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VERTEX INDUCED $k-$EDGE COLORING AND VERTEX INCIDENT $k-$EDGE COLORING OF GRAPHS

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Abstract. Let $k \geq 2$ be a natural number. Then the vertex induced $k-$edge coloring number $\psi'_{vik}(G)$ of a simple connected graph $G = (V, E)$ is the highest number of colors needed to color the edges of a graph G such that the edges of the subgraph induced by the closed neighborhood $N[v]$ of the vertex $v \in V(G)$ receives not more than k colors.

The vertex incident $k-$edge coloring number $\psi'_{vink}(G)$ of a simple connected graph $G = (V, E)$ is the highest number of colors required to color the edges of a graph G such that the edges incident to a vertex v in graph G receives not more than k colors. In this paper, we initiate the study on $\psi'_{vik}(G)$ and $\psi'_{vink}(G)$. We also determine the exact values of $\psi'_{vik}(G)$ and $\psi'_{vink}(G)$ for $k = 2$ for some special graphs.

1. Introduction

A wide range of studies has been explored in the area of graph coloring concepts where one has to maximize the number of colors under valid conditions. The notion of 3-consecutive vertex coloring number was first introduced by E. Sampathkumar in [9] and later its edge analog named 3-consecutive edge coloring number was studied in [2]. The findings in this paper have been inspired by the concepts studied in [9, 2, 7, 5].

For a graph $G = (V, E)$, the closed neighborhood of a vertex v in the graph G is denoted as $N[v]$ and is equal to $\{v\} \cup \{u : uv \in E\}$. Throughout this manuscript

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$\langle N[v]\rangle$ represents the subgraph induced by $N[v]$. The star subgraph at a vertex v is the subgraph of G containing all the edges incident at v.

The vertex induced $k-$edge coloring of a graph G is an edge coloring of G such that for each vertex v in $V(G)$, not more than k edges in the $\langle N[v]\rangle$ receives distinct colors, where $k \in \mathbb{N}$ and $k \geq 2$. The vertex induced $k-$edge coloring number is the highest number of colors conceded in such a coloring. Whereas, the vertex incident $k-$edge coloring number of a graph G is the maximum number of colors required to color the graph edges $E(G)$ such that for each vertex v in $V(G)$, not more than k edges, where $k \in \mathbb{N}$ and $k \geq 2$, in the star subgraph at a vertex v receive distinct colors. The vertex induced $k-$edge coloring and vertex incident $k-$edge coloring concepts are the generalized version of the edge coloring approach as introduced and studied in [7].

In this paper, we initiate a combinatorial study of vertex induced $k-$edge coloring number and vertex incident $k-$edge coloring number and obtain some bounds besides finding the exact value of these parameters for some graph classes. We have also determined the exact value of these parameters for $k = 2$ for some special graphs. For more definitions on graph theory refer to [4].

2. Results on $\psi'_{vik}(G)$ and $\psi'_{vink}(G)$

This section deals with some bounds and exact values of vertex induced $k-$edge coloring and vertex incident $k-$edge coloring of some graphs. Also we characterize simple connected graphs for which $\psi'_{vik}(G) = m$. The following bounds are obvious for a connected simple graph $G = (V, E)$ of order n and size m.

Theorem 2.1. For a connected graph G with size $m \geq 2$ and for $k \geq 3$,
(i) $2 \leq \psi'_{vik}(G) \leq m$.
(ii) $2 \leq \psi'_{vink}(G) \leq m$.

Next theorem provides a characterization of $\psi'_{vik}(G)$ in terms of the number of edges in a graph G. For a connected graph G of order $n \geq 2$ we denote the number of edges in the subgraph induced by $N[v_i]$ by $|E(\langle N[v_i]\rangle)|$. Let $q = \max_i |E(\langle N[v_i]\rangle)|$. For example, consider the graph G_1 depicted in the figure 1. In figure 1, $|E(\langle N[v_1]\rangle)| =$

$1, |E(\langle N[v_2]\rangle)| = 2, |E(\langle N[v_3]\rangle)| = 4, |E(\langle N[v_4]\rangle)| = 3$ and $|E(\langle N[v_5]\rangle)| = 3$. Therefore in graph G_1, $q = 4$.

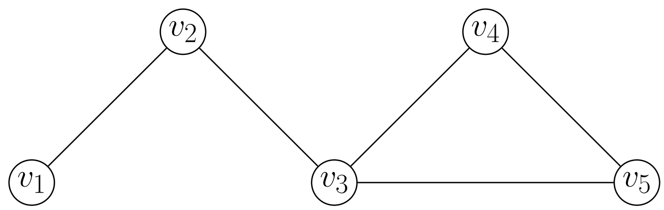

FIGURE 1. The graph G_1 with $q = 4$

Theorem 2.2. *Let G be a simple connected $(n, m)-$graph of order $n \geq 2$ and let $2 \leq k \leq m$. Then $\psi'_{vik}(G) = m$ if and only if $m \geq k \geq q$, where $q = \max_i |E(\langle N[v_i]\rangle)|$.*

Proof. Without loss of generality assume that G is a connected $(n, m)-$graph of order $n \geq 2$ and $2 \leq k \leq m$. To prove the necessary condition let $\psi'_{vik}(G) = m$. Suppose that $k < q$, where $q = \max_i |E(\langle N[v_i]\rangle)|$. Then, there exists at least one edge $e \in E(G)$ such that the edge e receives a repeated color. This implies $\psi'_{vik}(G) < m$, a contradiction. Therefore, $m \geq k \geq q$. Conversely, $m \geq k \geq q$. Since $k \geq q$, by the definition of vertex induced $k-$edge coloring, we can use distinct colors to color all the edges in $\langle N[v_i]\rangle$. Therefore, $\psi'_{vik}(G) = m$. $\square$

Corollary 2.1. *Let $G = (V, E)$ be a simple connected $(n, m)-$graph with $\Delta(G) \leq 2$. If G is either a path or a cycle, then for $k \geq 3$, $\psi'_{vik}(G) = \psi'_{vink}(G) = m$.*

Theorem 2.3. *If $G = (V, E)$ is a simple connected graph, then $\psi'_{vik}(G) \leq \psi'_{vink}(G)$ where $k \geq 3$.*

Proof. It can be noted that any vertex induced $k-$edge coloring of a graph G is a vertex incident $k-$edge coloring of the graph G. Since we are maximizing the number of colors to obtain vertex induced $k-$edge coloring number of G and vertex incident $k-$edge coloring number of G. Therefore, it follows that $\psi'_{vik}(G) \leq \psi'_{vink}(G)$. $\square$

The above result can be verified with an example. Consider the friendship graph F_n with n triangles T_i, where $1 \leq i \leq n$, with a common vertex. By construction, the friendship graph F_n is a simple planar graph having $2n+1$ vertices and $3n$ edges. Note that, $\psi'_{vik}(F_n) = k$ whereas, the vertex incident $k-$edge coloring number of graph F_n is $\psi'_{vink}(F_n) = k + n$.

For a triangle free graph, the subgraph induced by $N[v]$ is isomorphic to the star subgraph at vertex v. Hence the definition of the vertex induced $k-$edge coloring number and the vertex incident $k-$edge coloring number coincides for a triangle free graph. Therefore, if G is triangle free graph, then $\psi'_{vik}(G) = \psi'_{vink}(G)$. Similarly, the equality of $\psi'_{vik}(G)$ and $\psi'_{vink}(G)$ holds if $k = |E(G)|$.

Theorem 2.4. *Let $G = (V, E)$ be a simple connected graph with $V(G)$ and $E(G)$ as the vertex set and edge set of the graph G, respectively. If G is a graph with a universal vertex v, then $\psi'_{vik}(G) = k$.*

Proof. Since v is an universal vertex in G, i.e. $deg(v) = |V(G)| - 1$. So, $G = \langle N[v]\rangle$, where $\langle N[v]\rangle$ denotes the subgraph induced by $N[v]$. Therefore, one can use at most k colors to color the edges in graph G. Hence the result. □

The following graphs are some standard graphs with a universal vertex v.

Corollary 2.2. *i) For a complete graph K_n on $n \geq 3$ vertices , $\psi'_{vik}(K_n) = k$.*
ii) For a wheel graph W_n on $n \geq 4$ vertices, $\psi'_{vik}(W_n) = k$.
iii) For a star graph $K_{1,n}$ on $n \geq 3$ vertices, $\psi'_{vik}(K_{1,n}) = k$.

Theorem 2.5. *Let G be a simple connected $(n, m)-$graph. Then $k \geq \Delta(G)$ if and only if $\psi'_{vink}(G) = m$.*

Proof. If $k \geq \Delta(G)$, then it can be easily verified that $\psi'_{vink}(G) = m$. Conversely, assume for a simple connected $(n, m)-$graph with n vertices and m edges that $\psi'_{vink}(G) = m$. Suppose that $k < \Delta(G)$. Let v be a vertex in G with maximum degree $deg(v) = \Delta(G)$. Then there will be at least one edge incident to the vertex v, which receives a repeated color. Thus, $\psi'_{vink}(G) < m$, a contradiction. Therefore, $k \geq \Delta(G)$. □

Hereafter we assume that $k < \Delta(G)$, unless otherwise mentioned.

For integers n and r with $2 \leq 2r < n$, the generalized Petersen graph $G(n,r)$ as defined in [12] is a graph with vertex set $V(G(n,r)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E(G(n,r))$ consisting of all edges of the form $[u_iu_{i+1}], [u_iv_i], [v_iv_{i+r}]$, where i is an integer, $0 \leq i \leq n-1$ and the sums are *mod* n, $r < \frac{n}{2}$.

Theorem 2.6. *Vertex induced (incident) $k-$ edge coloring number of some standard graphs are listed below:*

(i) *For a complete graph K_n of order $n \geq 5$ and with $3 \leq k < n-1$, $n+1 \leq \psi'_{vink}(K_n) \leq \frac{n^2-3n+4}{2}$.*

(ii) *For a wheel graph W_n of order $n \geq 4$ and $3 \leq k < n$, $\psi'_{vink}(W_n) = k+n-1$.*

(iii) *For a generalized Petersen graph $G(n,r)$ where $n \geq 3$ and $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, $\psi'_{vik}(G(n,r)) = \psi'_{vink}(GP(n,r)) = 3n$.*

(iv) *For a complete bipartite graph $K_{m,n}$ with order $m,n \geq 1$,*

$$\psi'_{vik}(K_{m,n}) = \psi'_{vink}(K_{m,n}) = \begin{cases} k \cdot min\{m,n\}, & for\ \delta(K_{m,n}) \leq k < \Delta(K_{m,n}) \\ mn, & for\ k \geq \Delta(K_{m,n}) \end{cases}$$

A matching of a graph G is a set of mutually non-incident edges. A matching is said to be maximum if it has the largest number of edges among all matching of the graph G. The number of edges of a maximum matching of G is called the matching number of G and is denoted by $\nu(G)$ [1].

Theorem 2.7. *Let G be a $(n,m)-$graph with $m \geq 2$. Let M be a maximum matching set of G and S be an independent set of vertices in G such that $\sum\limits_{v \in S} deg(v)$ is maximum. Then, for $k \leq \Delta(G)$, $|M|+1 \leq \psi'_{vink}(G) \leq m - \sum\limits_{v \in S} deg(v) + k|S|$.*

Proof. Consider G to be a connected $(n,m)-$graph with $m \geq 2$. Let M be the maximum matching of graph G. Then, in any vertex incident $k-$edge coloring, each edge in the set M can be assigned with any of the $|M|$ distinct colors. So, the remaining uncolored edges in the graph G can be given at least one more color. Thus, for $k < \Delta(G)$, $|M|+1 \leq \psi'_{vink}(G)$.

Let S be an independent set of vertices in G. Then, at most k colors are used to color the edges incident at each vertex $v \in S$ where $k < \Delta(G)$. Thus, at most $k|S|$ colors are required to color all the edges incident to the vertices in S. From theorem 2.1, we know that the maximum number of colors used in coloring all the edges of graph G with distinct colors is m. Therefore, the remaining uncolored edges of graph G requires at most $m - \sum\limits_{v \in S} deg(v)$ colors. □

In the above theorem, the upper bound is attained in the case of cycle and tree when $k = 2$, while, the lower bound is satisfied for the graph P_3.

Corollary 2.3. *If G is a r-regular graph, $\nu(G)$ be the matching number of G and S is an independent set of vertices in G, then for $3 \leq k < r$, $\nu(G) + 1 \leq \psi'_{vink}(G) \leq m - |S|(r - k)$.*

Theorem 2.8. *Let $\langle N[v] \rangle$ be an induced subgraph of a simple connected graph $G = (V, E)$. If $s = max|E(\langle N[v] \rangle)|, v \in V(G)$, then $\psi'_{vik}(G) \leq m - s + k$.*

Proof. At most k colors are required to color the edges of induced subgraph $\langle N[v] \rangle$. From theorem 2.1, we know that $\psi'_{vik}(G) \leq m$. So, for the remaining uncolored edges of the graph G we can use maximum of $m - s$ colors. Therefore, at most $m - s + k$ colors are required to color the edges of a graph G with vertex induced $k-$edge coloring. □

Corollary 2.4. *Let $\nu(G)$ denote the matching number of a graph G. Then for a triangle-free connected $(n, m)-$graph G and for $2 \leq k \leq \Delta(G)$, $\nu(G) + 1 \leq \psi'_{vik}(G) \leq \psi'_{vink}(G) \leq m - \Delta(G) + k$.*

Proof. Since $\Delta(G) \leq s$. This result follows from theorem 2.1, theorem 2.3, theorem 2.7 and theorem 2.8. □

In a graph $G = (V, E)$, a set $A \subseteq V$ is called a neighbourhood set or an $n-$set if $G = \bigcup\limits_{v \in A} \langle N[v] \rangle$, where $\langle N[v] \rangle$ denotes the subgraph of G induced by the vertices in $N[v]$. The neighbourhood number, $n_0(G)$, of G is the minimum cardinality of an $n-$set of G [8].

Theorem 2.9. *For any connected graph G of order $n \geq 2$, $\psi'_{vik}(G) \leq k\ n_0(G)$.*

Proof. Since at most k colors are used to color the edges of subgraph $\langle N[v]\rangle$. Hence the result follows. $\square$

This bound is sharp and is attained for a graph with an universal vertex.

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is called vertex covering of G if every edge $uv \in E$ is incident with a vertex in S. The vertex covering number $\alpha_0(G)$ is the minimum cardinality of a vertex covering set of graph G.

- For a graph without isolated vertices, $n_0(G) \leq \alpha_0(G)$ (see [8]).
- For a graph G of order n, $n_0(G) = 1$ if and only if G has a vertex of degree $n-1$ (see [8]).

Corollary 2.5. *For any simple connected graph G of order $n \geq 2$, $\psi'_{vik}(G) \leq k\ \alpha_0(G)$.*

It is to be noted that, $k\ n_0(G)$ is not an upper bound for $\psi'_{vink}(G)$. For example consider, the friendship graph F_n. The neighbourhood number $n_0(F_n) = 1$, whereas, as discussed above $\psi'_{vink}(F_n) = k + n$.

However, the following result holds.

Theorem 2.10. *For a simple connected graph G of order $n \geq 2$ and for $k \leq \Delta(G)$, $\psi'_{vink}(G) \leq k\ \alpha_0(G)$.*

Proof. Consider S to be the minimum vertex covering set of graph G. Then, every edge in G is incident with a vertex in S. Since $k \leq \Delta(G)$, in a vertex incident $k-$edge coloring of G, at most k colors can be used to color the edges incident with a vertex $v \in S$. Thus, $\psi'_{vink}(G) \leq k\ \alpha_0(G)$. $\square$

3. $\psi'_{vi2}(G)$ and $\psi'_{vin2}(G)$ of some special graphs

The first result in this section is to find out the exact values for the vertex induced (incident) $2-$edge coloring number of multi r-bridge graph. Let r be a positive integer with $r \geq 3$ and let $\theta(l_1, l_2, \ldots, l_r)$ denote the graph obtained by connecting two distinct vertices with r independent (internally disjoint) paths of length $l_1, l_2, \ldots, l_r$ respectively. The graph is called the multi $r-$bridge graph where $l_1 \leq l_2 \leq \ldots \leq l_r$ and $l_1 \geq 2$; $l_i \in \mathbb{N}$ [6].

Theorem 3.1. *Let $\theta(l_1, l_2, \ldots, l_r)$, represent the multi $r-$bridge graph such that $l_1 \leq l_2 \leq \ldots \leq l_r$ and $r \geq 3$. Then, $\psi'_{vi2}(\theta(l_1, l_2, \ldots, l_r)) = \psi'_{vin2}(\theta(l_1, l_2, \ldots, l_r)) = \sum_{i=1}^{r} l_i - 2(r-2)$.*

Proof. Consider $G \equiv \theta(l_1, l_2, \ldots, l_p)$ where $p \geq 3$. We first prove the result for $p = 3$ and then the result can be extended for the case when $p = r$. When $p = 3$, i.e., $G \equiv \theta(l_1, l_2, l_3)$ is a graph obtained by connecting two distinct vertices with three independent paths of length l_1, l_2 and l_3 respectively. The vertex induced (incident) $2-$edge coloring number of a path P_n is $|E(P_n)|$ which is the length of P_n. Also, since the end vertices has three edges incident to it, at most two colors can be given to color these edges. Thus, $\sum_{i=1}^{3} l_i - 2$ gives the maximum vertex induced (incident) $2-$edge coloring number of G. Similarly, for r independent paths of the graph G, $\sum_{i=1}^{r} l_i$ distinct colors can be assigned to the edges. But, there are exactly two vertices in graph G with degree r and the edges incident to these vertices can be given at most two colors. Therefore, $2(r-2)$ colors has to be removed from the original color set else vertex induced $2-$edge coloring condition fails. Thus, $\psi'_{vi2}(\theta(l_1, l_2, \ldots, l_r)) = \sum_{i=1}^{r} l_i - 2(r-2)$. □

Let P_n be a path on n vertices. Then the snake-path multi $r-$bridge graph (or $s-$path $r-$bridge graph), denoted as $SP_\theta^n(l_1, l_2, \ldots, l_r)$, is a graph obtained from P_n by replacing each edge of the path by a multi $r-$bridge graph. An example of a $s-$path $r-$bridge graph, $SP_\theta^3(4, 3, 4)$, is shown in figure 2.

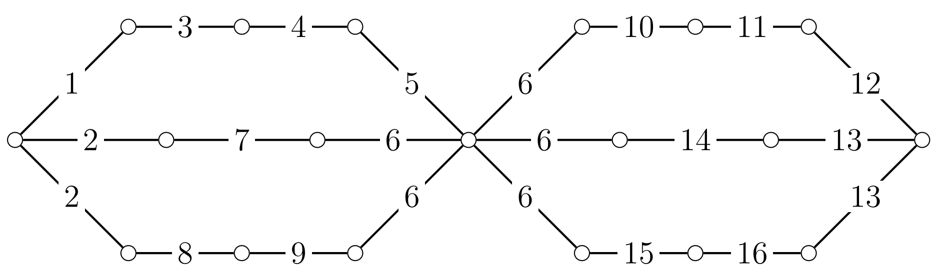

FIGURE 2. $SP_\theta^3(4, 3, 4)$

Corollary 3.1. *Let $r \geq 3$ be a positive integer. Consider $SP_\theta^n(l_1, l_2, \ldots, l_r)$ as the $s-$path $r-$bridge graph with r independent paths of length $l_1, l_2, \ldots, l_r$, respectively,* *such that* $l_1 \leq l_2 \leq \ldots \leq l_r$ *for* $l_1 \geq 2$; $l_i \in \mathbb{N}$. *Then* $\psi'_{vi2}(SP^n_\theta(l_1, l_2, \ldots, l_r)) = \psi'_{vin2}(SP^n_\theta(l_1, l_2, \ldots, l_r)) = (n-1)\sum\limits_{i=1}^{r} l_i - 2rn + 2n + 2r$.

Proof. Consider, $G \equiv SP^n_\theta(l_1, l_2, \ldots, l_r)$, where, for $1 \leq i \leq r$, l_i is the length of the $i-$path. From theorem 3.1, $\psi'_{vi2}(\theta(l_1, l_2, \ldots, l_r)) = \sum\limits_{i=1}^{r} l_i - 2(r-2)$. As each edge of the path P_n is identified as a multi $r-$bridge graph, so there are $n-1$ copies of multi $r-$bridge subgraph in the graph G. Thus, $(n-1)\sum\limits_{i=1}^{r} l_i - 2(r-2)$ distinct colors can be assigned to the edges of the multi $r-$bridge subgraph components of the graph G. Since, at most two different colors can be given to the edges incident to each vertex v with maximum degree, $\Delta(G)$. So, $(n-2)(2r-2)$ colors has to be removed from $(n-1)\sum\limits_{i=1}^{r} l_i - 2(r-2)$ in order to color the edges of the graph G. Therefore, $\psi'_{vi2}(G) = \psi'_{vin2}(G) = (n-1)\sum\limits_{i=1}^{r} l_i - 2(r-2) - (n-2)(2r-2) = (n-1)\sum\limits_{i=1}^{r} l_i - 2rn + 2n + 2r$. □

The vertex induced $2-$edge coloring number and vertex incident $2-$edge coloring number of graph $SP^3_\theta(4, 3, 4)$, shown in figure 2, is 16.

Let C_n be a cycle of order n. Then the snake-cycle multi $r-$bridge graph (or $s-$cycle $r-$bridge graph), denoted as $SC^n_\theta(l_1, l_2, \ldots, l_r)$, is a graph obtained from C_n by replacing each edge of the cycle graph by a multi $r-$bridge graph. An example of a $s-$cycle $r-$bridge graph, $SC^4_\theta(3, 3, 2)$, is shown in figure 3.

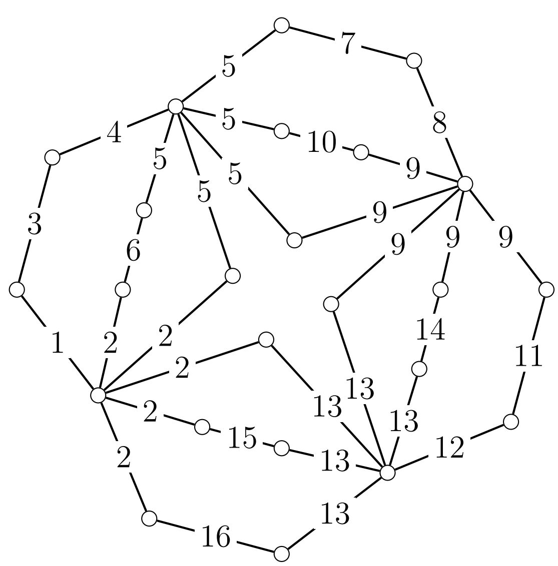

FIGURE 3. $SC^4_\theta(3, 3, 2)$

Corollary 3.2. *Let $r \geq 3$ be a positive integer. Consider $SC_\theta^n(l_1, l_2, \ldots, l_r)$ as the $s-$cycle $r-$bridge graph with r independent paths of length $l_1, l_2, \ldots, l_r$, respectively, such that $l_1 \leq l_2 \leq \ldots \leq l_r$ for $l_1 \geq 2$; $l_i \in \mathbb{N}$. Then $\psi'_{vi2}(SC_\theta^n(l_1, l_2, \ldots, l_r)) = \psi'_{vin2}(SC_\theta^n(l_1, l_2, \ldots, l_r)) = n \sum_{i=1}^{r} l_i - 2rn + 2n$.*

Proof. The result follows from theorem 3.1 and corollary 3.1. □

The vertex induced $2-$edge coloring number and vertex incident $2-$edge coloring number of graph $SC_\theta^4(3, 3, 2)$, shown in figure 3, is 16.

The vertex induced (incident) $2-$edge coloring number of Petersen graph $\mathcal{P}$ is 7 [7]. In the next result we find the exact values of the generalized Petersen graph $GP(n, r)$. The generalized Petersen graph $GP(n, r)$, also denoted as $P(n, r)$, for $n \geq 3$ and $1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ is a connected cubic graph consisting of an inner circulant graph $C_{i_n}(r)$ and an outer cycle graph C_n such that the corresponding vertices in the inner edges and the outer edges of the graph $GP(n, r)$ are connected with edges.

Theorem 3.2. *For the generalized Petersen graph $GP(n, r)$, $\psi'_{vi2}(GP(n, r)) = \psi'_{vin2}(GP(n, r)) = n + 2$ where $n \geq 3$.*

Proof. Consider $G \equiv GP(n, r)$ as the generalised Petersen graph, where $n \geq 3$, and $1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Each edge connecting the inner circulant subgraph and the outer cycle of the graph G can be assigned with n distinct colors. The remaining uncolored edges of the outer cycles of the graph G is assigned $(n+1)^{th}$ color, whereas, the inner cycle of G can be given $(n+2)^{th}$ color. Therefore, for $n \geq 3$, $\psi'_{vi2}(G) = \psi'_{vin2}(G) = n + 2$.

It is to be noted that the maximum vertex induced (incident) $2-$edge coloring number of the generalized Petersen graph $GP(n, r)$ is obtained by coloring the edges of G in the above-mentioned coloring procedure. Suppose if, n edges in the outer cycle of the graph G are given n different colors. Then, the uncolored edges joining the inner cycle and outer cycles of the graph G cannot be assigned with any new color. Further, at most one new color can be assigned to all the edges of the inner circulant subgraph of graph G. In this vertex induced (incident) $2-$edge coloring, we will get a maximum of $n+1$ colors which is less than that of the above-mentioned vertex induced (incident) 2-edge coloring number $n+2$, a contradiction. Similarly, a contradiction can be obtained when the coloring procedure is started by giving different colors to all the inner circulant edges of the graph G. □

Shadow of a graph G, denoted by $D_r(G)$, is obtained by taking r copies of a graph G and adding edges between the vertices of copies if their corresponding vertices are adjacent in G. In particular, a shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say H and $H^{'}$. Let $\{v_i\}$ and $\{v_i'\}$ be the vertex set of graph H and $H^{'}$ respectively. Join each vertex v in H to the neighbors of the corresponding vertex v' in $H^{'}$[11]. The next result will give a coloring procedure to get the lower bound of the vertex induced (incident) $2-$edge coloring of the graph $D_2(P_n)$.

Theorem 3.3. *Let $D_2(P_n)$ be the $2-$shadow of the path P_n, where $n \geq 3$. Then $\psi'_{vi2}(D_2(P_n)) = \psi'_{vin2}(D_2(P_n)) \geq n+1$.*

Proof. The graph $D_2(P_2)$ is a cycle of order 4. So, the vertex induced (incident) $2-$edge coloring number of graph $D_2(P_2)$ is 4 [7]. Consider $D_2(P_n)$ as the 2-shadow of the path P_n on n vertices with $n \geq 3$. Let $V(D_2(P_n)) = \{v_i, v_i' \mid i \in \{1, 2, ..., n\}\}$ and $E(D_2(P_n)) = \{v_iv_{i+1} \cup v_i'v_{i+1}' \cup v_{i+1}v_i' \cup v_iv_{i+1}' \mid i \in \{1, 2, ..., n-1\}\}$ be the vertex set and the edge set of the graph $D_2(P_n)$ respectively. The vertex induced (incident) $2-$edge coloring number of path on n vertices is $n-1$ (see [7]). So, the edges of the first copy of path in the graph $D_2(P_2)$ can be distinctively given $n-1$ colors, say $\mathcal{C} = \{c_1, c_2, \ldots, c_{n-1}\}$.

Now, consider the colorless edges of $D_2(P_n)$ that are incident to the vertices $\{v_1, v_2\}$. The edge v_1v_2' can be given a new color c_n whereas, the remaining two uncolored edges that are incident to vertices $\{v_1, v_2\}$ has to be colored with one of the colors from the set $\mathcal{C}$. Suppose if the edge v_2v_2' is given a new color, say c_n' then the vertex induced (incident) $2-$edge coloring condition fails at vertex v_2. Similarly, the edge v_nv_{n-1}' receives a new color c_{n+1} and all the remaining uncolored edges incident to vertex v_i where $2 \leq i \leq n-1$ has to be assigned with one of the colors from the set $\mathcal{C}$. It is to be noted that each vertex v_i' in $V(D_2(P_n))$, where $i \in \{2, ..., n-1\}$, is of degree

4. Further, since there are exactly two edges incident to each vertex v_i' that received two different colors from the coloring pattern discussed earlier. Therefore, the two uncolored edges incident at each of the vertex v_i', for $2 \leq i \leq n-1$, cannot be colored with any more new colors. So, $\psi'_{vi2}(D_2(P_n)) \geq n+1$. $\square$

The graph $D_2(C_n)$ is obtained by taking two copies of the cycle C_n such that each vertex v_j in the first copy of the cycle graph joins the neighbors of the corresponding vertex $v_j^{'}$ in the second copy, where $1 \leq j \leq n$ and $n \geq 4$. The next result gives the lower bound for vertex induced (incident) $2-$edge coloring of $D_2(C_n)$ graph.

Theorem 3.4. *Let $D_2(C_n)$ denote the $2-$shadow of cycle on n vertices where $n \geq 4$. Then $\psi'_{vi2}(D_2(C_n)) = \psi'_{vin2}(D_2(C_n)) \geq n+1$.*

Proof. Consider $D_2(C_n)$ as the $2-$shadow of cycle graph with order $2n$ and size $4n$. The coloring procedure can be started by giving each alternate edge in both copies of cycle of $D_2(C_n)$ by a new color. Whereas, all the remaining colorless edges of the graph $D_2(C_n)$ can be given one color. So, the $2n$ edges of cycle subgraphs of the graph $D_2(C_n)$ can be assigned with at least $n+1$ colors. Therefore, $\psi'_{vi2}(D_2(C_n)) = \psi'_{vin2}(D_2(C_n)) \geq n+1$.

$\square$

The coloring pattern discussed in the above theorem has been described in the figure 4. The vertex induced (incident) $2-$edge coloring number of $D_2(C_4)$ and $D_2(C_5)$ is shown in figure 4.(a) and figure 4.(b) respectively.

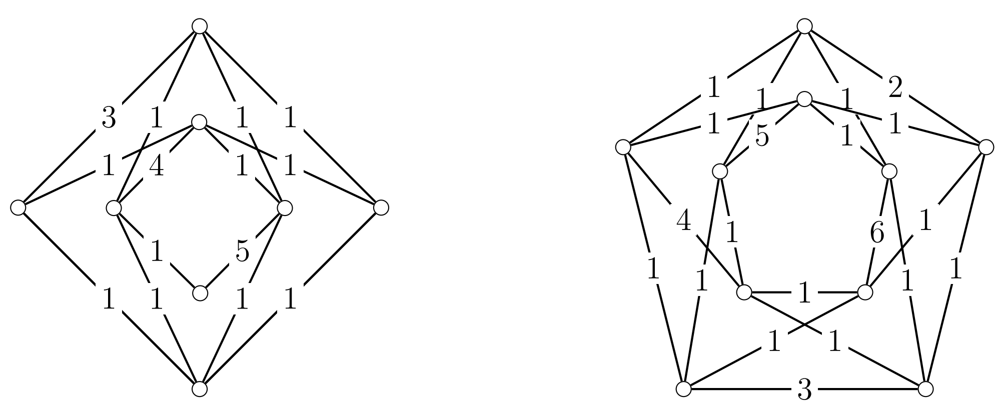

FIGURE 4. (a) $\psi'_{vi2}(D_2(C_4)) = 5$ and (b) $\psi'_{vi2}(D_2(C_5)) = 6$

Corollary 3.3. *Let $D_2(C_3)$ be the $2-$shadow of a cycle on three vertices. Then (i) $\psi'_{vi2}(D_2(C_3)) = 3$ and (ii) $\psi'_{vin2}(D_2(C_3)) = 4$.*

Proof. (i) Let $V = \{v_i, v'_i : i \in \{1, 2, 3\}\}$ be the vertex set of the graph $D_2(C_3)$. Since the vertex v_j and vertex v'_j, for $1 \leq j \leq 3$, are not adjacent in the graph $D_2(C_3)$. Thus, the induced subgraph generated by $N[v_j]$ (for any j) is not equivalent to the graph $D_2(C_3)$. Therefore, at most two colors can be given to the edges of the induced subgraph generated by $N[v_j]$, where $j \in \{1, 2, 3\}$. Moreover, as $\langle N[v_j]\rangle \cup \langle N[v'_j]\rangle \equiv D_2(C_3)$. So, the remaining uncolored edges of the $D_2(C_3)$ graph can be given at most one new color. Hence the result follows.
(ii) The result follows from theorem 3.4. □

The next few results are on the vertex induced (or incident) $2-$edge coloring number of $r-$splitting of some graphs. The splitting graph $S(G)$ of a graph $G = (V, E)$ is obtained by adding a new vertex v' to each vertex $v \in V$ such that v' is adjacent to every vertex that is adjacent to the vertex v in G [10]. The $r-$ splitting graph $spl_r(G)$ of a graph G is obtained by adding r new vertices, say $v^1, v^2, \ldots, v^r$, such that $v^j, 1 \leq j \leq r$ is adjacent to each vertex which is adjacent to v in G [11]. The Bistar graph $B_{n,n}$ is a graph obtained by joining the center vertices of two copies of $K_{1,n}$ by an edge [3].

Theorem 3.5. *Let G be a connected graph of order $n \geq 2$ with $\Delta(G) \leq 2$ and $G \neq K_3$. If $Spl_r(G)$ denotes the $r-$splitting graph of a graph G, then $\psi'_{vin2}(Spl_r(G)) = \psi'_{vi2}(Spl_r(G)) \geq n + 1$.*

Proof. Consider $V = \{v_1, v_2, \ldots, v_n\}$ to be the vertex set of the graph G such that $\Delta(G) \leq 2$ and $G \neq K_3$. Then, G is either a path graph or a cycle graph. Let $V' = \{v_1^j, v_2^j, \ldots, v_n^j : 1 \leq j \leq r\}$ be the vertex set of r new vertices which are added to the graph G to form the $r-$splitting graph of G. Then, $V \cup V'$ becomes the vertex set of the graph $spl_r(G)$. Clearly, $1 \leq deg(v_i^j) \leq 2$ and $r + 1 \leq deg(v_i) \leq 2(r + 1)$, for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, r\}$. The result is trivial for $n = 2$, as $Spl_r(P_2)$ is a bistar graph $B_{n,n}$ and so, ${\psi_{vin2}}'(Spl_r(P_2)) = 3 = 2 + 1$.

Assume that $r \geq 1$ and $n \geq 3$. Suppose we assign distinct colors to all the edges incident to the vertex v_i^j with degree 2. Then, there exists at least one colored edge incident to the vertex v_i with degree $2(r+1)$ where the vertex induced (incident) $2-$edge coloring condition fails. Thus, all edges incident to vertex v_i^j with degree 2 cannot be assigned with distinct colors. So, for a fixed j and $1 \leq i \leq n$, at least one of the edge which is incident to each vertex v_i^j in the set V', can be assigned with n distinct colors. Whereas, all the remaining uncolored edges of the graph $Spl_r(G)$ can be given at most one more new color. Hence, for $\Delta(G) \leq 2$ and $G \neq K_3$, $\psi'_{vi2}(Spl_r(G)) = \psi'_{vin2}(Spl_r(G)) \geq n+1$. ☐

Corollary 3.4. *Let $Spl_r(C_3)$ be the $r-$splitting of the cycle graph with order 3. Then,*

(i) $\psi'_{vi2}(Spl_r(C_3)) = 3$.
(ii) $\psi'_{vin2}(Spl_r(C_3)) = 4$.

Proof. This result can be easily verified. ☐

Theorem 3.6. *Let G be a connected $(n,m)-$graph without pendant edges such that $\Delta(G) = n-1$. Then,*

(i) $\psi'_{vi2}(Spl_r(G)) = 3$.
(ii) $\psi'_{vin2}(Spl_r(G)) \geq n+1$.

Proof. (i) Consider G to be a connected $(n,m)-$ graph without pendant edges. The graph $Spl_r(G)$ denotes the $r-$splitting of graph G. Let $V(Spl_r(G)) = \{v_i, v_i^j : 1 \leq i \leq n, 1 \leq j \leq r\}$ be the vertex set of the graph $Spl_r(G)$. Since $\Delta(G) = n-1$, there exists at least one vertex v with degree $n-1$ such that the vertex v forms an edge with all the vertices in vertex set $V(Spl_r(G))$ except the vertices in $\{v^1, v^2, \ldots, v^r\}$. It is to be noted that, no two vertices in the set $\{v, v^1, v^2, \ldots, v^r\}$ are adjacent to each other. Thus, all the edges in the induced subgraph generated by closed neighborhood $N[v]$ have to be allotted with 2 different colors. All the remaining colorless edges of the $Spl_r(G)$ graph can be colored with at most one new color to satisfy the vertex induced $2-$edge coloring condition. Hence the result.
(ii) similar to proof mentioned in proposition 3.5. ☐

4. Conclusion

In this paper, the vertex induced $k-$edge coloring number and vertex incident $k-$edge coloring number have been introduced. Section 2 gives few bounds and some exact values of $\psi'_{vik}(G)$ and $\psi'_{vink}(G)$ of some graph classes.

In section 3, the lower bound of vertex induced (incident) $2-$edge coloring of the $2-$shadow graph and the $r-$splitting of some graph classes has been mentioned. It is an open problem to find out the exact values of $\psi'_{vi2}(G)$ and $\psi'_{vin2}(G)$ of $r-$shadow graphs and $r-$splitting graphs. Section 3 also discusses some exact values of vertex induced (incident) 2-edge coloring of multi r-bridge graphs and the generalized Petersen graph.

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