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Clairaut Convolution Differential Equation

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Abstract: The Clairaut convolution differential equation, a variant of the more general Clairaut equation, has been a subject of considerable interest in the realm of mathematical analysis. Based on an infinite dimensional test space of holomorphic functions, the purposes of this work are to investigate the convolution calculus in white noise theory to introduce and study the solution of the analogue of the Clairaut equation, called Clairaut convolution differential equation $(CCDE)$. We study the Clairaut convolution differential equation as a natural counterpart of the classical Clairaut differential equation and give its general and singular solutions.

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1 Introduction

The Clairaut convolution differential equation, a variant of the more general Clairaut equation, has been a subject of considerable interest in the realm of mathematical analysis. First introduced by the French mathematician Alexis Clairaut in 1734, this equation takes the form

$$\eta = t\eta' + \Lambda(\eta'), \tag{1}$$

where Λ is a continuously differentiable non-linear function. Researchers studied various aspects of the Clairaut convolution differential equation, focusing on analytical methods, the existence, and the uniqueness of solutions, applications, and recent developments. Differentiate both sides to get

$$(t + \Lambda'(\eta'))\eta'' = 0.$$

Consequently, either

$$\eta'' = 0 \quad \text{or} \quad t + \Lambda'(\eta') = 0\cdot$$

Hence, $\eta' = Constant = \alpha$ which implies that

$$\eta(t) = \alpha t + \Lambda(\alpha).$$

In addition,

$$t + \Lambda'\left(\eta'\right) = 0,$$

defines a unique solution $\eta(t)$, known as the singular solution.

Recent research has extended the study of the Clairaut convolution differential equation to explore connections with other mathematical concepts. Investigations into higher-dimensional versions and applications in emerging scientific fields demonstrate the ongoing relevance and expanding scope of research in this area. In addition to analytical methods, numerical techniques play a crucial role in approximating solutions to the Clairaut convolution differential equation. Researchers explore numerical stability and accuracy, comparing different methods to find effective approaches for

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solving this equation. The Clairaut convolution differential equation finds applications in diverse scientific fields. Its solutions model phenomena where relationships exhibit both linear and nonlinear characteristics. Applications span from physics to biology, showcasing the versatility of this mathematical tool in understanding real-world phenomena. It is noteworthy that a function η is considered a singular solution of the differential equation $\Lambda(t,\eta,\eta')=0$ if the uniqueness of the solution is violated at each point (or at certain points) within the domain of the equation. Geometrically, this implies that more than one integral curve with a common tangent line passes through each point (t_0,η_0).

The parametric notation for the singular solution is give as $(x(p),y(p))$, where $p=y'(x)$ ([3,4,7,9,10,15,24,25]). Thus,

$$t=-\Lambda'(q) \quad ; \quad \eta = tq+\Lambda(q)\cdot$$

The purposes of this work are to investigate the convolution calculus in white noise theory to introduce and study the solution of the analogue of the Clairaut equation, called Clairaut convolution differential equation (CCDE) which is of the form

$$\varphi = a_\xi(\varphi)\star\psi_\eta + f(a_\xi(\varphi)) \tag{2}$$

where $f:\mathscr{F}_\gamma\to\mathscr{F}_\gamma$, $\psi_\eta(x)=\langle x,\eta\rangle, x\in\mathscr{S}'_{\mathbb{C}}$ and $\xi,\eta\in\mathscr{S}_{\mathbb{C}}$ such that $\langle\xi,\eta\rangle=1$.

In analogy with the classical Clairaut differential equation (1), the variable ψ_η plays the role of x in (1), the convolution plays the role of the multiplication and the convolution derivative a_ξ plays the role of the usual derivation ([1,2,5,6,8,12, 13,14]) and [36]-[46]. In Section 3, we give the general and the singular solutions of the CCDE (2).

2 Preliminaries

Let $\mathscr{S}(\mathbb{R})$ denote the Schwartz space [30,32], consisting of rapidly decreasing C^∞functions, and let $H=L^2(\mathbb{R},dt)$ be the Hilbert space with the norm $|\cdot|_0$. Now, we will define $\operatorname{proj}\lim_{n\to\infty}\mathscr{S}_n$ and $\operatorname{ind}\lim_{n\to\infty}\mathscr{S}_{-n}$. Let $\{S_q, q\geq 0\}$ be a sequence of Hilbert spaces the following embedding $S_{q+1}->S_q$ is continuous for each $q\geq 0$. Consider $S=\bigcap_{q=0}^{\infty} S_q$ equipped with the projective limit topology, i.e., the coarsest topology such that the inclusion $S\subset S_q$ is continuous for every q. This topological vector space S is called the projective limit of $\{S_q, q\geq 0\}$ and we write $\operatorname{proj}\lim_{n\to\infty}\mathscr{S}_n$. We define $\operatorname{ind}\lim_{n\to\infty}\mathscr{S}_{-n}$ by

$$\operatorname{ind}\lim_{n\to\infty}\mathscr{S}_{-n} = \bigcup_{n=0}^{\infty} S_{-n}.$$

The space $\mathscr{S}(\mathbb{R})$ can be reconstructed using the harmonic oscillator $A=1+t^2-d^2/dt^2$ and H in a standard way (see [27]). Specifically, $\mathscr{S}(\mathbb{R})$ is a nuclear space equipped with Hilbertian norms

$$|\beta|_n = |A^n\beta|_0, \qquad \beta\in\mathscr{S}(\mathbb{R}), \quad n\in\mathbb{R}$$

and we have

$$\mathscr{S}(\mathbb{R}) = \operatorname{proj}\lim_{n\to\infty}\mathscr{S}_n, \qquad \mathscr{S}'(\mathbb{R}) = \operatorname{ind}\lim_{n\to\infty}\mathscr{S}_{-n},$$

where, for $n\geq 0$, $\mathscr{S}p$ is the completion of $\mathscr{S}(\mathbb{R})$ with respect to the norm $|\cdot|p$ and $\mathscr{S}-p$ is the topological dual space of $\mathscr{S}p$. $\mathscr{S}'(\mathbb{R})$ is the topological dual of $\mathscr{S}(\mathbb{R})$. The canonical $\mathbb{C}$bilinear form from $\mathscr{S}'(\mathbb{R})\times\mathscr{S}(\mathbb{R})$ compatible with the inner product of H is denoted by $\langle\cdot,\cdot\rangle$, and $\mathscr{S}'(\mathbb{R})$ is the space of tempered distributions ([20,21,22,23,25,27,28,33]). The complexification of $\mathscr{S}(\mathbb{R})$ is denoted by $\mathscr{S}_{\mathbb{C}}=\mathscr{S}(\mathbb{R})+i\mathscr{S}(\mathbb{R})$, and $\mathscr{S}_{\mathbb{C},p}=\mathscr{S}_p(\mathbb{R})+i\mathscr{S}_p(\mathbb{R})$.

Consider a Young function $\gamma:\mathbb{R}+\to\mathbb{R}+$, which is a continuous, convex, increasing function satisfying $\lim_{x\to\infty}\frac{\gamma(x)}{x}=\infty$ and $\gamma(0)=0$. The polar function associated with γ, denoted by $\gamma^*(x)$, is given by

$$\gamma^*(x) = \sup_{t\geq 0}\{tx-\gamma(t)\}$$

and it is also a Young function, with $(\gamma^*)^*=\gamma$.

For a complex Banach space $(B,|\cdot|)$, let $\mathscr{H}(B)$ denote the space of all entire functions on B. For each $m>0$, let $\operatorname{Exp}(B,\gamma,m)$ denote the space of all entire functions on B with γ-exponential growth of finite type m, defined as

$$\operatorname{Exp}(B,\gamma,m) = \left\{f\in\mathscr{H}(B);\quad \|f\|_{\gamma,m} := \sup_{z\in B}|f(z)|e^{-\gamma(m\|z\|)} < \infty\right\}.$$

The projective system $\mathrm{Exp}(\mathscr{S}_{\mathbb{C},-p},\gamma,m);;p\in\mathbb{N},:m>0$ yields the space

$$\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}})=\operatorname*{proj\,lim}_{p\to\infty;m\downarrow 0}\mathrm{Exp}(\mathscr{S}_{\mathbb{C},-p},\gamma,m)\,. \tag{3}$$

On the other hand, $\{\mathrm{Exp}(\mathscr{S}_{\mathbb{C},p},\gamma,m);\ p\in\mathbb{N}\ ,m>0\}$ becomes an inductive system of Banach spaces, and we put

$$\mathscr{G}_\gamma(\mathscr{S}_{\mathbb{C}})=\operatorname*{ind\,lim}_{p\to\infty;m\downarrow\infty}\mathrm{Exp}(\mathscr{S}_{\mathbb{C},p},\gamma,m)\,.$$

It is known from [35] and [4] that every $\eta\in\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}})$ admits a Taylor expansion of the form

$$\eta(x)=\sum_{n=0}^{\infty}\langle x^{\otimes n},\eta_n\rangle,\quad x\in\mathscr{S}'_{\mathbb{C}},\eta_n\in\mathscr{S}_{\mathbb{C}}^{\widehat{\otimes}n}. \tag{4}$$

Let $F_\gamma(\mathscr{S}_{\mathbb{C}})$ be the space of all Taylor coefficients η_n as in (4). It is known that

$$F_\gamma(\mathscr{S}_{\mathbb{C}})=\operatorname*{proj\,lim}_{p\to\infty;m\downarrow 0}F_{\gamma,m}(\mathscr{S}_{\mathbb{C},p}),$$

where

$$F_{\gamma,m}(\mathscr{S}_{\mathbb{C},p})=\left\{\overrightarrow{\eta}=(\eta_n)_{n\geq 0};\ \eta_n\in\mathscr{S}_{\mathbb{C},p}^{\widehat{\otimes}n},\ \|\overrightarrow{\eta}\|_{\gamma,p,m}=\sum_{n=0}^{\infty}\gamma_n^{-2}m^{-n}|\eta_n|_p^2<\infty\right\}$$

and

$$\gamma_n=\inf_{r>0}\frac{e^{\gamma(r)}}{r^n},\quad n=0,1,2,\ldots.$$

Moreover, equipped with the projective limit topology, $F_\gamma(\mathscr{S}_{\mathbb{C}})$ is a nuclear Fréchet space and is isomorphic to $\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}})$. Let

$$G_\gamma(\mathscr{S}'_{\mathbb{C}})=\operatorname*{ind\,lim}_{p\to\infty;m\to\infty}G_{\gamma,m}(\mathscr{S}_{\mathbb{C},-p}),$$

where

$$G_{\gamma,m}(\mathscr{S}_{\mathbb{C},-p})=\left\{\overrightarrow{\eta}=(F_n)_{n\geq 0};F_n\in\mathscr{S}_{\mathbb{C},-p}^{\widehat{\otimes}n},\ \sum_{n=0}^{\infty}(n!\gamma_n)^2m^n|F_n|_{-p}^2<\infty\right\}.$$

By definition, $F_\gamma(\mathscr{S}_{\mathbb{C}})$ and $G_\gamma(\mathscr{S}'_{\mathbb{C}})$ are dual to each other. For any $n\in\mathbb{N}$, we say that $\eta\in\mathscr{F}^*_\gamma(\mathscr{S}'_{\mathbb{C}})$ belongs to the n-th chaos if η is of the form η_n, where $\eta_n\in\mathscr{S}_{\mathbb{C}}^{\prime\otimes n}$ is a symmetric distribution ([16,17,18,26,29,34]). The following theorem is useful.

Theorem 1.*[19] The Laplace transform $\mathscr{L}$ induces a topological isomorphism from $\mathscr{F}^*_\gamma(\mathscr{S}'_{\mathbb{C}})$ onto $\mathscr{G}_{\gamma^*}(\mathscr{S}_{\mathbb{C}})$, where*

$$\mathscr{L}\eta(\xi)=\langle\langle\eta,e_\xi\rangle\rangle,\quad \xi\in\mathscr{S}_{\mathbb{C}} \tag{5}$$

and for any $\xi\in\mathscr{S}_{\mathbb{C}}$, $x\in\mathscr{S}'_{\mathbb{C}}$ we have $e_\xi(x)=e^{\langle x,\xi\rangle}$.

Originally, a convolution operator on the test space $\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}})$ is a continuous linear operator from $\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}})$ into itself that commutes with all the translation operators, where the translation operator is defined by

$$\tau_x\varphi(y):=\varphi(x+y),\quad x,y\in\mathscr{S}'_{\mathbb{C}},\quad \varphi\in\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}}).$$

The convolution product of a distribution $\eta\in\mathscr{F}^*_\gamma(\mathscr{S}'_{\mathbb{C}})$ with a test function $\varphi\in\mathscr{F}_\gamma(\mathscr{S}'_{\mathbb{C}})$ is defined by

$$(\eta\star\varphi)(x)=\langle\langle\eta,\tau_{-x}\varphi\rangle\rangle,\quad x\in\mathscr{S}'_{\mathbb{C}}.$$

The convolution operator is denoted by C_η. It is noteworthy that the composition $C_{\eta_1}\circ C_{\eta_2}$ is also a convolution operator, so that there exists a unique element in $\mathscr{F}^*_\gamma(\mathscr{S}'_{\mathbb{C}})$ denoted by $\eta_1\star\eta_2$ such that

$$C_{\eta_1}\circ C_{\eta_2}=C_{\eta_1\star\eta_2},\quad \eta_1,\eta_2\in\mathscr{F}^*_\gamma(\mathscr{S}'_{\mathbb{C}}). \tag{6}$$

The distribution $\eta_1*\eta_2$ defined in (6) is called the convolution product of η_1 and η_2. The convolution product satisfies

$$\mathscr{L}(\eta_1*\eta_2)=\mathscr{L}(\eta_1)\mathscr{L}(\eta_2),$$

where $\mathscr{L}$ is the Laplace transform, giving

$$\mathscr{L}(\eta^{\star n})=(\mathscr{L}(\eta))^n,\quad n\in\mathbb{N}.$$

3 Clairaut's convolution differential equation

A convolution derivation, denoted as $\mathscr{D}$, is a continuous linear map from $\mathscr{F}\gamma(\mathscr{S}'_{\mathbb{C}})$ to $\mathscr{F}\gamma(\mathscr{S}'_{\mathbb{C}})$. It is defined as a derivation with respect to the convolution product. For $\eta, \Psi \in \mathscr{F}^*\gamma(\mathscr{S}'_{\mathbb{C}})$, it satisfies the following relation:

$$\mathscr{D}(\eta \star \Psi) = \mathscr{D}(\eta) \star \Psi + \eta \star \mathscr{D}(\Psi)\cdot$$

For more details, refer to [9]. If the Young function γ meets the condition given by

$$\lim_{r \to +\infty} \frac{\gamma(r)}{r^2} < +\infty, \tag{7}$$

then the resulting nuclear Gel'fand triple is

$$\mathscr{F}_{\gamma}(\mathscr{S}'_{\mathbb{C}}) \subset L^2(X', \mathscr{B}(\mathscr{S}(\mathbb{R})'), \mu) \subset \mathscr{F}^*_{\gamma}(\mathscr{S}'_{\mathbb{C}}),$$

where μ is the standard Gaussian measure on $(\mathscr{S}(\mathbb{R})', \mathscr{B}(\mathscr{S}(\mathbb{R})'))$, and its characteristic function is given by

$$\int_{\mathscr{S}(\mathbb{R})'} e^{i\langle y, \xi\rangle}\, d\mu(y) = e^{-|\xi|_0^2/2}, \quad \xi \in \mathscr{S}(\mathbb{R}).$$

As a standard example of the convolution derivative, we recall that the holomorphic derivative of $\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi_n\rangle$ at a point $x \in \mathscr{S}'_{\mathbb{C}}$ along the z axis

$$(a_z\varphi)(x) := \lim_{t \to 0} \frac{\varphi(x+tz) - \varphi(x)}{t}.$$

It is well-known that

$$(a_z\varphi)(x) = \sum_{n=0}^{\infty} (n+1)\langle x^{\otimes n}, z \otimes_1 \varphi_{n+1}\rangle,$$

where $z \otimes_1 \varphi_{n+1}$ stands for the contraction of the tensor products. Such a_z is called the annihilation operator associated to z, (see [11] and [27] for more details). We write $a_t := a(\delta_t)$ and call it the standard annihilation operator or Hida's differential operator. It is well–known from [31] and [32] that, for any $z \in \mathscr{S}'_{\mathbb{C}}$, the operator a_z is a continuous mapping from $\mathscr{F}\gamma(\mathscr{S}'_{\mathbb{C}})$ to itself. Moreover, for a fixed $\varphi \in \mathscr{F}_{\gamma}(\mathscr{S}'_{\mathbb{C}})$, the map $z \longrightarrow a_z(\varphi)$ is also continuous.

Let f be a function in the form

$$f(\varphi) = \alpha_0\delta_0 + \alpha_1\varphi + \alpha_2\varphi^{\star 2} + \cdots + \alpha_n\varphi^{\star n},$$

where $\varphi \in \mathscr{F}_{\gamma}(\mathscr{S}'_{\mathbb{C}})$, $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$. We introduce the CCDE as follows

$$\varphi = a_{\xi}(\varphi) \star \psi_{\eta} + f(a_{\xi}(\varphi)) \tag{8}$$

where $f : \mathscr{F}_{\gamma} \to \mathscr{F}_{\gamma}$, $\psi_{\eta}(x) = \langle x, \eta\rangle$, $x \in \mathscr{S}'_{\mathbb{C}}$ and $\xi, \eta \in \mathscr{S}_{\mathbb{C}}$ such that $\langle \xi, \eta\rangle = 1$.

Theorem 2. *The CCDE (8) have a singular solution with a parametric description given as follows: Setting $\eta_{\xi} = a_{\xi}(\varphi)$*

$$\begin{cases} \psi_{\eta} = -(a_{\xi}f)(\eta_{\xi}), \\ \varphi = \eta_{\xi} \star \psi_{\eta} + f(\eta_{\xi}). \end{cases} \tag{9}$$

Proof. Applying a_{ξ} on both sides in equation (8), we obtain

$$\begin{aligned} a_{\xi}(\varphi) &= a_{\xi}\Big(a_{\xi}(\varphi) \star \psi_{\eta} + f(a_{\xi}(\varphi))\Big) \\ &= a^2_{\xi}(\varphi) \star \psi_{\eta} + a_{\xi}(\varphi) \star a_{\xi}(\psi_{\eta}) + a_{\xi}\Big(f(a_{\xi}(\varphi))\Big). \end{aligned}$$

But $a_{\xi}(\psi_{\eta})$ is given by

$$\begin{aligned} a_{\xi}(\psi_{\eta})(x) &= \lim_{t \to 0} \frac{\langle x + t\xi, \eta\rangle - \langle x, \eta\rangle}{t} \\ &= \lim_{t \to 0} \frac{t\langle \xi, \eta\rangle}{t} \\ &= \langle \xi, \eta\rangle \\ &= 1. \end{aligned}$$

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Then, we obtain

$$a_\xi(\varphi) = a_\xi^2(\varphi)\star\psi_\eta + a_\xi(\varphi) + a_\xi\Big(f(a_\xi(\varphi))\Big).$$

This gives

$$a_\xi^2(\varphi)\star\psi_\eta + a_\xi\Big(f(a_\xi(\varphi))\Big) = 0. \tag{10}$$

Since

$$a_\xi\Big(f(a_\xi(\varphi))\Big) = (a_\xi f)(a_\xi(\varphi))\star a_\xi^2(\varphi),$$

Therefore, from (10), we get

$$a_\xi^2(\varphi)\star\Big(\psi_\eta + (a_\xi f)(a_\xi(\varphi))\Big) = 0\cdot$$

By applying the Laplace transform $\mathscr{L}$, we get

$$a_\xi^2(\varphi) = 0 \quad \text{or} \quad \psi_\eta + (a_\xi f)(a_\xi(\varphi)) = 0. \tag{11}$$

The singular solution of (8) corresponds to the right equation in (11).
Setting $\eta_\xi = a_\xi(\varphi)$, the right equation in (11) becomes

$$\psi_\eta = -(a_\xi f)(\eta_\xi).$$

Hence the parametric expression of the singular solution of (8) is given by

$$\psi_\eta = -(a_\xi f)(\eta_\xi) \quad \text{and} \quad \varphi = \eta_\xi\star\psi_\eta + f(\eta_\xi),$$

which is (16). This ends the proof.

Theorem 3.*The solution that includes all possibilities for equation (8) is expressed as:*

$$\varphi(x) = c\langle x,\eta\rangle + f(c),$$

where $c\in\mathbb{C}$.

*Proof.*By analogy with the solution that includes all possibilities the classical Clairaut equation (1), setting

$$\varphi(x) = c\langle x,\eta\rangle + f(c). \tag{12}$$

Then we get

$$\begin{aligned}
a_\xi(\varphi)(x) &= \lim_{t\to 0}\frac{c\langle x+t\xi,\eta\rangle + f(c) - c\langle x,\eta\rangle - f(c)}{t}\\
&= \lim_{t\to 0}\frac{c\langle x,\eta\rangle + tc\langle\xi,\eta\rangle - c\langle x,\eta\rangle}{t}\\
&= c\langle\xi,\eta\rangle\\
&= c.
\end{aligned}$$

Therefore

$$a_\xi(\varphi)\star\psi_\eta + f(a_\xi(\varphi)) = c\langle x,\eta\rangle + f(c) = \varphi(x).$$

which means that φ given in (12) satisfies the CCDE (8). By uniqueness of solution φ given in (12) is the general solution of (8).

As an application, we will study the following equation:

$$\frac{1}{2}\varphi^{\star 2} = a_\xi(\varphi)\star\psi_\eta\star\varphi + f(a_\xi(\varphi)\star\varphi). \tag{13}$$

We put

$$v = \frac{1}{2}\varphi^{\star 2}.$$

Then, we get

$$a_\xi(v) = \varphi\star a_\xi(\varphi).$$

Then, we obtain

$$v = a_\xi(v)\star\psi_\eta + f\left(a_\xi(v)\right).$$

Which leads to Clairaut equation.

*Example 1.*Let us study the following equation

$$\varphi = 2\psi_\eta \star a_\xi(\varphi) - 3\left(a_\xi(\varphi)\right)^{\star 2}. \tag{14}$$

Identifying Eq. (14) with Eq. (8), we can say that Eq. (14) is $(CCDE)$ such that:

$$\{\psi_\eta, f(\eta)\}$$

are replaced by

$$\{2\psi_\eta, -3\eta^{\star 2}\}.$$

Then, by Theorem (3.2), we obtain

$$\varphi(x) = 2c\langle x, \eta\rangle - 3c^2$$

as a general solution. Now, using Theorem (3.1), we get a singular solution as follows:

$$\begin{cases} 2\psi_\eta = 6\eta_\xi \\ \varphi = \eta_\xi \star \psi_\eta - 3f(\eta_\xi^{\star 2}) \end{cases} \tag{15}$$

which gives

$$\begin{cases} 2\psi_\eta = 3\eta_\xi \\ \varphi = \eta_\xi \star \psi_\eta - 3f(\eta_\xi^{\star 2}). \end{cases} \tag{16}$$

From which we obtain

$$\varphi = 0.$$

4 Conclusion

The Clairaut convolution differential equation, a variant of the more general Clairaut equation, has been a subject of considerable interest in the realm of mathematical analysis. Based on an infinite dimensional test space of holomorphic functions, In this work, using an infinite dimensional Gel'fand triple, we studied a new class of convolution differential equation which was called Clairaut convolution differential equation $(CCDE)$. Its general solution can be applied in harmonic analysis where the basic background is used in new context. We study the Clairaut convolution differential equation as a natural counterpart of the classical Clairaut differential equation and give its general and singular solutions.

Declarations

Competing interests: The authors declare that there is no conflict of interests regarding the publication of this article.
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