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A STUDY OF HYBRID HADAMARD FRACTIONAL DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS

HABIB DJOURDEM

Abstract. In this manuscript, we study the existence of solutions for a class of hybrid fractional Hadamard integro-differential inclusions supplemented with hybrid Hadamard integral boundary conditions. The results are obtained by applying the hybrid fixed point theorem for three operators in a Banach algebra due to Dhage. An example is also presented to illustrate our main results.

1. Introduction

Differential equations with fractional-order became an important field in analysis theory due their significations in mathematical modeling of many phenomena in real world related to engineering and scientific disciplines such as biology, chemistry, economics and numerous branches of physical sciences (see [21, 22, 25, 30, 33]). Boundary value problems of fractional differential equations implicit several kinds of fractional derivatives like Riemann-Liouville-type, Caputo-type, Hadamard-type, Caputo-Hadamard-type and Hilfer-Hadamard-type fractional derivative with different sorts of boundary conditions have studied by many authors (see [1, 5, 7, 8, 9, 17, 18, 26, 24]).

Hybrid differential equations have been considered more important and are more general and covers several dynamic systems as particular cases. First time, Dhage and Lakshmikantham in [16] proposed hybrid differential equations and showed some essential results on this kind of differential equations. In recent years, with the wide

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study of fractional differential equations, the theory of hybrid fractional differential equations were also studied by several researchers, see [3, 6, 28, 31] and the references therein. It's worth to mention some interesting works deal with solving integro-differential equations by using hybrid fixed point theorems or hybrid numerical methods, we quote for instance [13, 32, 37].

Integro-differential inclusions are models of many realistic problems in different fields, like economics, optimal control, stochastic analysis, we refer the reader to [11, 34, 36]. As an application to this kind of these problems, we cite as an example the integral inclusion for the temperature control by means of a thermostat, see [12]. Differential inclusions in ordinary forms or in hybrid forms have gained so much attention of many authors, see [2, 4, 10, 19, 29] and the references therein.

Motivated and inspired by the works mentioned above, we are concerned with the existence of solutions for the following nonlinear hybrid fractional differential inclusions

$$D^{\sigma}\left[\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right]\in G(t,\upsilon(\tau),I^{\gamma}\upsilon(\tau)),\quad 1<t<e, \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} &\upsilon(1)=0,\\ &D^{\sigma-1}\left(\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=1}=0,\\ &\left(\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=e}=\lambda\left(I^{p}\upsilon\right)(\eta), \end{aligned} \tag{1.2}$$

where D^{σ} denotes the Hadamard fractional derivative of order $2<\sigma\leq 3$. I^{γ}, I^{θ_i} and I^{p} are respectively the Hadamard fractional integrals of order $\gamma,\theta_i,p>0$ $(i=1,2,...,m)$, $\Psi\in C([1,e]\times\mathbb{R}^2,\mathbb{R}\setminus\{0\})$ and $G:\ [1,e]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\chi_i\in C([1,e]\times\mathbb{R},\mathbb{R})$ with $\chi_i(1,0)=0$, for $i=1,2,...,m$. λ, η are two real parameters with $\lambda>0$, $1<\eta<e$ and $\frac{\lambda\Gamma(\sigma-1)}{\Gamma(p+\sigma-1)}(\log\eta)^{p+\sigma-2}\neq 1$.

The rest of our work is divided into two sections. In the next section, we first recall some preliminary results that we need in the sequel. In section 3, an hybrid fixed point theorem for three operators in a Banach algebra due to Dhage is used to establish the existence results of the problem (1.1)-(1.2). Finally, an example is given to illustrate the obtained results.

2. Background and materials

Here, we give certain definitions and results which are needed to prove our main results.

Let $C(I,\mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$.

We begin by defining Hadamard fractional integrals and derivatives, and we introduce some properties that can be used thereafter.

Definition 2.1. [25] The Hadamard fractional integral of order $\sigma \in \mathbb{R}^+$ for a function $\Psi \in C[a,b]$, $0 \leq a \leq \tau \leq b \leq \infty$, is defined as

$$I^{\sigma}\Psi(\tau) = \frac{1}{\Gamma(\sigma)}\int_a^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\Psi(s)\frac{ds}{s},$$

where $\Gamma(.)$ is the Gamma function and $\log(.) = \log_e(.)$.

Definition 2.2. [25] Let $0 < a < b < \infty$ and $\delta = \tau\frac{d}{d\tau}$. The Hadamard derivative of fractional order $\sigma \in \mathbb{R}^+$ for a function $\Psi \in C^{n-1}([a,b],\mathbb{R})$ is defined as

$$D^{\sigma}\Psi(\tau) = \delta^n\left(I^{n-\sigma}\right)(\tau) = \frac{1}{\Gamma(n-\sigma)}\left(\tau\frac{d}{d\tau}\right)^n\int_a^{\tau}\left(\log\frac{\tau}{s}\right)^{n-\sigma-1}\frac{\Psi(s)}{s}ds,$$

where $n-1 < \sigma \leq n \in \mathbb{Z}^+$, $n = [\sigma]+1$ denotes the integer part of the real number q.

Lemma 2.1. *([25], Property 2.24) If $a, \alpha, \beta > 0$, then*

$$\left(D^{\sigma}\left(\log\frac{\tau}{a}\right)^{\beta-1}\right)(\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta-\sigma)}\left(\log\frac{\tau}{a}\right)^{\beta-\sigma-1},$$

$$\left(I^{\sigma}\left(\log\frac{\tau}{a}\right)^{\beta-1}\right)(\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta+\sigma)}\left(\log\frac{\tau}{a}\right)^{\beta+\sigma-1}.$$

Lemma 2.2. *([25]) Let $\sigma > 0$ and $\upsilon \in C[1,\infty) \cap L^1[1,\infty)$. Then the solution of Hadamard fractional differential equation $D^{\sigma}\upsilon(\tau) = 0$ is given by*

$$\upsilon(\tau) = \sum_{i=1}^{n} c_i(\log\tau)^{\sigma-i},$$

and the following formula holds:

$$I^{\sigma}D^{\sigma}\upsilon(\tau) = \upsilon(\tau) + \sum_{i=1}^{n} c_i(\log\tau)^{\sigma-i},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, *where* $n = [\sigma] + 1$.

Now, we put $\mathcal{Y} = C(I, \mathbb{R})$ where $I = [1, e]$. Define a norm $\|.\|$ and a multiplication in $\mathcal{Y}$ by

$$\|\upsilon\| = \sup_{\tau \in I} |\upsilon(\tau)| \ \textit{and} \ (\upsilon\varpi)(\tau) = \upsilon(\tau)\varpi(\tau), \ \forall \tau \in I.$$

Clearly $\mathcal{Y}$ is a Banach algebra with respect to above supremum norm and the multiplication in it.

In the rest of this section, we recall some material on multivalued analysis [14, 23] related to this research. For a normed space $(\mathcal{X}, \|.\|)$, let $\mathcal{P}_b(\mathcal{X}) = \{\mathcal{F} \in \mathcal{X} : \mathcal{F} \textit{ is bounded}\}$, $\mathcal{P}_{cl}(\mathcal{X}) = \{\mathcal{F} \in \mathcal{X} : \mathcal{F} \textit{ is closed}\}$, $\mathcal{P}_{cp}(\mathcal{X}) = \{\mathcal{F} \in \mathcal{X} : \mathcal{F} \textit{ is compact}\}$ and $\mathcal{P}_{cp,c}(\mathcal{X}) = \{\mathcal{F} \in \mathcal{X} : \mathcal{F} \textit{ is compact and convex}\}$.

Definition 2.3. A multivalued map $\mathcal{H} : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{X})$.
(1) is convex (closed) valued for all $\upsilon \in \mathcal{X}$ if $\mathcal{H}(\upsilon)$ is convex (closed) for all $\upsilon \in \mathcal{X}$;
(2) is bounded on bounded sets if $\mathcal{H}(\mathcal{B}) = \cup_{\upsilon \in \mathcal{B}} \mathcal{H}(\upsilon)$ is bounded in $\mathcal{X}$ for all $\mathcal{B} \in P_b(\mathcal{X})$ i.e $\sup_{\upsilon \in \mathcal{B}} \{\sup\{|\omega|, \omega \in \mathcal{H}(\upsilon)\}\} < \infty$.
(3) is called upper semi-continuous (u.s.c) on $\mathcal{X}$ if for each $\upsilon_0 \in \mathcal{X}$, the set $\mathcal{H}(\upsilon_0)$ is a nonempty closed subset of $\mathcal{X}$ and if for each open set $\mathcal{N}$ of $\mathcal{X}$ containing $\mathcal{H}(\upsilon_0)$ there exists an open neighborhood $\mathcal{N}_0$ of υ_0 such that $\mathcal{H}(\mathcal{N}_0) \subseteq \mathcal{N}$;
(4) is said to be completely continuous if $\mathcal{H}(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(\mathcal{X})$;
(5) has a fixed point if there is $\upsilon \in \mathcal{X}$ such that $\upsilon \in \mathcal{H}(\upsilon)$. The fixed point set of the multivalued operator $\mathcal{H}$ will be denote by $Fix\mathcal{H}$.

Remark 1 ([14], Proposition 1.2)**.** *It is well known that, if the multivalued map* $\mathcal{H}$ *is completely continuous with nonempty compact values, then* $\mathcal{H}$ *is u.s.c if and only if* $\mathcal{H}$ *has closed graph i.e.,* $\upsilon_n \longrightarrow \upsilon$, $\omega_n \longrightarrow \omega$, $\omega_n \in \mathcal{H}(\upsilon_n)$ *imply* $\omega \in G(\upsilon)$.

Definition 2.4. A multivalued map $\mathcal{H} : \ J \longrightarrow P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$ the function

$$t \longmapsto d(y, \mathcal{H}(t)) = \inf\{\|y - z\| : z \in \mathcal{H}(t)\},$$

is measurable.

Let $L^1(I,\mathbb{R})$ be the Banach space of measurable functions $\upsilon : I \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by $\|\upsilon\|_{L^1} = \int_1^e |\upsilon(\tau)|\, d\tau$

Definition 2.5. [14, 20] A multivalued map $\mathcal{H} : J \times \mathbb{R} \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is called L^1-Caratheodory if

(i) $t \longmapsto \mathcal{H}(t, \upsilon_1, \upsilon_2)$ is measurable for all $\upsilon_1, \upsilon_2 \in \mathbb{R}$,

(ii) $\tau \longmapsto \mathcal{H}(t, \upsilon_1, \upsilon_2)$ is upper semi-continuous for almost all $\tau \in [1, e]$, and

(iii) for each $\varsigma > 0$, there exists $f_\varsigma \in L^1(I, \mathbb{R}^+)$ such that

$$\|\mathcal{H}(\tau, \upsilon_1, \upsilon_2)\| = \sup\{|\omega|,\ \omega \in \mathcal{H}(\tau, \upsilon_1, \upsilon_2)\} \leq f_\varsigma(\tau),$$

for all $|\upsilon_1|, |\upsilon_2| \leq \varsigma$ and for a.e. $\tau \in I$.

The multivalued map $\mathcal{H}$ is said to be Caratheodory if it satisfies (i) and (ii).

For each $\upsilon \in C(I,\mathbb{R})$, we define the set of selections of G by

$$S_{\mathcal{H},\upsilon} = \left\{\omega \in L^1(I,\mathbb{R}) : \omega(\tau) \in \mathcal{H}(\tau, \upsilon(\tau)),\ for\ almost\ all\ \tau \in I\right\}.$$

Lemma 2.3. [27] *Let $\mathcal{Y}$ be a Banach space and let $\mathcal{H} : I \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{P}_{cp,c}(\mathcal{Y})$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(I,\mathcal{Y})$ to $C(I,\mathcal{Y})$. Then the operator $\Theta \circ S_{\mathcal{H},\upsilon} : C(I,\mathcal{Y}) \longrightarrow \mathcal{P}_{cp,c}(C(I,\mathcal{Y}))$ defined by $(\Theta \circ S_{\mathcal{H}})(\upsilon) = \Theta(S_{\mathcal{H},\upsilon})$ is a closed graph operator in $C(I,\mathcal{Y}) \times C(I,\mathcal{Y})$.*

3. Existence results

In this section, we will establish the existence results for the boundary value problem (1.1)-(1.2) by using the following hybrid fixed point theorem for three operators in a Banach algebra $\mathcal{Y}$ due to Dhage [15].

Lemma 3.1. *Let $\mathcal{Y}$ be a Banach algebra and let $A, C : \mathcal{Y} \longrightarrow \mathcal{Y}$ and $\mathcal{B} : \mathcal{Y} \longrightarrow \mathcal{P}_{cp,c}(\mathcal{Y})$ be three operators satisfying:*

(a_1) $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants r_1 and r_2, respectively,

(a_2) $\mathcal{B}$ is compact and upper semi-continuous,

(a_3) $r_1 M + r_2 < 1$, where $M = \|\mathcal{B}(T_\rho)\|_{\mathcal{P}}$.

Then, either (i) the operator inclusion $\upsilon \in \mathcal{A}\upsilon\mathcal{B}\upsilon + \mathcal{C}\upsilon$ has a solution, or

(ii) the set $\Delta = \{\upsilon \in \mathcal{Y} : \varrho\upsilon \in \mathcal{A}\upsilon\mathcal{B}\upsilon + \mathcal{C}\upsilon,\ \varrho > 1\}$ is unbounded.

For convenience we put

$$\Omega = 1 - \frac{\lambda\Gamma(\sigma-1)}{\Gamma(p+\sigma-1)}(\log\eta)^{p+\sigma-2}. \tag{3.1}$$

Lemma 3.2. *Let $h \in C([1,e],\mathbb{R})$. The solution function u_0 of the hybrid Hadamard equation*

$$D^{\sigma}\left[\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right] = h(\tau), \quad 1 < t < e, \ 2 < \sigma \leq 3, \tag{3.2}$$

subject to the boundary conditions

$$\begin{aligned} &\upsilon(1) = 0, \\ &D^{\sigma-1}\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=1} = 0 \\ &\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=e} = \lambda(I^{p}\upsilon)(\eta), \end{aligned} \tag{3.3}$$

if and only if the function υ_0 is a solution for the following Hadamard integral equation:

$$\begin{aligned} \upsilon(\tau) = \Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau)) &\left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right. \\ &+ \frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds\right. \\ &\left.\left.-\frac{1}{\Gamma(q)}\int_1^{e}\left(\log\frac{e}{s}\right)^{q-1}\frac{h(s)}{s}ds\right)\right] + \sum_{i=1}^{m}\chi_i(t,\upsilon(t)). \end{aligned} \tag{3.4}$$

Proof. Let υ_0 be a solution for hybrid equation (3.2) By virtue of the lemma 2.2, there exist constants c_1, c_2, $c_3 \in \mathbb{R}$ provided that

$$\begin{aligned} \left[\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right] &= \frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds \\ &+ c_1(\log\tau)^{\sigma-1} + c_2(\log\tau)^{\sigma-2} + c_3(\log\tau)^{\sigma-3}. \end{aligned} \tag{3.5}$$

Since $\chi_i(1,0) = 0$, $i = 1,2,...,m$ and $\Psi(1,0,0) \neq 0$, the use of boundary conditions $\upsilon(1) = 0$ and $D^{\sigma-1}\left(\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Big|_{\tau=1} = 0$ gives $c_1 = c_3 = 0$. Applying Hadamard fractional integral operator of order $p > 0$ on both sides of equality (3.5) and using Lemmas 2.1, we get that

$$I^{p}\left(\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)=\frac{1}{\Gamma(\sigma+p)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds + c_2\frac{\Gamma(\sigma-1)}{\Gamma(p+\sigma-1)}(\log\tau)^{p+\sigma-2}.$$

By using the Hadamard integral boundary condition $\left(\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Big|_{\tau=e}=\lambda(I^{p}\upsilon)(\eta)$, we get

$$c_2=\frac{1}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds-\frac{1}{\Gamma(\sigma)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right),$$

where Ω is defined in (3.1).

By inserting the values c_i for $i=1,2,3$ in (3.5), we get

$$\begin{aligned}\upsilon_0(\tau)=\Psi(\tau,\upsilon_0(\tau),I^{\gamma}\upsilon_0(\tau))&\left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right.\\&+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds\right.\\&\left.\left.-\frac{1}{\Gamma(q)}\int_1^{e}\left(\log\frac{e}{s}\right)^{q-1}\frac{h(s)}{s}ds\right)\right]+\sum_{i=1}^{m}\chi_i(t,\upsilon_0(t)).\end{aligned}$$

That is υ_0 a solution for integral equation (3.4). Conversely, one can easily see that υ_0 is a solution function for the hybrid boundary value problem of fractional order (3.2)-(3.3) whenever υ_0 is a solution function for the fractional integral equation (3.4). □

Definition 3.1. A function $\upsilon\in C([1,e],\mathbb{R})$ is called a solution for the problem (1.1)-(1.2) if there exists a function $\kappa\in S_{G,\upsilon}$ such that

$$\begin{aligned}\upsilon(\tau)=\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))&\left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{\kappa(s)}{s}ds\right.\\&+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{\kappa(s)}{s}ds\right.\\&\left.\left.-\frac{1}{\Gamma(q)}\int_1^{e}\left(\log\frac{e}{s}\right)^{q-1}\frac{\kappa(s)}{s}ds\right)\right]+\sum_{i=1}^{m}\chi_i(t,\upsilon(t)),\end{aligned}$$

where $S_{G,\upsilon}=\{\kappa\in L^1[1,e]:\ \kappa(\tau)\in G(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))\ \textit{for almost all}\ \tau\in[1,e]\}$.

Theorem 3.1. *Let us consider the continuous functions* $\varPsi : [1,e]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$, $\chi_i : [1,e]\times\mathbb{R} \longrightarrow \mathbb{R}$ *with* $\chi_i(1,0)=0$ *for* $i=1,2,...,m$), *and the measurable multi-valued map* $\tau \longmapsto G(\tau,x_1,x_2)$ *for all* $x_1,x_2\in\mathbb{R}$, *such that the following assumptions hold:*

$(\mathfrak{H}_1)$ *There exist two positive functions* $\varTheta, \varLambda_i$, $i=1,2,...,m$ *with bounds* $\|\varTheta\|$, $\|\varLambda_i\|$, $i=1,2,...,m$ *respectively, such that*

$$|\varPsi(\tau,\upsilon_1,\upsilon_2)-\varPsi(t,\overline{\upsilon}_1,\overline{\upsilon}_2)| \leq \varTheta(\tau)\left(|\upsilon_1-\overline{\upsilon}_1|+|\upsilon_2-\overline{\upsilon}_2|\right), \tag{3.6}$$

and

$$|\chi_i(\tau,\upsilon)-\chi_i(\tau,\overline{\upsilon})| \leq \varLambda_i(\tau)\,|\upsilon-\overline{\upsilon}|\,,\ i=1,2,...,m, \tag{3.7}$$

for $\tau\in[1,e]$ *and* $\upsilon,\ \overline{\upsilon},\ \upsilon_1,\ \upsilon_2,\ \overline{\upsilon}_1,\ \overline{\upsilon}_2\in\mathbb{R}$.

$(\mathfrak{H}_2)$ *There exists a continuous function* $\vartheta : [1,e]\longrightarrow(0,\infty)$ *such that*

$$\|G(\tau,\upsilon,\overline{\upsilon})\| \leq \vartheta(\tau),$$

for almost all $\tau\in[1,e]$ *and* $\upsilon,\ \overline{\upsilon}\in\mathbb{R}$.

If

$$\begin{aligned}\|\varTheta\|\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|\vartheta\|\left[\frac{1}{\Gamma(\sigma+1)}\left(1+\frac{1}{|\Omega|}\right)+\frac{\lambda(\log\eta)^{p+\sigma}}{|\Omega|\,\Gamma(p+\sigma+1)}\right]&\\ +\sum_{i=1}^{m}\frac{\|\varLambda_i\|}{\Gamma(\theta_i+1)}<1.&\end{aligned} \tag{3.8}$$

Then the problem (1.1)-(1.2) has at least one solution on $[1,e]$.

Proof. Let $\mathcal{Y}=C([1,e],\mathbb{R})$ and consider three operators $\mathcal{A} : \mathcal{Y}\longrightarrow\mathcal{Y}$ by

$$\mathcal{A}\upsilon(\tau)=\varPsi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau)),\ \tau\in[1,e], \tag{3.9}$$

$\mathcal{B} : \mathcal{Y}\longrightarrow\mathcal{Y}$ by

$$\begin{aligned}\mathcal{B}\upsilon(\tau)=\Bigg\{\Bigg[&\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{\kappa(s)}{s}ds\\ &+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{\kappa(s)}{s}ds\right.\\ &\left.-\frac{1}{\Gamma(q)}\int_1^{e}\left(\log\frac{e}{s}\right)^{q-1}\frac{\kappa(s)}{s}ds\right),\ \kappa\in S_{G,\upsilon}\Bigg\},\end{aligned} \tag{3.10}$$

and $\mathcal{C}: \mathcal{Y} \longrightarrow \mathcal{Y}$ by

$$
\begin{aligned}
Cv(\tau) &= \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau, v(\tau)) \\
&= \sum_{i=1}^{m} \int_0^{\tau} \left(\log\frac{\tau}{s}\right)^{\theta_i - 1} \chi_i(s, v(s)) \frac{ds}{s},\ \tau \in [1, e].
\end{aligned} \tag{3.11}
$$

Therefore, the problem (1.1)-(1.2) is equivalent to the problem

$$
v \in \mathcal{A}v\mathcal{B}v + \mathcal{C}v. \tag{3.12}
$$

We will prove that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy the assumptions of Lemma 3.1. We will divide the rest of the proof into five steps.

Step 1. We show that the opertors $\mathcal{A}, \mathcal{C}$ define single-valued operators $\mathcal{A}, \mathcal{C}$: $\mathcal{Y} \longrightarrow \mathcal{Y}$ and $\mathcal{B}: \mathcal{Y} \longrightarrow \mathcal{P}_{cp,c}(\mathcal{Y})$.

The operators $\mathcal{A}, \mathcal{C}$ are well defined because the functions Ψ and χ_i, $i = 1, 2, ..., m$ are continuous on $[1, e] \times \mathbb{R} \times \mathbb{R}$ and $[1, e] \times \mathbb{R}$ respectively. For this step, it only remains to prove the claim for the multi-valued operator $\mathcal{B}$. Note that, the operator $\mathcal{B}$ is equivalent to the composition $\mathcal{Q} \circ S_{G,v}$, where $\mathcal{Q}$ is the continuous linear operator on $L^1([1, e], \mathbb{R})$ into $C([1, e], \mathbb{R})$ defined by

$$
\begin{aligned}
\mathcal{Q}\kappa(\tau) = \frac{1}{\Gamma(\sigma)} \int_1^{\tau} \left(\log\frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \\
+ \frac{(\log\tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log\frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa(s)}{s} ds \right. \\
\left. - \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log\frac{e}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right).
\end{aligned}
$$

Let us consider an arbitrary element v in $\mathcal{Y}$ and $\{\kappa_n\}$ a sequence in $S_{G,v}$. Then, $\kappa_n \in G(\tau, v(\tau), I^{\gamma}v(\tau))$ for almost $\tau \in [1, e]$. Since $G(\tau, v(\tau), I^{\gamma}v(\tau))$ is compact for all $\tau \in [1, e]$, there exists a convergent subsequence of $\{\kappa_n(\tau)\}$(we denote it again $\{\kappa_n(\tau)\}$) to some $\kappa(\tau) \in S_{G,v}$. The continuity of $\mathcal{Q}$ gives that $\mathcal{Q}\kappa_n(\tau)$ converges to $\mathcal{Q}\kappa(\tau)$ pointwise on $[1, e]$. To show that this convergence is uniform, we must show that $\{\mathcal{Q}\kappa_n\}$ is an equi-continuous sequence. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$. So, we have

$$\begin{aligned}
\left|\mathcal{Q}\kappa\left(\tau_2\right)-\mathcal{Q}\kappa\left(\tau_2\right)\right| \leq \frac{1}{\Gamma\left(\sigma\right)}&\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\sigma-1}-\left(\log\frac{\tau_1}{s}\right)^{\sigma-1}\right]\frac{\kappa\left(s\right)}{s}ds\right.\\
&+\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\sigma-1}\frac{\kappa\left(s\right)}{s}ds\\
&+\left[\frac{\left(\log\tau_2\right)^{\sigma-2}-\left(\log\tau_1\right)^{\sigma-2}}{\Omega}\right]\\
&\times\left(\frac{\lambda}{\Gamma\left(\sigma+p\right)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{\kappa\left(s\right)}{s}ds\right.\\
&\left.\left.-\frac{1}{\Gamma\left(\sigma\right)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{\kappa\left(s\right)}{s}ds\right)\right|.
\end{aligned}$$

The right hand of the above inequality tends to zero when τ_2 tends to τ_1. Then, the sequence $\{\mathcal{Q}\kappa_n\}$ is equi-continuous. From Arzela-Ascoli, it follows that there is a uniformly convergent subsequence. Thus, there is a subsequence of $\{\kappa_n\}$ (we show it again by $\{\kappa_n\}$) such that $\mathcal{Q}\kappa_n \longrightarrow \mathcal{Q}\kappa$. Note that $\mathcal{Q}\kappa \in \mathcal{Q}\left(S_{G,\upsilon}\right)$. Then $\mathcal{B}=\mathcal{Q}\left(S_{G,\upsilon}\right)$ is compact for all $\upsilon \in \mathcal{Y}$.

Now, we shall show that $\mathcal{B}\left(\upsilon\right)$ is convex for all $\upsilon\in\mathcal{Y}$. Let $\upsilon\in\mathcal{Y}$ and $\upsilon_1,\upsilon_2\in\mathcal{B}\left(\upsilon\right)$. Select $\kappa_1,\kappa_2\in S_{G,\upsilon}$ such that

$$\begin{aligned}
\upsilon_i\left(\tau\right) &= \frac{1}{\Gamma\left(\sigma\right)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{\kappa_i\left(s\right)}{s}ds\\
&+\frac{\left(\log\tau\right)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma\left(\sigma+p\right)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{\kappa_i\left(s\right)}{s}ds\right.\\
&\left.-\frac{1}{\Gamma\left(\sigma\right)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{\kappa_i\left(s\right)}{s}ds\right),\ i=1,2,
\end{aligned}$$

for almost all $\tau\in[1,e]$. Let $0\leq\omega\leq1$. Then, we have

$$\begin{aligned}
&\left[\omega\upsilon_1+\left(1-\omega\right)\upsilon_2\right]\left(\tau\right)\\
&=\frac{1}{\Gamma\left(\sigma\right)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{\left[\omega\kappa_1\left(s\right)+\left(1-\omega\right)\kappa_2\left(s\right)\right]}{s}ds\\
&\quad+\frac{\left(\log\tau\right)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma\left(\sigma+p\right)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{\left[\omega\kappa_1\left(s\right)+\left(1-\omega\right)\kappa_2\left(s\right)\right]}{s}ds\right.\\
&\quad\left.-\frac{1}{\Gamma\left(\sigma\right)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{\left[\omega\kappa_1\left(s\right)+\left(1-\omega\right)\kappa_2\left(s\right)\right]}{s}ds\right).
\end{aligned}$$

Since G has convex valued, $S_{G,\upsilon}$ is convex, so $\omega\kappa_1(\tau)+(1-\omega)\kappa_2(\tau)\in S_{G,\upsilon}$. Hence

$$\omega\upsilon_1+(1-\omega)\upsilon_2\in\mathcal{Q}\upsilon.$$

This means that $\mathcal{B}$ is convex-valued.

Step 2. In this part, we show that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitz on $\mathcal{Y}$.

Let $\upsilon,\overline{\upsilon}\in\mathcal{Y}$. Then by $(\mathfrak{H}_1)$, for $\tau\in[1,e]$ we have

$$\begin{aligned}
\left|\mathcal{A}\upsilon(\tau)-\mathcal{A}\overline{\upsilon}(\tau)\right| &= \left|\Psi\left(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau)\right)-\Psi\left(\tau,\overline{\upsilon}(\tau),I^{\gamma}\overline{\upsilon}(\tau)\right)\right| \\
&\leq \Theta(\tau)\left(\left|\upsilon(\tau)-\overline{\upsilon}(\tau)\right|+\left|I^{\gamma}\upsilon(\tau)-I^{\gamma}\overline{\upsilon}(\tau)\right|\right) \\
&= \Theta(\tau)\left(\left|\upsilon(\tau)-\overline{\upsilon}(\tau)\right|\left(1+I^{\gamma}(1)\right)\right) \\
&\leq \|\Theta\|\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|\upsilon-\overline{\upsilon}\|,
\end{aligned}$$

which implies $\|\mathcal{A}\upsilon-\mathcal{A}\overline{\upsilon}\|\leq\|\Theta\|\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|\upsilon-\overline{\upsilon}\|$ for all $\upsilon,\overline{\upsilon}\in\mathcal{Y}$. Hence, $\mathcal{A}$ a Lipschitzian on $\mathcal{Y}$ with with Lipschitz constant $\|\Theta\|$.

Analogously, for any $\upsilon,\overline{\upsilon}\in\mathcal{Y}$, we have

$$\begin{aligned}
\left|\mathcal{C}\upsilon(\tau)-\mathcal{C}\overline{\upsilon}(\tau)\right| &= \left|\sum_{i=1}^{m} I^{\theta_i}\left[\chi_i(\tau,\upsilon(\tau))-\chi_i(\tau,\overline{\upsilon}(\tau))\right]\right| \\
&\leq \sum_{i=1}^{m}\frac{1}{\Gamma(\theta_i)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\theta_i-1}\Lambda_i(s)\left|\upsilon(s)-\overline{\upsilon}(s)\right|\frac{ds}{s} \\
&\leq \|\upsilon-\overline{\upsilon}\|\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)}.
\end{aligned}$$

Then

$$\|\mathcal{C}\upsilon-\mathcal{C}\overline{\upsilon}\|\leq\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)}\|\upsilon-\overline{\upsilon}\|.$$

Which means that $\mathcal{C}$ is a Lipschitzian on $\mathcal{Y}$ with Lipschitz constant $\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)}$.

Step 3. Now, we show that $\mathcal{B}$ is compact and upper semi-continuous.

We will show that $\mathcal{B}$ maps bounded sets into bounded sets in $\mathcal{Y}$. Let T_ρ be a bounded subset of $\mathcal{Y}$. Then, there is a constant $\rho>0$, such that $\|\upsilon\|\leq\rho$ for all $\upsilon\in T_\rho$. Then for $g \in \mathcal{B}$, $\upsilon \in T_\rho$, there exists $\kappa \in S_{G,\upsilon}$ such that

$$\begin{aligned} g(\tau) &= \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \\ &\quad + \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa(s)}{s} ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right). \end{aligned}$$

Then for $\tau \in [1, e]$, we have

$$\begin{aligned} |g(\tau)| &\leq \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{|\kappa(s)|}{s} ds \\ &\quad + \frac{(\log \tau)^{\sigma-2}}{|\Omega|} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{|\kappa(s)|}{s} ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{|\kappa(s)|}{s} ds \right) \\ &\leq \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\vartheta(s)}{s} ds \\ &\quad + \frac{(\log \tau)^{\sigma-2}}{|\Omega|} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\vartheta(s)}{s} ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\vartheta(s)}{s} ds \right) \\ &\leq \|\vartheta\| \left[\frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{ds}{s} \right. \\ &\quad + \frac{(\log \tau)^{\sigma-2}}{|\Omega|} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{ds}{s} \right. \\ &\quad \left. \left. + \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{ds}{s} \right) \right] \\ &\leq L_1, \quad \tau \in [1, e], \end{aligned}$$

where

$$\|\vartheta\| \left[\frac{1}{\Gamma(\sigma+1)} \left(1 + \frac{1}{|\Omega|}\right) + \frac{\lambda (\log \eta)^{\sigma+p}}{|\Omega| \Gamma(p+\sigma+1)} \right] = L_1, \tag{3.13}$$

for all $\tau \in [1, e]$. Hence, $\|g\| \leq L_1$, this means that $\mathcal{B}(T_\rho)$ is uniformly bounded on $\mathcal{Y}$.

Now, we will prove that $\mathcal{B}(T_\rho)$ is an equicontinuous set in $\mathcal{Y}$. Suppose that $t_1, t_2 \in [1,e]$ with $t_1 < t_2$ and $\upsilon \in T_\rho$. Then, we have

$$
\begin{aligned}
|g(t_2) - g(t_1)| &= \frac{1}{\Gamma(\sigma)} \left| \left[\int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds - \int_1^{t_1} \left(\log \frac{t_2}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right] \right. \\
&\quad + \frac{(\log t_2)^{\sigma-2} - (\log t_1)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa(s)}{s} ds \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right) \right| \\
&\leq \frac{1}{\Gamma(\sigma)} \left[\int_1^{t_1} \left| \left(\left(\log \frac{t_2}{s}\right)^{\sigma-1} - \left(\log \frac{t_1}{s}\right)^{\sigma-1} \right) \right| \frac{|\kappa(s)|}{s} ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\sigma-1} \frac{|\kappa(s)|}{s} ds \right] \\
&\quad + \frac{\left|(\log t_2)^{\sigma-2} - (\log t_1)^{\sigma-2}\right|}{|\Omega|} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{|\kappa(s)|}{s} ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{|\kappa(s)|}{s} ds \right) \\
&\leq \frac{1}{\Gamma(\sigma)} \left[\int_1^{t_1} \left| \left(\left(\log \frac{t_2}{s}\right)^{\sigma-1} - \left(\log \frac{t_1}{s}\right)^{\sigma-1} \right) \right| \frac{\|\vartheta\|}{s} ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\sigma-1} \frac{\|\vartheta\|}{s} ds \right] \\
&\quad + \frac{\left|(\log t_2)^{\sigma-2} - (\log t_1)^{\sigma-2}\right|}{|\Omega|} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\|\vartheta\|}{s} ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\|\vartheta\|}{s} ds \right),
\end{aligned}
$$

which is independent of υ. Then, the right-hand side of the above inequality tends to zero when $t_1 \longrightarrow t_2$. Hence, by using the Arzela-Ascoli theorem, $\mathcal{B}$ is completely continuous operator on T_ρ.

Here, we show that $\mathcal{B}$ has a closed graph. Suppose that $\upsilon_n \in T_\rho$ and $g_n \in \mathcal{B}\upsilon_n$ for all n such that $\upsilon_n \longrightarrow \upsilon^*$ and $g_n \longrightarrow g^*$. We show that $g^* \in \mathcal{B}\upsilon^*$. For each natural number n, select $\kappa_n \in S_{G,\upsilon_n}$ such that for each $\tau \in [1,e]$,

$$\begin{aligned} g_n(\tau) &= \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa_n(s)}{s} ds \\ &+ \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa_n(s)}{s} ds \right. \\ &\left. - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa_n(s)}{s} ds \right). \end{aligned}$$

Thus it suffices to show that there exists $\kappa^* \in S_{G,v^*}$ such that for each $\tau \in [1,e]$,

$$\begin{aligned} g^*(\tau) &= \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa^*(s)}{s} ds \\ &+ \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa^*(s)}{s} ds \right. \\ &\left. - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa^*(s)}{s} ds \right). \end{aligned}$$

Consider the linear operator $L: \ L^1([1,e],\mathbb{R}) \longrightarrow \mathcal{Y}$ defined by

$$\begin{aligned} \kappa \longmapsto L(\kappa)(\tau) &= \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \\ &+ \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa(s)}{s} ds \right. \\ &\left. + \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right). \end{aligned}$$

Note that

$$\begin{aligned} \|g_n(\tau) - g^*(\tau)\| &= \frac{1}{\Gamma(\sigma)} \left\| \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{(\kappa_n(s) - \kappa^*(s))}{s} ds \right. \\ &+ \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{(\kappa_n(s) - \kappa^*(s))}{s} ds \right. \\ &\left.\left. + \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{(\kappa_n(s) - \kappa^*(s))}{s} ds \right) \right\| \to 0, \end{aligned}$$

as $n \longrightarrow \infty$.

Thus, it follows by Lemma 2.3 that $L \circ S_G$ is a closed graph operator. Further, we have $g_n(\tau) \in L(S_{G,\upsilon_n})$. As $\upsilon_n \longrightarrow \upsilon^*$, we get

$$\begin{aligned} g^*(\tau) &= \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa^*(s)}{s} ds \\ &+ \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa^*(s)}{s} ds \right. \\ &\left. - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa^*(s)}{s} ds \right), \end{aligned}$$

for some $\kappa^* \in S_{G,\kappa^*}$. Hence $\mathcal{B}$ has a closed graph. Consequently, the operator $\mathcal{B}$ is upper semi-continuous.

Step 4. We show that the condition (a_3) of Lemma 3.1 holds. Since

$$\begin{aligned} M = \|\mathcal{B}(T_\rho)\| &= \sup_{\upsilon \in T_\rho} \left\{ \sup_{\tau \in [1,e]} (\mathcal{B}\upsilon(\tau)) \right\} \\ &\leq \|\vartheta\| \left[\frac{1}{\Gamma(\sigma+1)} \left(1 + \frac{1}{|\Omega|}\right) + \frac{\lambda (\log \eta)^{p+\sigma}}{|\Omega| \Gamma(p+\sigma+1)} \right], \end{aligned} \tag{3.14}$$

as

$$\begin{aligned} \|\Theta\| \left(1 + \frac{1}{\Gamma(\gamma+1)}\right) \|\vartheta\| \left[\frac{1}{\Gamma(\sigma+1)} \left(1 + \frac{1}{|\Omega|}\right) + \frac{\lambda (\log \eta)^{p+\sigma}}{|\Omega| \Gamma(p+\sigma+1)} \right] \\ + \sum_{i=1}^m \frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)} < 1, \end{aligned}$$

where $r_1 = \|\Theta\| \left(1 + \frac{1}{\Gamma(\gamma+1)}\right)$ and $r_2 = \sum_{i=1}^m \frac{\|\|\Lambda_i\|\|}{\Gamma(\theta_i+1)}$. Then condition (a_3) of Lemma 3.1 is satisfied. So, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy the conditions of Lemma 3.1.

Step 5. In this last step, we show that the conclusion (ii) of Lemma 3.1 is unachievable.

Let υ any solution of the problem (1.1)-(1.2) such that $\varrho\upsilon \in \mathcal{A}\upsilon\mathcal{B}\upsilon + \mathcal{C}\upsilon$ for some $\varrho > 1$. Then there exists $\kappa \in S_{G,\upsilon}$ such that

$$\begin{aligned} \upsilon(\tau) &= \epsilon\Psi(\tau, \upsilon(\tau), I^\gamma \upsilon(\tau)) \left[\frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right. \\ &+ \frac{(\log \tau)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{\kappa(s)}{s} ds \right. \\ &\left.\left. - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{\kappa(s)}{s} ds \right) \right] \\ &+ \epsilon \sum_{i=1}^m I^{\theta_i} \chi_i(\tau, \upsilon(\tau)), \quad \tau \in [1, e], \end{aligned}$$

where $\epsilon = \frac{1}{\varrho} < 1$. Therefore, we have

$$
\begin{aligned}
|\upsilon(\tau)| \leq{}& |\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))| \left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{|\kappa(s)|}{s}ds\right.\\
&+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{|\kappa(s)|}{s}ds\right.\\
&\left.\left.-\frac{1}{\Gamma(q)}\int_1^{e}\left(\log\frac{e}{s}\right)^{q-1}\frac{|\kappa(s)|}{s}ds\right)\right]\\
\leq{}& \left(|\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))-\Psi(\tau,0,0)|+|\Psi(\tau,0,0)|\right)\|\vartheta\|\left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{ds}{s}\right.\\
&\left.+\frac{(\log\tau)^{\sigma-2}}{|\Omega|}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{ds}{s}+\frac{1}{\Gamma(\sigma)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{ds}{s}\right)\right]\\
&+\sum_{i=1}^{m}\frac{1}{\Gamma(\theta_i+1)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{s}\right)\left(|\chi_i(s,\upsilon(s))-\chi_i(s,0)|+|\chi_i(s,0)|\right)\frac{ds}{s}\\
\leq{}& \left[\|\Theta\|\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|\upsilon\|+\Psi_0\right]L_1+\left(\|\upsilon+X_0\|\right)\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)},
\end{aligned}
$$

where L_1 defined in (3.13), $\Psi_0 = \sup_{\tau\in[1,e]}|\Psi(\tau,0,0)|$ and $X_0 = \sup_{\tau\in[1,e]}|\chi_i(\tau,0)|$, $i = 1,2,...,m$, this gives that

$$
\|\upsilon\| \leq \frac{\Psi_0 L_1 + X_0\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)}}{1-\left[L_1\|\Theta\|\left(1+\frac{1}{\Gamma(\gamma+1)}\right)+\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)}\right]}.
$$

Consequently, the conclusion (ii) of Lemma 3.1 does not hold. Thus, the conclusion (i) holds and consequently the problem (1.1)-(1.2) has at least one solution on $[1, e]$. □

In order to illustrate the obtained results, we give an example in the next.

Example 3.1. *Consider the following hybrid fractional differential inclusion*

$$
(3.15)\qquad D^{\frac{5}{2}}\left[\frac{\upsilon(\tau)-\sum_{i=1}^{4}I^{\frac{2i-1}{2}}\chi_i(\tau,\upsilon(t))}{\Psi\left(\tau,\upsilon(\tau),I^{\frac{9}{2}}\upsilon(\tau)\right)}\right]\in G\left(\tau,\upsilon(\tau),I^{\frac{9}{2}}\upsilon(\tau)\right),\quad \tau\in[1,e],
$$

with the boundary conditions

$$\begin{aligned}
&\upsilon(1) = 0,\\
&D^{\frac{3}{2}}\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau), I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=1} = 0,\\
&\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi\left(\tau,\upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right)}\right)\Bigg|_{\tau=e} = 0,02\left(I^{2,5}\upsilon\right)(\eta),
\end{aligned} \tag{3.16}$$

where

$$\chi_i(\tau,\upsilon(\tau)) = \frac{1-|\upsilon(\tau)|}{3(6+i+\tau)(1+|\upsilon(\tau)|)}, \quad i = 1,2,3,4,$$

$$\Psi\left(\tau,\upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right) = \frac{\sin|\upsilon(\tau)|}{1+\cosh(\sqrt{\tau})} + \frac{1}{12(\tau+1)}\left(\frac{3}{1+\left|I^{\frac{9}{2}}\upsilon(\tau)\right|}+3\right)\left|I^{\frac{9}{2}}\upsilon(\tau)\right|,$$

and $G\left(\tau,\upsilon(\tau), I^{\frac{9}{2}}\right) = \left[\frac{1+\cos|\upsilon(\tau)|}{\cosh(\upsilon(\tau))(2+\tau^3)}, \frac{\sin\left(I^{\frac{9}{2}}\upsilon(\tau)\right)}{2\sqrt{\tau+1}}\right]$

Here, $\sigma = \frac{5}{2}$, $m = 4$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{3}{2}$, $\theta_3 = \frac{5}{2}$, $\theta_4 = \frac{7}{2}$, $\gamma = \frac{9}{2}$, $p = \frac{5}{2}$ *and* $\lambda = 0,02$. *We obtain*

$$|\Psi(\tau,\upsilon_1,\overline{\upsilon}_1) - \Psi(\tau,\upsilon_2,\overline{\upsilon}_2)| \leq \frac{1}{2(\tau+1)}\left(|\upsilon_1-\upsilon_2| + |\overline{\upsilon}_1-\overline{\upsilon}_2|\right),$$

and

$$|\chi_i(\tau,\upsilon) - \chi_i(\tau,\overline{\upsilon})| \leq \frac{2}{3(6+i+\tau)}|\upsilon-\overline{\upsilon}|, \ i = 1,2,3,4,$$

for $\upsilon,\overline{\upsilon},\upsilon_j,\overline{\upsilon}_j \in \mathbb{R}$, $j = 1,2$. *Set* $\Theta(\tau) = \frac{1}{2(\tau+1)}$ *and* $\Lambda_i(\tau) = \frac{2}{3(6+i+\tau)}$, *we get* $\|\Theta\| = \frac{1}{2}$ *and* $\|\Lambda_i\| = \frac{2}{3(7+i)}$, $i = 1,2,3,4$. *For* $g \in G$, *we have*

$$|g| \leq \max\left(\frac{1+\cos|\upsilon(\tau)|}{\cosh(\upsilon(\tau))(2+\tau^3)}, \frac{\sin\left(I^{\frac{9}{2}}\upsilon(\tau)\right)}{2\sqrt{\tau+1}}\right) \leq \frac{1}{2\sqrt{\tau+1}}.$$

Then

$$\left\|G\left(\tau,\upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right)\right\|_{\mathcal{P}} \leq \vartheta(\tau),$$

where $\vartheta(\tau) = \frac{1}{2\sqrt{\tau+1}}$, *thus* $\|\vartheta\| = \frac{1}{2\sqrt{2}}$.
By direct computation, we have

$$\|\Theta\|\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|\vartheta\|\left[\frac{1}{\Gamma(q+1)}\left(1+\frac{1}{\Omega}\right)+\frac{\lambda(\log\eta)^{p+q}}{\Omega\Gamma(p+q+1)}\right]$$
$$+\sum_{i=1}^{m}\frac{\|\Lambda_i\|}{\Gamma(\theta_i+1)} = 0,3113141 < 1.$$

By Theorem 3.1, we claim that problem (3.15)-(3.16) has at least one solution on $[1,e]$.

4. Conclusion

In this article, we study a class of hybrid integro-differential inclusions involving Hadamard-type derivative supplemented with hybrid Hadamard integral boundary conditions. We investigate the existence results of the suggested problem by means of hybrid fixed point theorem of Schaefer type for three operators in Banach algebra due to Dhage. We justify our obtained results by giving an example.

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Department of Mathematics, Faculty of science and technology, Relizane University, Relizane 48000, Algeria

Email address: djourdem.habib7@gmail.com