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EDGE IRREGULARITY STRENGTH OF SOME RELATED GRAPHS TO T_p-TREE

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Abstract. For a graph $G = (V, E)$, let $\phi : V(G) \to \{1, 2, \cdots k\}$ to be vertex k-labeling. An edge irregular k-labeling is defined as follows: for each edge $e = xy$ a weight $wt(xy)$ is given by $wt(xy) = \phi(x) + \phi(y)$ such that $wt(xy) \neq wt(x^{'}y^{'})$ for every two different edges xy and $x^{'}y^{'}$. The edge irregularity strength of G, written by $es(G)$, is the least k for which the graph G has an edge irregular k-labeling. In this paper we find the edge irregularity strength of a T_p-tree (transformed tree) and some related graphs to T_p-trees.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph. Chartrand et al. [8] proposed irregular strength of graphs as a result of irregular assignments. Ahmad et al. [1] defined the edge irregular k-labeling as $\phi : V(G) \to \{1, 2, \cdots, k\}$ such that for each edge xy a weight $wt(xy)$ is given by $\phi(x)+\phi(y)$. In this assignment the condition to be satisfied is $wt(xy) \neq wt(x^{'}y^{'})$ for every two different edges xy and $x^{'}y^{'}$. The edge irregularity strength of G, given by $es(G)$, is the smallest k such that G possess an edge irregular k-labeling.

The edge irregularity strength of cartesian product of specific families of graphs with path P_2 was calculated by Al-Mushayt [3]. Using an algorithmic technique, Asim et al. [4] estimated the improved upper bound for complete graphs. For the toeplitz graphs, Ahmad [2] found the edge irregular k-labeling. Tarawneh et al. [15]

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studied this parameter for disjoint union of star graph and subdivision graph. Further research works on edge irregularity strength were discussed in [5, 8, 13, 14, 16].

Theorem 1.1. [1] For a simple graph G with maximum degree $\Delta(G)$, $es(G) \geq \max\{\frac{|E(G)|+1}{2}, \Delta(G)\}$.

Definition 1.1. [9] Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Suppose there are two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge u_0v_0 is called transformable edge.

If by the sequence of ept's, T can be reduced to a path, then T is called a T_p-tree (transformed tree) and such a sequence regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$.

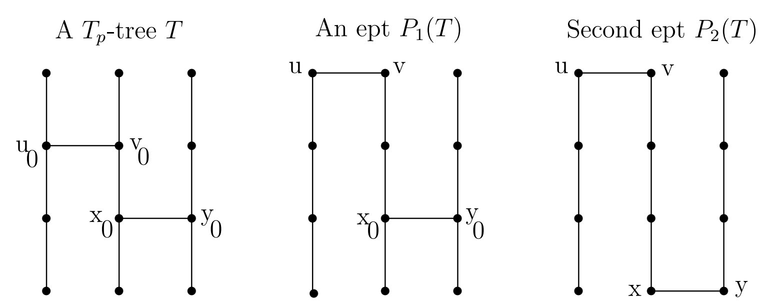

FIGURE 1. A T_p-tree and a sequence of two ept's reducing it to a path

Definition 1.2. [11] Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1\widehat{O}G_2$ is obtained from G_1 and p copies of G_2 by identifying one vertex of i^{th} copy of G_2 with the i^{th} vertex of G_1.

Definition 1.3. [11] Let G_1 be a graph of order p and G_2 be any graph. The corona product of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained by taking one copy of G_1 and p copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Definition 1.4. [11] Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1\tilde{O}G_2$ is obtained from G_1 and p copies of G_2 by joining one vertex of i^{th} copy of G_2 with the i^{th} vertex of G_1 by an edge.

Definition 1.5. If every vertex of V_1 is joined to all the vertices of V_2, then G is called a complete bipartite graph. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a star graph.

Definition 1.6. [12] The quadrilateral snake QS_n is obtained from a path $u_1, u_2, \cdots, u_n$ by joining u_i, u_{i+1} to new vertices x_i, y_i respectively and then joining x_i and y_i. That is every edge of the path is replaced by a cycle C_4.

2. Main Results

In this section, we will find the edge irregularity strength of a T_p-tree and some related graphs to T_p-trees such as $T\widehat{O}P_n$, $T\widehat{O}K_{1,n}$, $T \odot nK_1$, $T\widehat{O}QS_n$ and $T\widetilde{O}QS_n$.

Theorem 2.1. If T is a T_p-tree on m vertices, then $es(T) = \lceil \frac{m}{2} \rceil$.

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have, $(i)V(P(T)) = V(T)$ and $(ii)E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other.
Define $\phi : V(T) \to \{1, 2, \cdots, \lceil \frac{m}{2} \rceil\}$ as follows:

$$\phi(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq m \\ \frac{i}{2} & \text{if } i \text{ is even and } 1 \leq i \leq m. \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1=j-t$ which implies $j=i+2t+1$. Therefore, i and j are of opposite parity.
The value of the edge v_iv_j is given by

$$\begin{aligned} wt(v_iv_j) &= wt(v_iv_{i+2t+1}) \\ &= \phi(v_i)+\phi(v_{i+2t+1}) \\ &= i+t+1. \end{aligned}$$

The value of the edge $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} wt(v_{i+t}v_{j-t}) &= wt(v_{i+t}v_{i+t+1}) \\ &= \phi(v_{i+t})+\phi(v_{i+t+1}) \\ &= i+t+1. \end{aligned}$$

Therefore, $wt(v_iv_j)=wt(v_{i+t}v_{j-t})$.
The induced edge labels are

$wt(v_iv_{i+1})=i+1,\ 1\le i\le m-1.$

It is a routine matter to verify that all vertex and edge labels are at most $\lceil \frac{m}{2} \rceil$ and the edge weights are distinct. Hence $es(T)=\lceil \frac{m}{2} \rceil$. □

Theorem 2.2. If T is a T_p-tree on m vertices with identifying each vertex of T to the last vertex of P_n, then $es(T\widehat{O}P_n)=\lceil \frac{mn}{2} \rceil$.

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T))=V(T)$ and (ii) $E(P(T))=(E(T)-E_d)\bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P=(P_1,P_2,\cdots,P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1,v_2,\cdots,v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j,u_2^j,\cdots,u_n^j(1\le j\le m)$ be the vertices of the j^{th} copy of P_n with $u_n^j=v_j$. Then $V(T\widehat{O}P_n)=\{v_j,u_i^j : 1\le i\le n, 1\le j\le m \text{ with } u_n^j=v_j\}$ and

$E(T\widehat{O}P_n) = E(T) \bigcup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$.

By Theorem (1.1), we have $es(T\widehat{O}P_n) \geq \lceil \frac{mn}{2} \rceil$. For the reverse inequality, we define the labeling $\phi : V(T\widehat{O}P_n) \to \{1, 2, 3, \ldots, \lceil \frac{mn}{2} \rceil\}$ as follows:

For $1 \leq j \leq m$ and $1 \leq i \leq n$,

$$\phi(u_i^j) = \begin{cases} \frac{n(j-1)}{2} + \lceil \frac{i}{2} \rceil & \text{if } j \text{ is odd} \\ \frac{nj}{2} - \lfloor \frac{i-1}{2} \rfloor & \text{if } j \text{ is even }. \end{cases}$$

Let $v_i v_j$ be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity.

The weight of edge $v_i v_j$ is given by

$$\begin{aligned} wt(v_i v_j) &= wt(v_i v_{i+2t+1}) \\ &= \phi(v_i) + \phi(v_{i+2t+1}) \\ &= n(i + t) + 1. \end{aligned}$$

The weight of edge $v_{i+t} v_{j-t}$ is given by

$$\begin{aligned} wt(v_{i+t} v_{j-t}) &= wt(v_{i+t} v_{i+t+1}) \\ &= \phi(v_{i+t}) + \phi(v_{i+t+1}) \\ &= n(i + t) + 1. \end{aligned}$$

Therefore, $wt(v_i v_j) = wt(v_{i+t} v_{j-t})$.

The edge weights are as follows:

$$wt(v_j v_{j+1}) = nj + 1,\ 1 \leq j \leq m - 1;$$

for $1 \leq i \leq n - 1$ and $1 \leq j \leq m$,

$$wt(u_i^j u_{i+1}^j) = \begin{cases} n(j-1) + i + 1 & \text{if } j \text{ is odd} \\ nj - i + 1 & \text{if } j \text{ is even }. \end{cases}$$

It is a routine matter to verify that all vertex and edge labels are at most $\lceil \frac{mn}{2} \rceil$ and the edge weights are distinct. Hence $es(T\widehat{O}P_n) = \lceil \frac{mn}{2} \rceil$. □

Theorem 2.3. If T is a T_p-tree on m vertices with identifying each vertex of T to one of the vertices of degree one of $K_{1,n}$, then $es(T\widehat{O}K_{1,n}) = \left\lceil \frac{m(n+1)}{2} \right\rceil$.

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have $(i)V(P(T)) = V(T)$ and $(ii)E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_0^j, u_1^j, \cdots, u_n^j (1 \le j \le m)$ be the vertices of the i^{th} copy of $K_{1,n}$ with $u_1^j = v_j$. Then $V(T\widehat{O}K_{1,n}) = \{v_j, u_0^j, u_i^j : 1 \le i \le n, 1 \le j \le m \text{ with } v_j = u_1^j\}$ and $E(T\widehat{O}K_{1,n}) = E(T) \bigcup \{u_0^j u_i^j : \ 1 \le j \le m, \ 1 \le i \le n\}$.

By Theorem (1.1), we have $es(T\widehat{O}K_{1,n}) \ge \left\lceil \frac{m(n+1)}{2} \right\rceil$. For the reverse inequality, it is enough to show that $es(T\widehat{O}K_{1,n}) \le \left\lceil \frac{m(n+1)}{2} \right\rceil$. Define $\phi : V(T\widehat{O}K_{1,n}) \to \left\{1, 2, 3, \ldots, \left\lceil \frac{m(n+1)}{2} \right\rceil \right\}$ as follows:

Case 1. m is even.

$$\phi(v_j) = \begin{cases} \frac{(n+1)(j-1)}{2} + 1 & \text{if } \ j \text{ is odd and } 1 \le j \le m \\ \frac{(n+1)j}{2} & \text{if } \ j \text{ is even and } 1 \le j \le m \ ; \end{cases}$$

$$\phi(u_0^j) = \begin{cases} \frac{(n+1)(j-1)}{2} + 1 & \text{if } \ j \text{ is odd and } 1 \le j \le m \\ \frac{(n+1)j}{2} & \text{if } \ j \text{ is even and } 1 \le j \le m \ ; \end{cases}$$

$$\phi(u_i^j) = \begin{cases} \frac{(n+1)(j-1)}{2} + i & \text{if } \ j \text{ is odd and } 1 \le j \le m, \ 2 \le i \le n \\ \frac{(n+1)j}{2} - i + 1 & \text{if } \ j \text{ is even and } 1 \le j \le m, \ 2 \le i \le n \ . \end{cases}$$

Case 2. m is odd.

$$\phi(v_j) = \begin{cases} \frac{(n+1)(j-1)}{2} + 1 & \text{if } \ j \text{ is odd and } 1 \le j \le m \\ \frac{(n+1)j}{2} & \text{if } \ j \text{ is even and } 1 \le j \le m \ ; \end{cases}$$

$$\phi(u_0^j) = \begin{cases} \frac{(n+1)(j-1)}{2} + 1 & \text{if } j \text{ is odd and } 1 \leq j \leq m-1 \\ \frac{(n+1)j}{2} & \text{if } j \text{ is even and } 1 \leq j \leq m-1 \\ \left\lceil \frac{(n+1)m}{2} \right\rceil & \text{if } j = m; \end{cases}$$

$$\phi(u_i^j) = \begin{cases} \frac{(n+1)(j-1)}{2} + i & \text{if } j \text{ is odd and } 1 \leq j \leq m-1,\ 2 \leq i \leq n \\ \frac{(n+1)j}{2} - i + 1 & \text{if } j \text{ is even and } 1 \leq j \leq m-1,\ 2 \leq i \leq n; \end{cases}$$

When n even, $$\phi(u_i^m) = \begin{cases} \left\lceil \frac{(n+1)m}{2} \right\rceil - \left\lceil \frac{i}{2} \right\rceil & \text{if } 2 \leq i \leq \frac{n}{2}, \\ \left\lceil \frac{(n+1)m}{2} \right\rceil - i & \text{if } \frac{n}{2} + 1 \leq i \leq n\ ; \end{cases}$$

When n odd, $$\phi(u_i^m) = \begin{cases} \left\lceil \frac{(n+1)m}{2} \right\rceil - \left\lceil \frac{i-2}{2} \right\rceil & \text{if } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{(n+1)m}{2} \right\rceil - i + 1 & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n\ . \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity.

The weight of the edge v_iv_j is given by

$$\begin{aligned} wt(v_iv_j) &= wt(v_iv_{i+2t+1}) \\ &= \phi(v_i) + \phi(v_{i+2t+1}) \\ &= (n+1)(i+t) + 1. \end{aligned}$$

The weight of the edge $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} wt(v_{i+t}v_{j-t}) &= wt(v_{i+t}v_{i+t+1}) \\ &= \phi(v_{i+t}) + \phi(v_{i+t+1}) \\ &= (n+1)(i+t) + 1. \end{aligned}$$

Therefore, $wt(v_iv_j) = wt(v_{i+t}v_{j-t})$.

The edge weights are as follows:

$wt(v_jv_{j+1}) = (n+1)j + 1,\ 1 \leq j \leq m-1;$

When m is even and $1 \leq j \leq m$.

$$wt(v_j u_0^j) = \begin{cases} (n+1)(j-1)+2 & \text{if } j \text{ is odd} \\ (n+1)j & \text{if } j \text{ is even;} \end{cases}$$

$$wt(u_0^j u_i^j) = \begin{cases} (n+1)(j-1)+i+1 & \text{if } j \text{ is odd and } 2 \leq i \leq n \\ (n+1)j-i+1 & \text{if } j \text{ is even and } 2 \leq i \leq n\ . \end{cases}$$

When m is odd and $1 \leq j \leq m$.

$$wt(v_j u_0^j) = \begin{cases} (n+1)(j-1)+2 & \text{if } j \text{ is odd and } 1 \leq j \leq m-1 \\ (n+1)j & \text{if } j \text{ is even and } 1 \leq j \leq m-1; \end{cases}$$

$$wt(v_m u_0^m) = (n+1)m - \left\lfloor \frac{n+1}{2} \right\rfloor + 1;$$

$$wt(u_0^j u_i^j) = \begin{cases} (n+1)(j-1)+i+1 & \text{if } j \text{ is odd and } 2 \leq i \leq n,\ 1 \leq j \leq m-1 \\ (n+1)j-i+1 & \text{if } j \text{ is even and } 2 \leq i \leq n,\ 1 \leq j \leq m-1\ ; \end{cases}$$

$$\text{for } n \text{ even, } wt(u_0^m u_i^m) = \begin{cases} (n+1)m - \left\lceil \frac{i}{2} \right\rceil + 1 & \text{if } 2 \leq i \leq \frac{n}{2}, \\ (n+1)m-i+1 & \text{if } \frac{n}{2}+1 \leq i \leq n\ ; \end{cases}$$

$$\text{for } n \text{ odd, } wt(u_0^m u_i^m) = \begin{cases} (n+1)m - \left\lceil \frac{i-2}{2} \right\rceil & \text{if } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ (n+1)m-i+1 & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n\ . \end{cases}$$

It is easy to verify that all the vertex and edge labels are at most $\left\lceil \frac{m(n+1)}{2} \right\rceil$ and the edge weights are distinct. Hence $es(T\widehat{O}K_{1,n}) = \left\lceil \frac{m(n+1)}{2} \right\rceil$. □

An illustration of edge irregular k-labeling of $T\widehat{O}K_{1,3}$ where T is a T_p-tree with 11 vertices is shown in Figure 2.

Theorem 2.4. If T is a T_p-tree with even number of vertices, then $es(T \odot nK_1) = \left\lceil \frac{mn+m}{2} \right\rceil$.

Proof. Let T be a T_p-tree with m vertices where m is even. By the definition of T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up

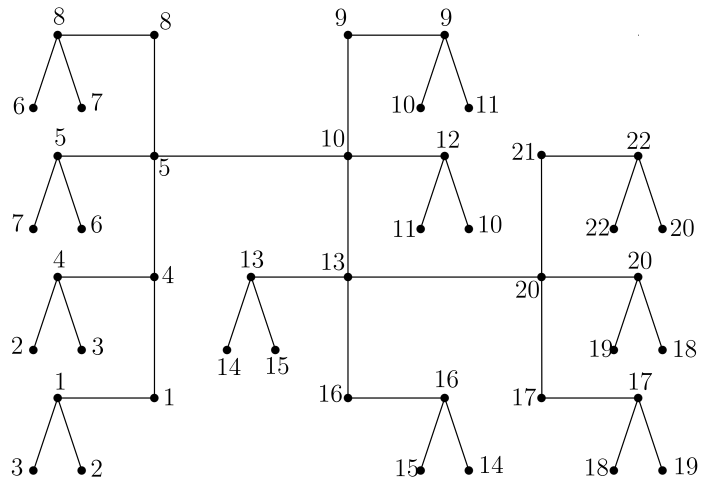

FIGURE 2

to the other. Let $u_1^j, u_2^j, \cdots, u_n^j (1 \leq j \leq m)$ be the pendant vertices joined with $v_j (1 \leq j \leq m)$ by an edge. Then $V(T \odot nK_1) = \{v_j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T \odot nK_1) = E(T) \bigcup \{v_j u_i^j :\ 1 \leq j \leq m,\ 1 \leq i \leq n\}$.

By Theorem (1.1), we have $es(T \odot nK_1) \geq \lceil \frac{mn+m}{2} \rceil$. For the reverse inequality, we define the labeling $\phi : V(T \odot nK_1) \to \{1, 2, 3, \ldots, \lceil \frac{mn+m}{2} \rceil\}$ as follows:
For $1 \leq j \leq m$,

$$\phi(v_j) = \begin{cases} \frac{(n+1)(j-1)}{2} + 1 & \text{if } j \text{ is odd} \\ \frac{(n+1)j}{2} & \text{if } j \text{ is even ;} \end{cases}$$

$$\phi(u_i^j) = \begin{cases} \frac{(n+1)(j-1)}{2} + i & \text{if } j \text{ is odd } 1 \leq i \leq n \\ \frac{(n+1)(j-2)}{2} + i + 1 & \text{if } j \text{ is even } 1 \leq i \leq n \text{ ;} \end{cases}$$

Let $v_i v_j$ be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity.

The weight of edge v_iv_j is given by

$$\begin{aligned} wt(v_iv_j) &= wt(v_iv_{i+2t+1}) \\ &= \phi(v_i) + \phi(v_{i+2t+1}) \\ &= (n+1)(i+t)+1. \end{aligned}$$

The weight of edge $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} wt(v_{i+t}v_{j-t}) &= wt(v_{i+t}v_{i+t+1}) \\ &= \phi(v_{i+t}) + \phi(v_{i+t+1}) \\ &= (n+1)(i+t)+1. \end{aligned}$$

Therefore, $wt(v_iv_j) = wt(v_{i+t}v_{j-t})$.

The edge weights are as follows:

$$wt(v_jv_{j+1}) = (n+1)j+1, \ 1 \le j \le m-1;$$
$$wt(v_ju_i^j) = (n+1)(j-1)+i+1, \ 1 \le j \le m, \ 1 \le i \le n.$$

Thus the edge weights are distinct. Hence $es(T \odot nK_1) = \left\lceil \frac{mn+m}{2} \right\rceil$. □

An illustration of edge irregular k-labeling of $T \odot 4K_1$ where T is a T_p-tree with 10 vertices is shown in Figure 3.

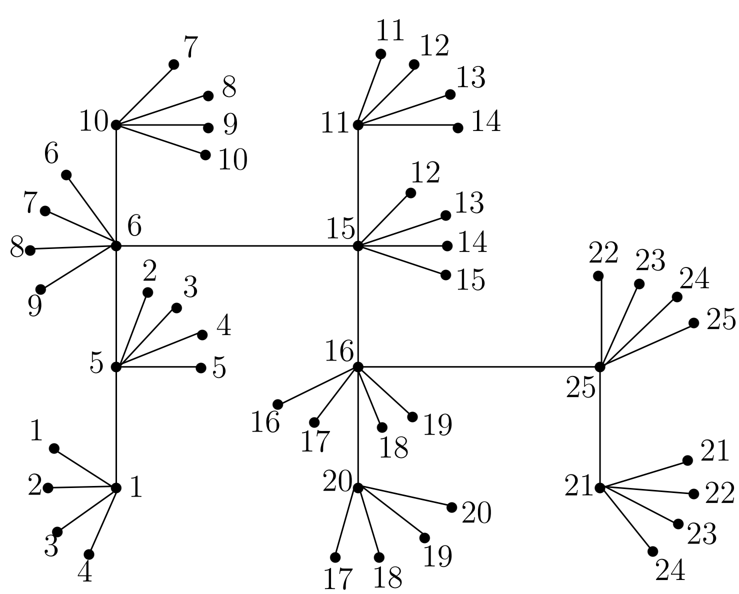

FIGURE 3

Theorem 2.5. If T is a T_p-tree on m vertices with identifying each vertex of T to the last vertex of QS_n, then $es(T\widehat{O}QS_n) = \lceil \frac{4mn+m}{2} \rceil$.

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j, u_{n+1}^j (1 \le j \le m)$ be the vertices of j^{th} copy of QS_n with $u_{n+1}^j = v_j$. Then $V(T\widehat{O}QS_n) = \{u_i^j : 1 \le i \le n+1, 1 \le j \le m\} \bigcup \{x_i^j, y_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(T\widehat{O}QS_n) = E(T) \bigcup E(QS_n)$. We note that $\left| V(T\widehat{O}QS_n) \right| = 3nm + m$ and $\left| E(T\widehat{O}QS_n) \right| = 4mn + m - 1$.

By Theorem (1.1), we have $es(T\widehat{O}QS_n) \ge \lceil \frac{4mn+m}{2} \rceil$. For the reverse inequality, it is enough to show that $es(T\widehat{O}QS_n) \le \lceil \frac{4mn+m}{2} \rceil$. Define $\phi : V(T\widehat{O}QS_n) \to \{1, 2, 3, \ldots, \lceil \frac{4mn+m}{2} \rceil\}$ as follows:

For $1 \le i \le n+1$,

$$\phi(u_i^j) = \begin{cases} \frac{(4n+1)(j-1)}{2} + 2(i-1) + 1 & \text{if } j \text{ is odd and } 1 \le j \le m \\ \frac{(4n+1)j}{2} - 2(i-1) & \text{if } j \text{ is even and } 1 \le j \le m . \end{cases}$$

$$\phi(v_j) = \phi(u_{n+1}^j).$$

For $1 \le i \le n$,

$$\phi(x_i^j) = \begin{cases} \frac{(4n+1)(j-1)}{2} + 2i & \text{if } j \text{ is odd and } 1 \le j \le m \\ \frac{(4n+1)j}{2} - 2i + 2 & \text{if } j \text{ is even and } 1 \le j \le m; \end{cases}$$

$$\phi(y_i^j) = \begin{cases} \frac{(4n+1)(j-1)}{2} + 2(i-1) + 1 & \text{if } j \text{ is odd and } 1 \le j \le m, \\ \frac{(4n+1)j}{2} - 2i + 1 & \text{if } j \text{ is even and } 1 \le j \le m. \end{cases}$$

Let $v_i v_j$ be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and

j are of opposite parity.

The weight of the edge v_iv_j is given by

$$\begin{aligned} wt(v_iv_j) &= wt(v_iv_{i+2t+1}) \\ &= \phi(v_i) + \phi(v_{i+2t+1}) \\ &= (4n+1)(i+t)+1. \end{aligned}$$

The weight of edge $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} wt(v_{i+t}v_{j-t}) &= wt(v_{i+t}v_{i+t+1}) \\ &= \phi(v_{i+t}) + \phi(v_{i+t+1}) \\ &= (4n+1)(i+t)+1. \end{aligned}$$

Therefore, $wt(v_iv_j) = wt(v_{i+t}v_{j-t})$.

The edge weights are as follows:

$$wt(v_jv_{j+1}) = (4n+1)j+1, \ 1 \leq j \leq m-1;$$

For $1 \leq i \leq n$,

$$wt(u_i^jx_i^j) = \begin{cases} (4n+1)(j-1)+4i-1 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+1)j-4i+4 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(u_i^jy_i^j) = \begin{cases} (4n+1)(j-1)+4(i-1)+2 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+1)j-4i+3 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(x_i^ju_{i+1}^j) = \begin{cases} (4n+1)(j-1)+4i+1 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+1)j-4i+2 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(y_i^ju_{i+1}^j) = \begin{cases} (4n+1)(j-1)+4i & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+1)j-4i+1 & \text{if } j \text{ is even and } 1 \leq j \leq m. \end{cases}$$

Thus the edge weights are distinct. Hence $es(T\widehat{O}QS_n) = \lceil \frac{4mn+m}{2} \rceil$. □

An illustration of edge irregular k-labeling of $T\widehat{O}QS_2$ where T is a T_p-tree with 8 vertices is shown in Figure 4.

Theorem 2.6. If T is a T_p-tree on m vertices with joining each vertex of T to the last vertex of QS_n by an edge, then $es(T\widetilde{O}QS_n) = \lceil \frac{4mn+2m}{2} \rceil$.

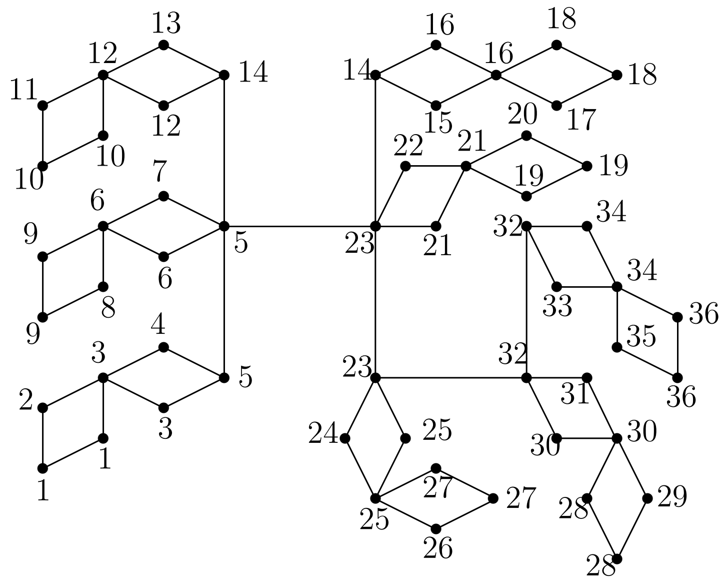

FIGURE 4

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d)\bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j, u_{n+1}^j (1 \le j \le m)$ be the vertices of j^{th} copy of QS_n. Then $V(T\widetilde{O}QS_n) = \{v_j, u_i^j : 1 \le i \le n+1, 1 \le j \le m\}\bigcup\{x_i^j, y_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(T\widetilde{O}QS_n) = E(T)\bigcup E(QS_n)\bigcup\{v_j u_{n+1}^j : 1 \le j \le m\}$. We note that $\left|V(T\widetilde{O}QS_n)\right| = m(3n+2)$ and $\left|E(T\widetilde{O}QS_n)\right| = 4mn + 2m - 1$.

By Theorem (1.1), we have $es(T\widetilde{O}QS_n) \ge \left\lceil \frac{4mn+2m}{2} \right\rceil$. For the reverse inequality, it is enough to show that $es(T\widetilde{O}QS_n) \le \left\lceil \frac{4mn+2m}{2} \right\rceil$. Define $\phi : V(T\widetilde{O}QS_n) \to \left\{1, 2, 3, \ldots, \left\lceil \frac{4mn+2m}{2} \right\rceil\right\}$ as follows:

$$\phi(v_j) = \begin{cases} (2n+1)j & \text{if } j \text{ is odd and } 1 \le j \le m \\ (2n+1)(j-1)+1 & \text{if } j \text{ is even and } 1 \le j \le m \; . \end{cases}$$

For $1 \le i \le n+1$,

$$\phi(u_i^j) = \begin{cases} (2n+1)(j-1)+2(i-1)+1 & \text{if } j \text{ is odd and } 1 \leq j \leq m \\ (2n+1)j-2(i-1) & \text{if } j \text{ is even and } 1 \leq j \leq m\ . \end{cases}$$

For $1 \leq i \leq n$,

$$\phi(x_i^j) = \begin{cases} (2n+1)(j-1)-2i & \text{if } j \text{ is odd and } 1 \leq j \leq m \\ (2n+1)j-2i+2 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$\phi(y_i^j) = \begin{cases} (2n+1)j+2(i-1)+1 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (2n+1)j-2i+1 & \text{if } j \text{ is even and } 1 \leq j \leq m. \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.

The weight of the edge v_iv_j is given by

$$\begin{aligned} wt(v_iv_j) &= wt(v_iv_{i+2t+1}) \\ &= \phi(v_i) + \phi(v_{i+2t+1}) \\ &= (4n+2)(i+t)+1. \end{aligned}$$

The weight of edge $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} wt(v_{i+t}v_{j-t}) &= wt(v_{i+t}v_{i+t+1}) \\ &= \phi(v_{i+t}) + \phi(v_{i+t+1}) \\ &= (4n+2)(i+t)+1. \end{aligned}$$

Therefore, $wt(v_iv_j) = wt(v_{i+t}v_{j-t})$.

The edge weights are as follows:

$$wt(v_jv_{j+1}) = (4n+2)j+1,\ 1 \leq j \leq m-1;$$

For $1 \leq i \leq n$,

$$wt(u_i^jx_i^j) = \begin{cases} (4n+2)(j-1)+4i-1 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+2)j-4i+4 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(u_i^j y_i^j) = \begin{cases} (4n+2)(j-1)+4i-2 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+2)j-4i+3 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(x_i^j u_{i+1}^j) = \begin{cases} (4n+2)(j-1)+4i+1 & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+2)j-4i+2 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(y_i^j u_{i+1}^j) = \begin{cases} (4n+2)(j-1)+4i & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+2)j-4i+1 & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

$$wt(v_j u_{n+1}^j) = \begin{cases} (4n+2)j & \text{if } j \text{ is odd and } 1 \leq j \leq m, \\ (4n+2)j-4n & \text{if } j \text{ is even and } 1 \leq j \leq m; \end{cases}$$

Thus the edge weights are distinct. Hence $es(T\widetilde{O}QS_n)) = \lceil \frac{4mn+2m}{2} \rceil$. □

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