[2u(x)-\lim_{w\to-\infty}u(w)-\lim_{w\to\infty}u(w)\right].\end{aligned}\tag{12}$$

Fourth case: $x < y < z$. Here, we have $x < y < w < z$ and, as in the second case, $u(y) - u(z) = \int_y^z -u'(w)\,\mathrm{d}w$ rather than $\int_z^y u'(w)\,\mathrm{d}w$ as written in (7). Thus,

$$\begin{aligned}
\iiint &= \int_x^\infty -u'(w) \int_x^w (y-x)^{2s-1} \int_w^\infty (z-y)^{-2s-1}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}w \\
&= \int_x^\infty \frac{-1}{2s} u'(w) \int_x^w (y-x)^{2s-1} (w-y)^{-2s}\,\mathrm{d}y\,\mathrm{d}w \\
&= \int_x^\infty \frac{-1}{2s} u'(w) \int_0^1 v^{2s-1}(1-v)^{-2s}(w-x)^{2s-1-2s+1}\,\mathrm{d}v\,\mathrm{d}w \\
&= \int_x^\infty \frac{-1}{2s} u'(w) B(2s, 1-2s)\,\mathrm{d}w \\
&= \frac{\Gamma(2s)\Gamma(1-2s)}{-2s} \int_x^\infty u'(w)\,\mathrm{d}w \\
&= \frac{\pi}{2s\sin 2\pi s}\left(u(x) - \lim_{w\to\infty} u(w)\right),
\end{aligned} \tag{13}$$

where in the third line we used the substitution $v = \frac{y-w}{x-w}$.

Combining all cases together. Summing the expressions for the triple integrals from (8) and (12) and (13), we have the overall result as:

$$\begin{aligned}
(-\Delta)^{-s}(-\Delta)^s u(x) &= \left(\frac{s}{\pi}\tan s\pi\right)\left(\frac{\pi}{2s\sin 2\pi s} + \frac{\Psi(1-2s)-\Psi(2s)}{2s}\right) \\
&\qquad \times [2u(x) - u(-\infty) - u(\infty)] \\
&= \frac{\tan \pi s}{\pi}[\Gamma(2s)\Gamma(1-2s) + \Psi(1-2s) - \Psi(2s)] \\
&\qquad \times \left(u(x) - \frac{u(-\infty)+u(\infty)}{2}\right) \\
&= u(x) - \frac{u(-\infty)+u(\infty)}{2},
\end{aligned}$$

using some further well-known facts on the gamma and digamma functions.

3.2 Function Space

As discussed above, to be able to apply Mikusiński's method for the fractional Laplacian, we need to find a suitable function space which is closed under the operations of pointwise addition and Fourier-type convolution (6) and which consists of functions defined on the whole real line $(-\infty,\infty)$.

The function space C_α defined in Subsection 2.1, which is usually applicable in the study of Mikusiński's operational calculus for fractional operators, cannot be suitable for us now, as it consists of functions defined only on the positive half-line $[0,\infty)$.

Fractional Laplacians are usually studied on $\mathbf{L}^p$ spaces, but would such spaces have suitable properties under Fourier convolution? The key fact here is Young's convolution inequality [16], which states that, if $p,q,r \in [1,\infty)$ satisfy $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$ and $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then the convolution $f \star g \in L^r(\mathbb{R})$ satisfies

$$\|f\star g\|_r \le \|f\|_p \cdot \|g\|_q.$$

By this fact, the study of Fourier convolutions in L^p spaces, where $1 \le p \le \infty$, is well established. In particular, the space $L^1(\mathbb{R})$ is closed under convolution. For a more comprehensive coverage of L^p spaces, convolution operators, and the behaviour of these operators on functions in the L^p spaces, including L^p for various endpoint cases, we recommend referring to [16,38,32].

However, it is important to note that L^p spaces do not necessarily contain all power functions. This is important because the inverse fractional Laplacian is defined by convolution with the function $\frac{K}{|x|^{1-2s}}$ for some constant K. Mikusiński's method requires the integration operator to correspond to multiplication (convolution) with an element in the same function space that we have given a rng structure. Here, our integration operator is given by convolution with $\frac{K}{|x|^{1-2s}}$, so this function must be contained in our chosen function space.

Power functions $\frac{K}{|x|^{1-\alpha}}$ are not contained in $L^1(\mathbb{R})$, nor in any $L^p(\mathbb{R})$ for any value of p. Hence, these spaces are not a suitable setting for Mikusiński's operational calculus for the fractional Laplacian.

The space of continuous functions bounded at $\pm\infty$ cannot be used either, as this space is not closed under convolution. So, what properties do we need our function space to satisfy?

Assumption 1. Suppose there exists a function space, say $C_\star$, such that:

– Functions in $C_\star$ are defined almost everywhere on $\mathbb{R}$.
– $C_\star$ contains the function $\frac{K_\alpha}{|x|^{1-\alpha}}$ for at least one α with $0<\alpha<1$, where $K_\alpha \in \mathbb{R}$ is a non-zero constant.
– $C_\star$ is closed under the operations of pointwise addition and Fourier-type convolution of functions.

If a function space $C_\star$ exists satisfying Assumption 1, then this, together with the operations of addition and Fourier-type convolution, will form a commutative rng $(C_\star,+,\star)$ without multiplicative identity. The function $k_\alpha(x)=\frac{K_\alpha}{|x|^{1-\alpha}}$, which is the kernel function for the inverse fractional Laplacian operator $(-\Delta)^{-\frac{\alpha}{2}}$, will play a similar role as the functions $\ell(x)=1$ in the classical Mikusiński setting or $h_\alpha(x)=\frac{x^{\alpha-1}}{\Gamma(\alpha)}$ in the fractional Mikusiński setting. The conditions in Assumption 1 are necessary in order to be able to apply Mikusiński's method to the fractional Laplacian operator, because we need a space that can form a rng (hence it must be closed under the two operations) and in which the inverse fractional Laplacian can be interpreted as multiplication (hence the kernel functions must be contained in the rng).

Being a rng, $C_\star$ will embed naturally into its own field of fractions, say $\mathscr{M}_\star$, as long as we can prove that the rng has no zero divisors. Also, the algebraic inverse of the inverse fractional Laplacian operator $(-\Delta)^{-\frac{\alpha}{2}}$ will be the multiplicative inverse in the field $\mathscr{M}_\star$ of the previously defined element k_α, namely:

$$S_\alpha=\frac{I}{k_\alpha},$$

where $I=\frac{k_\alpha}{k_\alpha}$ denotes the multiplicative identity element in the field $\mathscr{M}_\star$. Then, using the fundamental theorem of calculus constructed in Theorem 1, we can interpret the fractional Laplacian operator in terms of the algebraic element S_α, as follows:

$$\begin{aligned}
(-\Delta)^{-s}(-\Delta)^{s}u(x)&=u(x)-\frac{u(-\infty)+u(\infty)}{2}\\
\Longrightarrow k_\alpha\star\left((-\Delta)^{s}u\right)&=u-\left(\frac{u(-\infty)+u(\infty)}{2}\right)\ell\\
\Longrightarrow (-\Delta)^{s}u&=S_\alpha\star u-\left(\frac{u(-\infty)+u(\infty)}{2}\right)\ell * S_\alpha,
\end{aligned}$$

thus giving a direct description of the fractional Laplacian in terms of S_α. After this, differential equations involving the fractional Laplacian can be solved in the same way as was done in [19,26] for fractional differential equations involving Riemann–Liouville and Caputo operators.

All of this can be done under the conditions of Assumption 1. If such a function space can be found, we can construct a Mikusiński's operational calculus for the fractional Laplacian and the associated differential equations. On the other hand, if such a function space does not exist, then Mikusiński's method cannot be applied to fractional Laplacian operators.

3.3 Proof of Impossibility

We will now show that it is impossible for a function space $C_\star$ to satisfy all of the required conditions listed in Assumption 1.

To justify our claim, we need to consider first the convolution of the functions $\frac{K_\alpha}{|x|^{1-\alpha}}$ with each other. If all finite convolution-products of such functions exist, then the set of all finite linear combinations of finite convolution-products of such functions (the algebra generated by them) would be a minimal possibility for the space $C_\star$.

Theorem 2. *Let $f(x) = \frac{K_1}{|x|^{1-\alpha}}$ and $g(x) = \frac{K_2}{|x|^{1-\beta}}$, where $K_1, K_2 \in \mathbb{R}$ are nonzero constants. If $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha, \beta > 0$ and $\alpha + \beta < 1$, then*

$$(f \star g)(x) = \frac{K_{\alpha,\beta}}{|x|^{1-(\alpha+\beta)}},$$

where $K_{\alpha,\beta}$ is another nonzero constant depending only on α and β.

Proof. We simplify the convolution integral by substituting $y = xt$:

$$\begin{aligned}(f \star g)(x) &= \int_{-\infty}^{\infty} \frac{K_1}{|x-y|^{1-\alpha}} \cdot \frac{K_2}{|y|^{1-\beta}} \,\mathrm{d}y \\ &= K_1, K_2 \int_{-\infty}^{\infty} |x-y|^{\alpha-1} |y|^{\beta-1} \,\mathrm{d}y \\ &= \frac{K_1 K_2}{|x|^{1-(\alpha+\beta)}} \int_{-\infty}^{\infty} |1-t|^{\alpha-1} |t|^{\beta-1} \,\mathrm{d}t \\ &= \frac{K_{\alpha,\beta}}{|x|^{1-(\alpha+\beta)}},\end{aligned}$$

provided that the last integral converges. Note that this integral depends only on α and β, and it is clearly non-zero, so finding convergence conditions for the integral will be enough to complete the proof.

For which conditions on α and β does this integral converge? The integrand is continuous almost everywhere, with just four possible "problem points" (singularities or points at infinity) that might lead to divergence, namely $-\infty, 0, 1, \infty$. So we split the integral as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty} |1-t|^{\alpha-1} |t|^{\beta-1} \,\mathrm{d}t \\ &\quad = \int_{-\infty}^{0} |1-t|^{\alpha-1} |t|^{\beta-1} \,\mathrm{d}t + \int_{0}^{1} |1-t|^{\alpha-1} |t|^{\beta-1} \,\mathrm{d}t + \int_{1}^{\infty} |1-t|^{\alpha-1} |t|^{\beta-1} \,\mathrm{d}t \\ &\quad = \int_{-\infty}^{0} (1-t)^{\alpha-1} t^{\beta-1} \,\mathrm{d}t + \int_{0}^{1} (1-t)^{\alpha-1} t^{\beta-1} \,\mathrm{d}t + \int_{1}^{\infty} (1-t)^{\alpha-1} t^{\beta-1} \,\mathrm{d}t.\end{aligned}$$

Now we just need to check for convergence at each of the possible problem points, the two endpoints of each of the three integrals. Since the integrands are all powers of t and $1-t$, we only need to check what overall power is obtained near each endpoint, and check the requirements on its sign.

- As $t \to -\infty$, we have $(1-t)^{\alpha-1}(-t)^{\beta-1} \sim (-t)^{\alpha+\beta-2}$, so $\int \sim c|t|^{\alpha+\beta-1}$ and we need $\alpha+\beta-1 < 0$.
- As $t \to 0^-$, we have $(1-t)^{\alpha-1}(-t)^{\beta-1} \sim (-t)^{\beta-1}$, so $\int \sim c|t|^{\beta}$ and we need $\beta > 0$.
- As $t \to 0^+$, we have $(1-t)^{\alpha-1} t^{\beta-1} \sim t^{\beta-1}$, so $\int \sim c|t|^{\beta}$ and we need $\beta > 0$.
- As $t \to 1^-$, we have $(1-t)^{\alpha-1}(-t)^{\beta-1} \sim (1-t)^{\alpha-1}$, so $\int \sim c|t-1|^{\alpha}$ and we need $\alpha > 0$.
- As $t \to 1^+$, we have $(t-1)^{\alpha-1}(-t)^{\beta-1} \sim (t-1)^{\alpha-1}$, so $\int \sim c|t-1|^{\alpha}$ and we need $\alpha > 0$.
- As $t \to \infty$, we have $(t-1)^{\alpha-1} t^{\beta-1} \sim t^{\alpha+\beta-2}$, so $\int \sim ct^{\alpha+\beta-1}$ and we need $\alpha+\beta-1 < 0$.

So the convolution converges iff $\alpha > 0$ and $\beta > 0$ and $\alpha+\beta < 1$, which were the conditions stated in the theorem. The result is now proved.

From the conditions on α and β in Theorem 2, we begin to see a problem arising. The conditions $\alpha > 0$ and $\beta > 0$ will still be true no matter how many times we add together the numbers α and β, but the same cannot be said for the condition $\alpha + \beta < 1$. If we keep combining functions k_α and k_β, which is equivalent to adding the numbers α and β, eventually we will reach a point where the sum of two parameters is no longer less than 1. In order to show that this problem is reached even from the most minimal of starting points, we consider convolution of a single function $\frac{K}{|x|^{1-\alpha}}$ with itself arbitrarily many times, which is required to remain in the space $C_\star$ if such a space exists according to Assumption 1.

Theorem 3. *Let $f(x) = \frac{K}{|x|^{1-\alpha}}$ where $0 < \alpha < 1$ and $K \in \mathbb{R}$ is a nonzero constant. Then, for N large enough, the finite convolution-product of f with itself N times must diverge.*

Proof. Based on Theorem 2, it follows that:

$$\left(\frac{K}{|x|^{1-\alpha}}\right)\star\left(\frac{K}{|x|^{1-\alpha}}\right)=\frac{K_\alpha^{(2)}}{|x|^{1-2\alpha}}$$
$$\left(\frac{K}{|x|^{1-\alpha}}\right)\star\left(\frac{K}{|x|^{1-\alpha}}\right)\star\left(\frac{K}{|x|^{1-\alpha}}\right)=\frac{K_\alpha^{(3)}}{|x|^{1-3\alpha}}$$
$$\vdots$$
$$\left(\frac{K}{|x|^{1-\alpha}}\right)\star\left(\frac{K}{|x|^{1-\alpha}}\right)\star\ldots\star\left(\frac{K}{|x|^{1-\alpha}}\right)=\frac{K_\alpha^{(N)}}{|x|^{1-N\alpha}},\quad N\in\mathbb{N}.$$

The problem is that, when $N\alpha>1$, this is a positive power of $|x|$, and the convolution of two positive powers of $|x|$ will definitely diverge:

$$\left(\frac{K_\alpha^{(N)}}{|x|^{1-N\alpha}}\right)\star\left(\frac{K_\alpha^{(N)}}{|x|^{1-N\alpha}}\right)=\left(K_\alpha^{(N)}\right)^2\int_{-\infty}^{\infty}|x-y|^{N\alpha-1}|y|^{N\alpha-1}\,\mathrm{d}y=\infty,$$

since the integrand is positive and continuous everywhere and tends to infinity at both ends of the interval.

Finally, we are now in a position to prove the main result of this paper, as follows.

Theorem 4. *There is no function space $C_\star$ satisfying Assumption 1.*

Proof. If there exists a function space $C_\star$ that contains a single function $\frac{K_\alpha}{|x|^{1-\alpha}}$ and is closed under convolution, then it would also contain all conceivable functions resulting from convolving the aforementioned function with itself. But Theorem 3 established that, upon repeated convolutions of said function with itself a considerable number of times, we would ultimately encounter a divergent convolution integral. Consequently, there does not exist a conceivable function space that includes such a function and remains closed under convolution. Hence, we can deduce that the existence of $C_\star$ is negated.

4 Conclusions

Our work above has resulted in two major outcomes. Firstly, our Theorem 1 is a fundamental theorem of calculus for the fractional Laplacian which we could not find (at this level of generality) elsewhere in the literature. Of course, inversion relations between the fractional Laplacian and inverse fractional Laplacian are well known in the existing literature [37], but they usually take place on spaces such as $L^p(\mathbb{R})$ which require decay of all functions at infinity. We find it interesting that we were able to prove a fundamental theorem of calculus in a space which does not require decay at infinity, and which takes account of possible boundary terms $u(\infty)$ and $u(-\infty)$.

Such a result also ties into the deeper philosophy of Mikusiński's operational calculus, which solves differential equations in larger spaces than would otherwise be possible. For example, the results of Mikusiński's method for ordinary differential equations [28] look formally identical to the results of applying the Laplace transform to the same equations, but the advantage of Mikusiński's method is that it applies on larger function spaces: functions such as e^{x^2} have no Laplace transform but can still be studied within the space $C[0,\infty)$ that Mikusiński used. So, even though we have failed to produce a full operational calculus for the fractional Laplacian, we have at least succeeded in extending some results (inversion relations) beyond the usual function spaces to larger ones. Many problems concerning the fractional Laplacian can be solved using Fourier transforms, but what happens if we need to extend to a larger space of functions whose Fourier transforms do not necessarily exist? Perhaps our work will be helpful in answering questions like this.

The other major strand of research here has culminated in a negative result, Theorem 4, showing that it is impossible to extend the same Mikusiński's operational calculus used in other types of fractional calculus to the one-dimensional fractional Laplacian. To contextualise and discuss this result, we must consider a number of different points of view.

There are different ways to do operational calculus. The approach of Bengochea [3] constructs a different sort of algebraic formalism from Mikusiński's, which can still be used to rigorously solve differential equations, including fractional ones [4]. Some variants of Mikusiński's operational calculus have used different types of convolution operators [39]. Perhaps another operational calculus approach can be successfully applied to the fractional Laplacian operator; our

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result concerns only the specific approach of Mikusiński which has been so successfully applied to many other fractional-order operators.

Finally, we note that there are also variants of the fractional Laplacian operator which may be considered separately. One of these is the Riesz–Feller derivative, a generalisation of the fractional Laplacian which also has a useful connection with the Fourier transform and which is used in the study of Lévy processes and probability density functions [27] and space-time fractional differential equations of various types [35]. The positivity of the inverse fractional Laplacian's kernel led to its convolutions ultimately diverging, but perhaps an operator like the Riesz–Feller derivative, involving kernels that are not always positive, may be more amenable to operational calculus techniques. This remains as the subject of a future investigation.

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