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On Some Fundamental Subclasses of Analytic Bi-univalent Functions

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Abstract: The aim of the current paper is to establish a general coefficient estimations for fundamental subclasses of bi-univalent functions. These subclasses encompass bi-starlike, bi-convex, strongly bi-starlike, and strongly bi-convex functions. Some new estimates are sharp and other findings involve several enhancements and extensions of well-known results.

Keywords: Bi-univalent functions; Coefficient estimates; Bi-starlike functions; Bi-convex functions; Strongly bi-starlike functions; Strongly bi-convex functions.

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1 Introduction and preliminary

Let $\mathcal{A}$ represent the class of analytic functions $f(z)$, defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let $\mathcal{S}$ represents the subclass of $\mathcal{A}$ that consists of univalent functions in $\mathbb{U}$ and the class $\mathcal{P}$ denotes the set of all analytic functions p fulfilling $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$, $(z \in \mathbb{U})$ defined by $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. For $n \geq 1$, it is well known that $|c_n| \leq 2$ according to the Carathéodory lemma, see [1]. The disk of radius $1/4$ is guaranteed to be presented in the image of $\mathbb{U}$ under any $f \in \mathcal{S}$ according to the Koebe one quarter theorem, see [1]. As a result, $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f(f^{-1}(v)) = v$, $(|v| < 1/4)$ define the inverse function f^{-1} for every function $f \in \mathcal{S}$, in the form (1.1). Indeed, the inverse of f is $g = f^{-1}$ given by

$$\begin{aligned} g(v) &= v - a_2 v^2 + (2a_2^2 - a_3)v^3 - (5a_2^3 - 5a_2 a_3 + a_4)v^4 + \cdots \\ &=: v + \sum_{k=2}^{\infty} b_k v^k, \quad \left(|v| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right). \end{aligned}$$

If f and its inverse function f^{-1} are both univalent in $\mathbb{U}$, then f, a function in the class $\mathcal{A}$, is said to be bi-univalent in $\mathbb{U}$. The symbol Σ denotes the set of bi-univalent functions in $\mathbb{U}$. The Taylor-Maclaurin coefficients of functions in Σ and $\mathcal{S}$ have been the subject of extensive investigation (see, for example, [2–20]). Non-sharp estimates for the initial two coefficients of the bi-starlike and bi-convex functions of order γ, as well as the strongly bi-starlike and the strongly bi-convex functions of order γ, were introduced by Brannan and Taha [21]. When f belongs to specific subclasses of Σ,

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many authors (see for examples [22–26]) have used the Faber polynomials [27] to estimate the first non-zero coefficient (a_n) for the bi-univalent function

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k; \ (z \in \mathbb{U},\ n \geq 2), \tag{1.2}$$

For five decades, finding bound of $|a_n|$ to the functions of Σ remains an open problem. Recently, Al-Refai and Ali [28] have found a solution for this problem. Indeed, they proved the following interesting theorem

Theorem 1. *Let $f(z)$, defined by (1.2), be univalent in $\mathbb{U}$ with*

$$f^{-1}(u) = u + \sum_{k=j}^{\infty} b_k u^k; \left(|u| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right). \tag{1.3}$$

Then, $b_{2j-1} = ja_j^2 - a_{2j-1}$ and $b_k = -a_k$, for $(j \leq k \leq 2j-2)$.

This result implies that $|a_j| \leq \sqrt{4-2/j}$ and $|ja_j^2 - a_{2j-1}| \leq 2j-1$. Bounds of $|a_k|$, $(j \leq k \leq 2j-1)$, $|ja_j^2 - a_{2j-1}|$, and other functional bounds can be studied for different subclasses of Σ, motivated by Theorem 1. In this work, we use the above new method of estimating coefficients to study these constraints for the following broad Σ subclasses that we find interesting. The results improve and generalize well known estimates. Other finding are also obtained.

Definition 1. *A function $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \Sigma$, $(n \geq 2)$ is said to be in the class $\mathcal{B}_{\Sigma,n}^{h,p}$ if it satisfies*

$$\frac{zf'(z)}{f(z)} \in H(\mathbb{U}), \quad (z \in \mathbb{U}) \tag{1.4}$$

and

$$\frac{vg'(v)}{g(v)} \in P(\mathbb{U}), \quad (v \in \mathbb{U}), \tag{1.5}$$

where $H, P \in \mathcal{P}$ and $g = f^{-1}$ is defined by (1.3).

Definition 2. *A function $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \Sigma$, $(n \geq 2)$ is said to be in the class $\mathcal{A}_{\Sigma,n}^{h,p}$ if it satisfies*

$$\frac{zf''(z)}{f'(z)} + 1 \in H(\mathbb{U}), \quad (z \in \mathbb{U}) \tag{1.6}$$

and

$$\frac{vg''(v)}{g'(v)} + 1 \in P(\mathbb{U}), \quad (v \in \mathbb{U}), \tag{1.7}$$

where $H, P \in \mathcal{P}$ and $g = f^{-1}$ is defined by (1.3).

We can specialize the functions $H(z)$ and $P(z)$ to provide interesting subclasses of bi-univalent functions. If we set

$$H(z) = 1 + \sum_{k=n}^{\infty} h_k z^{k-1} \ \text{and} \ P(z) = 1 + \sum_{k=n}^{\infty} p_k z^{k-1}, \quad (z \in \mathbb{U}, n \geq 2), \tag{1.8}$$

where $|h_k| \leq 2 - 2\gamma$ and $|p_k| \leq 2 - 2\gamma$, for $(k \geq n, 0 \leq \gamma < 1)$, then the hypotheses of Definitions 1 and 2 are satisfied, so we have the following subclasses of Σ.

Definition 3. *A function $f \in \mathcal{A}$, defined by (1.2), is said to be in the class $\mathcal{S}_{\Sigma,n}^{*}(\gamma)$, $(n \geq 2)$, if it satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad (z \in \mathbb{U}, 0 \leq \gamma < 1)$$

and

$$\operatorname{Re}\left(\frac{vg'(v)}{g(v)}\right) > \gamma, \quad (v \in \mathbb{U}, 0 \leq \gamma < 1),$$

where the function g is given by (1.3).

Definition 4. *A function $f \in \mathcal{A}$, defined by (1.2), is said to be in the class $C_{\Sigma,n}(\gamma)$, $(n \geq 2)$, if it satisfies*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \gamma, \quad (z \in \mathbb{U}, 0 \leq \gamma < 1)$$

and

$$\operatorname{Re}\left(\frac{vg''(v)}{g'(v)}+1\right) > \gamma, \quad (v \in \mathbb{U}, 0 \leq \gamma < 1),$$

where the function g is given by (1.3).

The class $\mathcal{S}^*_{\Sigma,2}(\gamma)$ is consisting of the bi-starlike functions of order γ. However, $C_{\Sigma,2}(\gamma)$ is called the class of bi-convex functions of order γ. Similarly, the hypotheses of Definitions 1 and 2 hold for the choice

$$H(z) = (\mu(z))^{\zeta} \quad \text{and} \quad P(z) = (\chi(z))^{\zeta}, \ (0 < \zeta \leq 1),$$

where μ and χ are given by

$$\mu(z) = 1 + \sum_{k=n}^{\infty} c_k z^{k-1} \text{ and } \chi(z) = 1 + \sum_{k=n}^{\infty} d_k z^{k-1}, \quad (z \in \mathbb{U}, n \geq 2), \tag{1.9}$$

such that $|c_k| \leq 2$ and $|d_k| \leq 2$, $\forall k \geq n$. This provides the following subclasses of Σ.

Definition 5. *A function $f \in \mathcal{A}$, defined by (1.2), is said to be in the class $\mathcal{S}^*_{\Sigma,n}[\zeta]$, $(n \geq 2)$, if it satisfies*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \frac{\zeta\pi}{2}, \quad (z \in \mathbb{U}, 0 < \zeta \leq 1)$$

and

$$\left|\arg\left(\frac{vg'(v)}{g(v)}\right)\right| \leq \frac{\zeta\pi}{2}, \quad (v \in \mathbb{U}, 0 < \zeta \leq 1)$$

where the function g is given by (1.3).

Definition 6. *A function $f \in \mathcal{A}$, defined by (1.2), is said to be in the class $C_{\Sigma,n}[\zeta]$, $(n \geq 2)$, if it satisfies*

$$\left|\arg\left(\frac{zf''(z)}{f'(z)}+1\right)\right| \leq \frac{\zeta\pi}{2}, \quad (z \in \mathbb{U}, 0 < \zeta \leq 1)$$

and

$$\left|\arg\left(\frac{vg''(v)}{g'(v)}+1\right)\right| \leq \frac{\zeta\pi}{2}, \quad (v \in \mathbb{U}, 0 < \zeta \leq 1)$$

where the function g is given by (1.3).

The class $\mathcal{S}^*_{\Sigma,2}[\zeta]$ is consisting of the strongly bi-starlike functions of order α. However, $C_{\Sigma,2}[\zeta]$ is called the class of strongly bi-convex functions of order ζ. When $n = 2$, the class $\mathcal{B}^{h,p}_{\Sigma,n}$ reduces to the class $\mathcal{B}^{h,p}_{\Sigma}$, which was studied by Bulut [5]. The definitions 3, 4 and 5 were defined by Brannan and Taha [21], for the special case whenever $n = 2$.

2 General coefficient estimates for the class $\mathcal{B}^{h,p}_{\Sigma,n}$

Firstly, given functions in the class $\mathcal{B}^{h,p}_{\Sigma,n}$, we obtain generic coefficient bounds.

Theorem 2. *Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \geq 2)$ be in the class $\mathcal{B}^{h,p}_{\Sigma,n}$. Then,*

$$|a_n| \leq \min\left\{\sqrt{\frac{|H^{(2n-2)}(0)| + |P^{(2n-2)}(0)|}{2(n-1)^2(2n-2)!}}, \frac{|H^{(n-1)}(0)|}{(n-1)!(n-1)}\right\}, \tag{2.1}$$

$$|a_j| \leq \frac{|H^{(j-1)}(0)|}{(j-1)!(j-1)}, \quad (n \leq j \leq 2n-2) \tag{2.2}$$

$$|a_{2n-1}| \leq \min\left\{\frac{(2n-1)|H^{(2n-2)}(0)| + |P^{(2n-2)}(0)|}{4(n-1)^2(2n-2)!}, \quad \frac{|H^{(2n-2)}(0)|}{(2n-2)!(2n-2)} + \frac{1}{2}\left(\frac{|H^{(n-1)}(0)|}{(n-1)!(n-1)}\right)^2\right\}, \tag{2.3}$$

$$|na_n^2 - a_{2n-1}| \leq \min\left\{\frac{(2n-1)|P^{(2n-2)}(0)| + |H^{(2n-2)}(0)|}{4(n-1)^2(2n-2)!}, \quad \frac{|P^{(2n-2)}(0)|}{(2n-2)!(2n-2)} + \frac{1}{2}\left(\frac{|H^{(n-1)}(0)|}{(n-1)!(n-1)}\right)^2\right\}, \tag{2.4}$$

$$|a_n^2 - 2a_{2n-1}| \leq \frac{|H^{(2n-2)}(0)|}{(2n-2)!(n-1)}, \tag{2.5}$$

$$|(2n-1)a_n^2 - 2a_{2n-1}| \leq \frac{|P^{(2n-2)}(0)|}{(2n-2)!(n-1)} \tag{2.6}$$

and

$$|na_n^2 - 2a_{2n-1}| \leq \frac{|H^{(2n-2)}(0)| + |P^{(2n-2)}(0)|}{(2n-2)!(2n-2)} \tag{2.7}$$

*Proof.*By virtue of conditions (1.4) and (1.5), we have

$$zf'(z) = f(z)H(z) \quad \text{and} \quad wg'(w) = g(w)P(w), \quad (z, w \in \mathbb{U}), \tag{2.8}$$

where $g(w) = w + \sum_{k=n}^{\infty} b_k w^k$; $(n \geq 2)$ is the inverse of f,

$$H(z) = 1 + \sum_{k=n}^{\infty} h_k z^{k-1} \text{ and } P(w) = 1 + \sum_{k=n}^{\infty} p_k w^{k-1}.$$

According to (2.8), we can write

$$z + \sum_{k=n}^{\infty} k a_k z^k = z + \sum_{k=n}^{\infty} (a_k + h_k) z^k + \left(\sum_{k=n}^{\infty} a_k z^k\right)\left(\sum_{k=n}^{\infty} h_k z^{k-1}\right) \tag{2.9}$$

and

$$w + \sum_{k=n}^{\infty} k b_k w^k = w + \sum_{k=n}^{\infty} (b_k + p_k) w^k + \left(\sum_{k=n}^{\infty} b_k w^k\right)\left(\sum_{k=n}^{\infty} p_k w^{k-1}\right). \tag{2.10}$$

For $j \geq n$, a computation shows that

$$H^{(j-1)}(z) = \sum_{k=j}^{\infty} (k-1)(k-2)\cdots(k-j+1) h_k z^{k-j} \tag{2.11}$$

and so

$$H^{(j-1)}(0) = (j-1)! h_j, \quad (j \geq n). \tag{2.12}$$

Now, equating the coefficients of z^j,$(n \leq j \leq 2n-2)$ in both sides of (2.9), implies that

$$h_j = (j-1) a_j, \tag{2.13}$$

which is, by (2.12), equivalent to

$$a_j = \frac{H^{(j-1)}(0)}{(j-1)!(j-1)}, \quad (n \leq j \leq 2n-2). \tag{2.14}$$

This gives the estimate (2.2). In particular, we have

$$h_n = (n-1) a_n \tag{2.15}$$

and

$$a_n = \frac{H^{(n-1)}(0)}{(n-1)!(n-1)} \tag{2.16}$$

Further, equating the coefficients of z^{2n-1} in both sides of (2.9), implies that

$$(2n-2)a_{2n-1} = h_{2n-1} + a_n h_n. \tag{2.17}$$

In view of (2.12) and (2.15), equation (2.17) can be written as

$$(2n-2)a_{2n-1} = \frac{H^{(2n-2)}(0)}{(2n-2)!} + (n-1)a_n^2. \tag{2.18}$$

Similarly, we have

$$(2n-2)b_{2n-1} = \frac{P^{(2n-2)}(0)}{(2n-2)!} + (n-1)b_n^2. \tag{2.19}$$

By Theorem 1, equation (2.19) can be expressed as

$$(2n-2)(na_n^2 - a_{2n-1}) = \frac{P^{(2n-2)}(0)}{(2n-2)!} + (n-1)a_n^2. \tag{2.20}$$

Equations (2.18) and (2.20), by addition, give

$$a_n^2 = \frac{H^{(2n-2)}(0) + P^{(2n-2)}(0)}{2(n-1)^2(2n-2)!}. \tag{2.21}$$

Combining (2.21) along with (2.16), we get the estimate (2.1). By substituting a_n, defined by (2.16), in (2.18), we reach at

$$a_{2n-1} = \frac{H^{(2n-2)}(0)}{(2n-2)!(2n-2)} + \frac{1}{2}\left(\frac{H^{(n-1)}(0)}{(n-1)!(n-1)}\right)^2. \tag{2.22}$$

On the other hand, substituting a_n, defined by (2.21), in (2.18) yields

$$a_{2n-1} = \frac{(2n-1)H^{(2n-2)}(0) + P^{(2n-2)}(0)}{4(n-1)^2(2n-2)!}. \tag{2.23}$$

Combining (2.22) along with (2.23) yields the estimate (2.3). Similarly, substituting a_n, defined by (2.16) or (2.21), in (2.20) ensure that the estimate (2.4) holds true. Next, the estimates (2.5) and (2.6) follow respectively from (2.18) and (2.20). Finally, an application of the triangle inequality to (2.5) and (2.6) gives (2.7). The proof is complete.

Setting $n = 2$ in Theorem 2 gives the following corollary.

Corollary 1. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be in the class $\mathcal{B}_{\Sigma,2}^{h,p}$. Then,*

$$|a_2| \le \min\left\{\frac{\sqrt{|H''(0)| + |P''(0)|}}{2}, |H'(0)|\right\},$$

$$|a_3| \le \min\left\{\frac{3|H''(0)| + |P''(0)|}{8}, \frac{|H''(0)|}{4} + \frac{|H'(0)|^2}{2}\right\},$$

$$|2a_2^2 - a_3| \le \min\left\{\frac{3|P''(0)| + |H''(0)|}{8}, \frac{|P''(0)|}{4} + \frac{|H'(0)|^2}{2}\right\},$$

$$|a_2^2 - 2a_3| \le \frac{|H''(0)|}{2},$$

$$|3a_2^2 - 2a_3| \le \frac{|P''(0)|}{2}$$

and

$$|a_2^2 - a_3| \le \frac{|H''(0)| + |P''(0)|}{8}$$

Corollary 2. *Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$ be in the class $\mathcal{S}^*_{\Sigma,n}(\gamma)$, $(n \geq 2)$. Then*

$$|a_n| \leq \begin{cases} \frac{\sqrt{2(1-\gamma)}}{n-1}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}, \\ \frac{2(1-\gamma)}{n-1}, & \text{if } \frac{1}{2} \leq \gamma < 1, \end{cases}$$

$$|a_j| \leq \frac{2(1-\gamma)}{j-1}, \quad (n \leq j \leq 2n-2)$$

$$|a_{2n-1}| \leq \begin{cases} \frac{n(1-\gamma)}{(n-1)^2}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}, \\ \frac{(n-2\gamma+1)(1-\gamma)}{(n-1)^2}, & \text{if } \frac{1}{2} \leq \gamma < 1, \end{cases}$$

$$|na_n^2 - a_{2n-1}| \leq \begin{cases} \frac{n(1-\gamma)}{(n-1)^2}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}, \\ \frac{(n-2\gamma+1)(1-\gamma)}{(n-1)^2}, & \text{if } \frac{1}{2} \leq \gamma < 1, \end{cases}$$

$$|a_n^2 - 2a_{2n-1}| \leq \frac{2(1-\gamma)}{n-1}$$

$$|(2n-1)a_n^2 - 2a_{2n-1}| \leq \frac{2(1-\gamma)}{n-1}$$

and

$$|na_n^2 - 2a_{2n-1}| \leq \frac{2(1-\gamma)}{n-1}.$$

Proof. Assume that H and P have the definitions found in (1.8). For any $k \geq n$, a calculation demonstrates that

$$|H^{(k-1)}(0)| = (k-1)!|h_k| \leq (k-1)!(2-2\gamma) \tag{2.24}$$

and

$$|P^{(k-1)}(0)| = (k-1)!|p_k| \leq (k-1)!(2-2\gamma). \tag{2.25}$$

The proof is complete by applying (2.24) and (2.25) to Theorem 2.

Remark 1. The estimation of $|a_n|$ in Corollary 2 improves [23, Corollary 3] and [24, Theorem 1]. Moreover, for the special case, when $n = 2$ in Corollary 2, the estimates

$$|a_2| \leq \begin{cases} \sqrt{2(1-\gamma)}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}, \\ 2(1-\gamma), & \text{if } \frac{1}{2} \leq \gamma < 1, \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\gamma), & \text{if } 0 \leq \gamma \leq \frac{1}{2}, \\ (3-2\gamma)(1-\gamma), & \text{if } \frac{1}{2} \leq \gamma < 1 \end{cases}$$

improve those given by Bulut [5, Corollary 3.2] and Brannan and Taha [21, Theorem 3.1].
Setting $n = 2$ and $\gamma = 1/2$ in Corollary 2 gives the following

Remark 2. Let f defined by (1.2) be in the class $\mathcal{S}^*_{\Sigma,2}(1/2)$. Then $|a_2|$, $|a_3|$, $|2a_2^2 - a_3|$, $|a_2^2 - 2a_3|$, $|3a_2^2 - 2a_3|$ are all bounded above by 1 and all the estimates are sharp, where the equality attained in each case for the function

$$f(z) = \frac{z}{1-z}$$

and its rotations.
The following theorem introduces general coefficient bounds for functions in the class $\mathcal{S}^*_{\Sigma,n}[\zeta]$.

Theorem 3. *Let f, defined by (1.2), be in the class $\mathcal{S}^*_{\Sigma,n}[\zeta]$, $(n \geq 2)$. Then,*

$$|a_n| \leq \frac{\sqrt{2}\zeta}{n-1},$$

$$|a_k| \le \frac{2\zeta}{k-1}, \ \ (n \le k \le 2n-2),$$

$$|a_{2n-1}| \le \frac{n\zeta^2}{(n-1)^2},$$

$$|na_n^2 - a_{2n-1}| \le \frac{n\zeta^2}{(n-1)^2},$$

$$|a_n^2 - 2a_{2n-1}| \le \frac{2\zeta^2}{n-1},$$

$$|(2n-1)a_n^2 - 2a_{2n-1}| \le \frac{2\zeta^2}{n-1}$$

and

$$|na_n^2 - 2a_{2n-1}| \le \frac{2\zeta^2}{n-1}$$

*Proof.*Let $H(z) = (\mu(z))^\zeta$ and $P(z) = (\chi(z))^\zeta$, $(0 < \zeta \le 1)$, where μ and χ are defined by (1.9). For $j \ge n$, it follows

$$\mu^{(j-1)}(z) = \sum_{k=j}^{\infty}(k-1).(k-2)....(k+1-j)c_k z^{k-j}.$$

A computation yields that $\mu(z)H'(z) = \zeta H(z)\mu'(z)$, which means

$$\left(1+\sum_{k=n}^{\infty}c_k z^{k-1}\right)\left(\sum_{k=n}^{\infty}(k-1)h_k z^{k-2}\right) = \zeta\left(1+\sum_{k=n}^{\infty}h_k z^{k-1}\right)\left(\sum_{k=n}^{\infty}(k-1)c_k z^{k-2}\right). \tag{2.26}$$

Comparing the coefficients of z^j; $(n \le j \le 2n-2)$ in both sides of (2.26) yields that $h_j = \zeta c_j$. This implies that

$$\frac{|H^{(j-1)}(0)|}{(j-1)!} = |c_j|\zeta \le 2\zeta. \tag{2.27}$$

In particular, we have

$$h_n = \zeta c_n \tag{2.28}$$

and

$$\frac{|H^{(n-1)}(0)|}{(n-1)!} = |c_n|\zeta \le 2\zeta. \tag{2.29}$$

Now, equating the coefficients of z^{2n-3} in both sides of (2.26) implies that

$$(n-1)h_n c_n + (2n-2)h_{2n-1} = \zeta(2n-2)c_{2n-1} + \zeta(n-1)h_n c_n. \tag{2.30}$$

Therefore, by substituting (2.28) in (2.30), we obtain

$$h_{2n-1} = \zeta c_{2n-1} + \frac{\zeta(\zeta-1)}{2}c_n^2. \tag{2.31}$$

Equation (2.31) can be written as

$$\frac{H^{(2n-2)}(0)}{(2n-2)!} = \zeta c_{2n-1} + \frac{\zeta(\zeta-1)}{2}c_n^2.$$

It follows that

$$|H^{(2n-2)}(0)| \le \zeta(2n-2)!|c_{2n-1}| + \frac{\zeta(\zeta-1)(2n-2)!}{2}|c_n|^2 \le (2n-2)!(2\zeta^2). \tag{2.32}$$

Similarly,

$$|P^{(2n-2)}(0)| \le (2n-2)!(2\zeta^2). \tag{2.33}$$

By applying (2.27), (2.29), (2.32) and (2.33) to Theorem 2, we obtain the desired estimates and the proof is complete.

*Remark.*For $n = 2$ in Theorem 3, the estimates of $|a_2|$ and $|a_3|$ were given by Bulut [2, Corollary 3.1]. Indeed, they improve those given by Brannan and Taha [21].

3 General coefficient estimates for the class $\mathcal{A}^{h,p}_{\Sigma,n}$

Theorem 4. *Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \geq 2)$ be in the class $\mathcal{A}^{h,p}_{\Sigma,n}$. Then,*

$$|a_n| \leq \min\left\{\sqrt{\frac{|H^{(2n-2)}(0)| + |P^{(2n-2)}(0)|}{2n(n-1)^2(2n-2)!}}, \frac{|H^{(n-1)}(0)|}{n!(n-1)}\right\}, \tag{3.1}$$

$$|a_j| \leq \frac{|H^{(j-1)}(0)|}{j!(j-1)}, \quad (n \leq j \leq 2n-3, \ n \geq 3) \tag{3.2}$$

$$|a_{2n-1}| \leq \min\left\{\frac{(3n-2)|H^{(2n-2)}(0)| + n|P^{(2n-2)}(0)|}{4(n-1)^2(2n-1)!}, \ \frac{|H^{(2n-2)}(0)|}{(2n-1)!(2n-2)} + \frac{1}{2(2n-1)}\left(\frac{|H^{(n-1)}(0)|}{(n-1)!(n-1)}\right)^2\right\}, \tag{3.3}$$

$$|na_n^2 - a_{2n-1}| \leq \min\left\{\frac{(3n-2)|P^{(2n-2)}(0)| + n|H^{(2n-2)}(0)|}{4(n-1)^2(2n-1)!}, \ \frac{|P^{(2n-2)}(0)|}{(2n-1)!(2n-2)} + \frac{1}{2(2n-1)}\left(\frac{|H^{(n-1)}(0)|}{(n-1)!(n-1)}\right)^2\right\}, \tag{3.4}$$

$$|n^2a_n^2 - 2(2n-1)a_{2n-1}| \leq \frac{|H^{(2n-2)}(0)|}{(2n-2)!(n-1)}, \tag{3.5}$$

$$|n(3n-2)a_n^2 - 2(2n-1)a_{2n-1}| \leq \frac{|P^{(2n-2)}(0)|}{(2n-2)!(n-1)} \tag{3.6}$$

and

$$|na_n^2 - 2a_{2n-1}| \leq \frac{|H^{(2n-2)}(0)| + |P^{(2n-2)}(0)|}{(2n-1)!(2n-2)} \tag{3.7}$$

Proof. The relations (1.6) and (1.7) imply that

$$zf''(z) = f'(z)(H(z)-1) \quad \text{and} \quad vg''(v) = g'(v)(P(v)-1), \quad (z, v \in \mathbb{U}), \tag{3.8}$$

where $g(v) = v + \sum_{k=n}^{\infty} b_k v^k$; $(n \geq 2)$ is the inverse of f,

$$H(z) = 1 + \sum_{k=n}^{\infty} h_k z^{k-1} \ \text{ and } \ P(w) = 1 + \sum_{k=n}^{\infty} p_k w^{k-1}.$$

By (3.8), we can write

$$\sum_{k=n}^{\infty} k(k-1)a_k z^{k-1} = \left(1 + \sum_{k=n}^{\infty} k a_k z^{k-1}\right)\sum_{k=n}^{\infty} h_k z^{k-1} \tag{3.9}$$

and

$$\sum_{k=n}^{\infty} k(k-1)b_k v^{k-1} = \left(1 + \sum_{k=n}^{\infty} k b_k v^{k-1}\right)\sum_{k=n}^{\infty} p_k v^{k-1}$$

For $j \geq n$, a computation shows that

$$H^{(j-1)}(z) = \sum_{k=j}^{\infty} (k-1)(k-2)\cdots(k-j+1)h_k z^{k-j}$$

and so

$$H^{(j-1)}(0) = (j-1)!h_j, \ \ (j \geq n). \tag{3.10}$$

Now, equating the coefficients of z^j,$(n \leq j \leq 2n-3)$ in both sides of (3.9), implies that

$$h_j = j(j-1)a_j,$$

which is, by (3.10), equivalent to

$$a_j = \frac{H^{(j-1)}(0)}{j!(j-1)}, \quad (n \leq j \leq 2n-3).$$

This gives the estimate (3.2). In particular, we have

$$h_n = n(n-1)a_n \tag{3.11}$$

and

$$a_n = \frac{H^{(n-1)}(0)}{n!(n-1)} \tag{3.12}$$

Further, equating the coefficients of z^{2n-2} in both sides of (3.9), implies that

$$(2n-1)(2n-2)a_{2n-1} = h_{2n-1} + na_nh_n. \tag{3.13}$$

In view of (3.10) and (3.11), equation (3.13) can be written as

$$(2n-1)(2n-2)a_{2n-1} = \frac{H^{(2n-2)}(0)}{(2n-2)!} + (n-1)n^2a_n^2. \tag{3.14}$$

Similarly, we have

$$(2n-1)(2n-2)b_{2n-1} = \frac{P^{(2n-2)}(0)}{(2n-2)!} + (n-1)n^2b_n^2. \tag{3.15}$$

By Theorem 1, equation (3.15) can be expressed as

$$(2n-1)(2n-2)(na_n^2 - a_{2n-1}) = \frac{P^{(2n-2)}(0)}{(2n-2)!} + (n-1)n^2a_n^2. \tag{3.16}$$

Equations (3.14) and (3.16), by addition, gives

$$a_n^2 = \frac{H^{(2n-2)}(0) + P^{(2n-2)}(0)}{2n(n-1)^2(2n-2)!}. \tag{3.17}$$

Combining (3.17) along with (3.12), we get the estimate (3.1). By substituting a_n, defined by (3.12), in (3.14), we reach at

$$a_{2n-1} = \frac{H^{(2n-2)}(0)}{(2n-1)!(2n-2)} + \frac{1}{2(2n-1)}\left(\frac{H^{(n-1)}(0)}{(n-1)!(n-1)}\right)^2. \tag{3.18}$$

On the other hand, substituting a_n, defined by (3.17), in (3.14) yields

$$a_{2n-1} = \frac{(3n-2)H^{(2n-2)}(0) + nP^{(2n-2)}(0)}{4(n-1)^2(2n-1)!}. \tag{3.19}$$

Combining (3.18) along with (3.19) yields the estimate (3.3). Similarly, substituting a_n, defined by (3.12) or (3.17), in (3.16) ensure that the estimate (3.4) holds true. Next, the estimates (3.5) and (3.6) follow respectively from (3.14) and (3.16). Finally, an application of the triangle inequality to (3.5) and (3.6) gives (3.7). This completes the proof of Theorem 4.

Letting $n = 2$ in Theorem 4 yields the following corollary.

Corollary 3. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be in the class $\mathcal{A}_{\Sigma,2}^{h,p}$. Then,*

$$|a_2| \le \min\left\{\frac{\sqrt{|H''(0)| + |P''(0)|}}{8}, \frac{|H'(0)|}{2}\right\},$$

$$|a_3| \le \min\left\{\frac{2|H''(0)| + |P''(0)|}{12}, \frac{|H''(0)| + |H'(0)|^2}{6}\right\},$$

$$|2a_2^2 - a_3| \le \min\left\{\frac{2|H''(0)| + |P''(0)|}{12}, \frac{|H''(0)| + |H'(0)|^2}{6}\right\},$$

$$|2a_2^2 - 3a_3| \le \frac{|H''(0)|}{4},$$

$$|4a_2^2 - 3a_3| \le \frac{|P''(0)|}{4}$$

and

$$|a_2^2 - a_3| \le \frac{|H''(0)| + |P''(0)|}{24}$$

The following corollaries are stated without proofs, since the proofs are respectively similar to those of Corollary 2 and Theorem 3.

Corollary 4.*Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$ be in the class $C_{\Sigma,n}[\zeta]$, $(n \geq 2)$. Then*

$$|a_n| \leq \frac{2\zeta}{n(n-1)},$$

$$|a_{2n-1}| \leq \frac{\zeta^2}{(n-1)^2}$$

$$|na_n^2 - a_{2n-1}| \leq \frac{\zeta^2}{(n-1)^2}$$

$$|n^2 a_n^2 - 2(2n-1)a_{2n-1}| \leq \frac{2\zeta^2}{n-1}$$

$$|n(3n-2)a_n^2 - 2(2n-1)a_{2n-1}| \leq \frac{2\zeta^2}{n-1}$$

and

$$|na_n^2 - 2a_{2n-1}| \leq \frac{2\zeta^2}{(2n-1)(n-1)}.$$

For $n \leq j \leq 2n-3$ with $n \geq 3$, we have

$$|a_j| \leq \frac{2\zeta}{j(j-1)}.$$

Corollary 5.*Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$ be in the class $C_{\Sigma,n}(\gamma)$, $(n \geq 2)$. Then*

$$|a_n| \leq \frac{2(1-\gamma)}{n(n-1)}, \tag{3.20}$$

$$|a_{2n-1}| \leq \frac{(n-2\gamma+1)(1-\gamma)}{(2n-1)(n-1)^2} \tag{3.21}$$

$$|na_n^2 - a_{2n-1}| \leq \frac{(n-2\gamma+1)(1-\gamma)}{(2n-1)(n-1)^2} \tag{3.22}$$

$$|n^2 a_n^2 - 2(2n-1)a_{2n-1}| \leq \frac{2(1-\gamma)}{n-1} \tag{3.23}$$

$$|n(3n-2)a_n^2 - 2(2n-1)a_{2n-1}| \leq \frac{2(1-\gamma)}{n-1} \tag{3.24}$$

and

$$|na_n^2 - 2a_{2n-1}| \leq \frac{2(1-\gamma)}{(2n-1)(n-1)}.$$

For $n \leq j \leq 2n-3$ with $n \geq 3$, we have

$$|a_j| \leq \frac{2(1-\gamma)}{j(j-1)}.$$

The estimates (3.20), (3.21), (3.22), (3.23) and (3.24) are all sharp when $n = 2$ and $\gamma = 0$, where the equality attained in each estimate for the function

$$f(z) = \frac{z}{1-z}$$

and its rotations.

Remark 3. For the special case when $n = 2$ in Corollary 5, the estimates

$$|a_2| \leq 1-\gamma \tag{3.25}$$

and

$$|a_3| \leq \frac{(3-2\gamma)(1-\gamma)}{3} \tag{3.26}$$

are sharp, where the equality attained for the function

$$f(z) = \begin{cases} \frac{1-(1-z)^{(2\gamma-1)}}{2\gamma-1}, & \gamma \neq \frac{1}{2}, \\ -\log(1-z), & \gamma = \frac{1}{2} \end{cases}$$

and its rotations. We emphasize that the estimates (3.25) and (3.26) are much better than those given by Brannan and Taha [21, Theorem 4.1].

4 Conclusion

The upper bounds for the general coefficient $|a_n|$ and the first two coefficients $|a_2|$, $|a_3|$, to different classes of analytic bi-univalent functions have been explored by numerous writers. This work uses a new method inspired by a recent result of Al-Refai and Ali [28] to examine constraints for different coefficient functionals to fundamental subclasses of bi-univalent functions. While some boundaries are tight, others enhance and expand upon well-known approximations. The novel method may enhance known results and encourage other scholars to look at bounds for different subclasses of Σ.

Declarations

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