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SOME OPERATOR PRESERVING INEQUALITIES INVOLVING FUNCTIONS OF EXPONENTIAL TYPE

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Abstract. Let f be an entire function of exponential type τ such that $|f(x)| \leq M$ on the real axis. Then, it was proved by Bernstein [8, Theorem 14.1.7] that

$$|f'(x)| \leq M\tau.$$

In the literature, there exist several results estimating $|f'(x)|$ in terms of $\sup_{-\infty<x<\infty} |f(x)|$. Here we prove some Bernstein-type inequalities for an operator L defined by $L[f(z)] = e^{\beta z}[e^{-\beta z} f(z)]'$, where β is any complex number. These inequalities generalise some well-known inequalities for entire functions of exponential type.

1. Introduction

An entire function $f(z)$ is said to be of exponential type τ, if it is of order less than 1 or it is of order 1 and type less than or equal to τ. The indicator function $h_f(\theta)$ of f is defined by

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r}.$$

It is important to note that if $f(z)$ is an entire function of exponential type τ, then the indicator function $h_f(\theta) \leq \tau$, for all $\theta, 0 \leq \theta < 2\pi$.
The following result [1, Theorem 6.2.4] serves as a basic tool in the study of functions of exponential type.

Theorem 1.1. *Let $f(z)$ be a function of exponential type in the open upper half-plane such that $h_f(\frac{\pi}{2}) \leq c$. Furthermore, let f be continuous in the closed upper half-plane*

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and suppose that $|f(x)| \leq M$ *on the real axis. Then*

$$|f(x+iy)| < Me^{cy} \ (-\infty < x < \infty, y > 0) \tag{1.1}$$

unless $f(z) \equiv Me^{i\gamma}e^{-icz}$ *for some real* γ.

A classical result due to Bernstein [8, Theorem 14.1.7], famously known as Bernstein's Inequality for entire functions of exponential type can be stated as:

Theorem 1.2. *Let* $f(z)$ *be a function of exponential type* $\tau > 0$ *such that* $|f(x)| \leq M$ *on the real axis. Then for real* x

$$|f'(x)| \leq M\tau. \tag{1.2}$$

As a refinement of Theorem 1.2, Boas [2] proved the following result:

Theorem 1.3. *Let* $f(z)$ *be an entire function of exponential type* τ *such that* $|f(x)| \leq M$*, for real* x*. Also suppose that* $f(z)$ *does not vanish in the upper half-plane, that is,* $\Im z > 0$ *and* $h_f(\frac{\pi}{2}) = 0$*. Then, for* $y = \Im z < 0$

$$|f(x+iy)| \leq \frac{M}{2}(e^{\tau|y|} + 1) \tag{1.3}$$

and for real x

$$|f'(x)| \leq M\frac{\tau}{2}. \tag{1.4}$$

Definition 1.1. An entire function f of exponential type is said to belong to the class $\mathbb{P}$ if it has no zeros in the open lower half-plane and satisfies one of the conditions:
1. $h_f(-\theta) \geq h_f(\theta)$, for some θ, $0 < \theta < \pi$
2. $|f(z)| \geq |f(\bar{z})|$ for $\Im z < 0$.

Definition 1.2. An additive homogeneous operator $B[f(z)]$ which carries entire functions of exponential type into entire functions of exponential type and leaves the class $\mathbb{P}$ invariant is called a B-operator.

It may be added that an operator B is additive if $B[f+g] = B[f] + B[g]$ and homogeneous if $B[cf] = cB[f]$, where c denotes the constant. Concerning the entire functions belonging to class $\mathbb{P}$, we have the following result due to Boas [2, Theorem 11.7.2].

Theorem 1.4. *Let $f(z)$ be an entire function of exponential type τ, B is a B-operator, and $\omega(z)$ is an entire function of class $\mathbb{P}$ and of order 1, type $\sigma \geq \tau$, then*

$$|f(x)| \leq |\omega(x)|, \quad -\infty < x < \infty, \tag{1.5}$$

implies

$$|B[f(x)]| \leq |B[\omega(x)]|, \quad -\infty < x < \infty. \tag{1.6}$$

In this paper, we prove some basic results concerning an operator L (see for reference [2, p.226]) defined by

$$L[f(z)] = e^{\beta z} D[e^{-\beta z} f(z)] = f'(z) - \beta f(z) = (D - \beta I) f(z) \tag{1.7}$$

where f is an entire function of exponential type τ and β is any complex number.

Remark 1. *Boas* [2, Theorem 11.7.5] *proved that the operator L as defined above is a B-operator, if $\Im(\beta) \leq 0$.*

2. Lemmas

For proofs of the theorems, we need the following lemmas.

Lemma 2.1. *Let f be an entire function of exponential type τ belonging to class $\mathbb{P}$. Furthermore, let g be an entire function of exponential type $\sigma \leq \tau$ such that $|g(x)| \leq |f(x)|$ for real x. Then for $\psi_\alpha(z) := g(z) - \alpha f(z)$ belongs to $\mathbb{P}$ for any α, $|\alpha| > 1$.*

The above lemma is due to Levin [6] (see also [1, Theorem 7.8.6]).

Lemma 2.2. *Let f be an entire function of exponential type τ having no zeros in the open upper half-plane, that is, $\Im z > 0$ and $h_f(\frac{\pi}{2}) = 0$. Then for $\Im z < 0$ we have*

$$|L[f(z)]| \leq |L[g(z)]| \tag{2.1}$$

where $g(z) = e^{i\tau z}\overline{f(\bar{z})}$.

Proof. Since $f(z)$ does not vanish in the open upper half-plane and $h_f(\frac{\pi}{2}) = 0$, therefore $g(z) = e^{i\tau z}\overline{f(\bar{z})}$ belongs to class $\mathbb{P}$ (see [7, Proof of Lemma 2]). Also, for real x

$$|f(x)| = |g(x)|.$$

Therefore, by Lemma 2.1 for every $\alpha \in \mathbb{C}, |\alpha| > 1$, $f(z) - \alpha g(z)$ belongs to class $\mathbb{P}$. By using the definition of B-operators, $L[f(z)] - \alpha L[g(z)]$ belongs to class $\mathbb{P}$, that is, $L[f(z)] - \alpha L[g(z)]$ does not vanish in $\Im z < 0$.
Now, if (2.1) is false, then there exists z_0 such that $\Im z_0 < 0$ and $|Lf(z)|_{z=z_0} > |Lg(z)|_{z=z_0}$. We choose

$$\alpha = \frac{|Lf(z)|_{z=z_0}}{|Lg(z)|_{z=z_0}}$$

so that $|\alpha| > 1$. For this value of α, we have $\{L[f(z)] - \alpha L[g(z)]\}_{z=z_0} = 0$, for $z_0 \in \Im z < 0$.
This is a contradiction and therefore (2.1) holds true. $\square$

3. Main Results

There exist several results estimating $\sup\limits_{-\infty<x<\infty} |f'(x)|$ in terms of $|f(x)|$ in the literature. Since *differentiation is also a B-operator* (see [2, p.226]), it is interesting to find such estimates for $L[f]$ which in turn provides more general results. As an attempt in this case, we first prove the following:

Theorem 3.1. *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq M$, for real x . Then for $\Im z < 0$*

$$|L[f(z)]| \leq M|L[e^{i\tau z}]|. \tag{3.1}$$

Proof. Since $|f(x)| \leq M$ for real x, therefore, it follows from Lemma 2.1 that $f(z) - \alpha M e^{i\tau z}$ belongs to $\mathbb{P}$ for any $\alpha \in \mathbb{C}, |\alpha| > 1$.
Now, by using the properties of a B-operator, we can say that $L[f(z) - \alpha M e^{i\tau z}]$ also belongs to class $\mathbb{P}$.

This implies that $L[f(z)]-\alpha ML[e^{i\tau z}]$ has no zeros in the open lower half-plane. Now, if (3.1) is false, then for some $z_0 \in \Im z < 0$

$$|L[f(z)]|_{z=z_0} > M|L[e^{i\tau z}]|_{z=z_0}.$$

Since $L[e^{i\tau z}]$ does not vanish in the open lower half-plane, therefore, we can choose

$$\alpha = \frac{|L[f(z)]|_{z=z_0}}{M|L[e^{i\tau z}]|_{z=z_0}}$$

so that $|\alpha| > 1$ and $[L[f(z)] - \alpha ML[e^{i\tau z}]]_{z=z_0} = 0$, which is a contradiction. Therefore, (3.1) is true and this completes the proof. □

Remark 2. *Substituting for $L[f]$, we equivalently have for $\Im z < 0$*

$$|f'(z) - \beta f(z)| \leq M|(i\tau - \beta)e^{i\tau z}|. \tag{3.2}$$

If we take $\beta = 0$ in (3.2), which is clearly possible, we get for $y = \Im z < 0$

$$|f'(z)| \leq M|i\tau e^{i\tau z}| = M\tau e^{\tau|y|}. \tag{3.3}$$

Inequality (1.2) is a special case of inequality (3.3).

We next prove:

Theorem 3.2. *Let f be an entire function of exponential type τ such that $|f(x)| \leq M$ for real x and $h_f(\frac{\pi}{2}) \leq 0$. Then for $\Im z < 0$ we have*

$$|L[f(z)]| + |L[g(z)]| \leq M\{|L[e^{i\tau z}]| + |\beta|\}. \tag{3.4}$$

where $g(z) = e^{i\tau z}\overline{f(\bar{z})}$.

Proof. Since f is an entire function of exponential type τ, $h_f(\frac{\pi}{2}) \leq 0$ and $|f(x)| \leq M$ for real x, therefore by Theorem 1.1, $|f(z)| < M$ in the open upper half-plane. Hence $F(z) := f(z) - Me^{-i\gamma}, \gamma \in \mathbb{R}$ is an entire function of exponential type τ which has no zeros in the open upper half-plane. Consequently, the function

$$G(z) := e^{i\tau z}\overline{F(\bar{z})} = g(z) - Me^{i\gamma}e^{i\tau z}$$

belongs to class $\mathbb{P}$. Moreover $|F(x)| = |G(x)|$ for all real x. Therefore, by Lemma 2.2, we have for $\Im z < 0$

$$|L[F(z)]| \leq |L[G(z)]|.$$

This gives for $\Im z < 0$

$$|L[f(z) - Me^{-i\gamma}]| \leq |L[g(z) - Me^{i\gamma}e^{i\tau z}]|.$$

Equivalently

$$|L[f(z)] - L[Me^{-i\gamma}]| \leq |L[g(z)] - Me^{i\gamma}L[e^{i\tau z}]|.$$

As an application of Theorem 3.1, we can choose γ such that

$$|L[f(z)]| - |L[Me^{-i\gamma}]| \leq |Me^{i\gamma}L[e^{i\tau z}]| - |L[g(z)]|$$

which implies for $\Im z < 0$

$$|L[f(z)]| + |L[g(z)]| \leq |Me^{i\gamma}L[e^{i\tau z}]| + |L[Me^{-i\gamma}]|$$

or

$$|L[f(z)]| + |L[g(z)]| \leq M\{|L[e^{i\tau z}]| + |\beta|\}.$$

Hence the proof of Theorem 3.2. $\square$

Remark 3. *Substituting for $L[f]$ and $L[g]$ and taking $\beta = 0$ in* (3.4), *which is clearly possible, we get for $y = \Im z < 0$*

$$|f'(z)| + |g'(z)| \leq M\tau e^{\tau|y|}. \tag{3.5}$$

This inequality is due to Hachani [5, Lemma B].

If we choose $f(z) \equiv g(z) = e^{i\tau z}\overline{f(\bar{z})}$, then $|L[f(z)]| = |L[g(z)]|$. Therefore, we have the following:

Remark 4. *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq M$, for real x. If $f(z) \equiv e^{i\tau z}\overline{f(\bar{z})}$, then for $\Im z < 0$*

$$|L[f(z)]| \leq \frac{M}{2}\{|L[e^{i\tau z}]| + |\beta|\}. \tag{3.6}$$

Also, if we take $\beta = 0$ in inequality (3.6), we obtain the following:

Remark 5. *If $f(z)$ is an entire function of exponential type τ such that $|f(x)| \leq M$, for real x. If $f(z) \equiv e^{i\tau z}\overline{f(\bar{z})}$, then for $y = \Im z < 0$*

$$|f'(z)| \leq M\frac{\tau}{2}e^{\tau|y|}. \tag{3.7}$$

Now, if we suppose that $f(z)$ does not vanish in the half-plane $\Im z > 0$ then we get the following improvement of Theorem 3.1.

Theorem 3.3. *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq M$, for real x. Also suppose that $f(z)$ does not vanish in the upper half-plane, that is, $\Im z > 0$ and $h_f(\frac{\pi}{2}) = 0$. Then, for $\Im z < 0$*

$$|L[f(z)]| \leq \frac{M}{2}\{|L[e^{i\tau z}]| + |\beta|\}. \tag{3.8}$$

Proof. Combining Lemma 2.2 and Theorem 3.2, we have

$$\begin{aligned} 2|L[f(z)]| &\leq |L[f(z)]| + |L[g(z)]| \\ &\leq M\{|L[e^{i\tau z}]| + |\beta|\} \end{aligned}$$

which gives inequality (3.8) and Theorem 3.3 is completely proved. □

Remark 6. *From inequality* (3.8)*, we have for $\Im z < 0$*

$$|f'(z) - \beta f(z)| \leq \frac{M}{2}\{|(i\tau - \beta)e^{i\tau z}| + |\beta|\}.$$

If we take $\beta = 0$ in above, which is clearly possible, we get for $y = \Im z < 0$

$$|f'(z)| \leq M\frac{\tau}{2}e^{\tau|y|},$$

which is a particular case of a result due to Gardner and Govil [3, Corollary 1].

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