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NEW FIXED-POINT THEOREMS ON PARTIALLY E-CONE METRIC SPACES

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Abstract. In this paper, we extend the definition of partial metric space to a partially E-cone metric space partially ordered with a non-normal positive cone E^+ of a real normed space E having empty interior. We prove an extension of some fixed point theorems of certain contractive maps in partially E-cone metric space to a larger class of cone metric spaces. Moreover, we establish some convergence properties of a sequence of elements in the sense partially E-cone metric space.

1. Introduction

Banach, 1922, [13], presented a method for finding the fixed point of a self operator in complete metric spaces in a systematic manner. Later on, a lot of work on variants and generalizations was published to improve the Banach Contraction Principle by modifying the topology of the space or acting on the contraction requirement,[1]-[14],[29]-[37], and references therein.

Over the past decades, nonlinear functional analysis, especially fixed point theory in ordered normed spaces, had covered a large number of applications in optimization theory, game theory, dynamical systems, fractals, models in economy, computer science and many other fields. Among them a partial ordering, is given by utilizing vector cones crude estimates via a norm by substituting an ordered Banach space instead of the real line, see [30]-[38]. In 2007, Huang and Zhang, [2] introduced the

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notion of cone metric spaces and afterwards was characterized to what is called E-normed metric spaces by Al-Rawashdeh-Shatanawi in [6]. Many mathematicians followed Huang's lead and focused on fixed point problems in such spaces (see, [[2]-[25]]) . Most fixed point issues in cone metric spaces are embedded in solid cones, which are cones with non empty interior. Unfortunately there were just a few results that took non-solid cones into account. Fortunately in 2017 Basile et al. [15] established the concept of the semi-interior point which took fixed point results in E-metric spaces into consideration by embedding non-solid cones that contain semi-interior points in E-metric spaces. Embedding such cones in the setting of $E-$metric spaces, Mehmood et al. [28] and Huang [21], obtained some fixed theorems in 2019.

In this paper we define the concept of partially E-cone metric space and prove some fixed point results with reference to a class of cones in normed spaces, i.e. cones with semi-interior points generalizing some of the existence results in [6, 28]. Our results would constitute a base to develop the theory of non-solid cones in this direction in mathematical analysis. The following definitions and results will be needed in this paper.

Definition 1.1. [23] *An ordered space E is a vector space over the real numbers, with a partial order relation "$\preceq$" such that*

(1) *for all x, y and $z \in E$, $x \preceq z$ implies $x + y \preceq z + y$.*
(2) *for all $\alpha \in \mathbb{R}^+$ and for all $x \in E$ with $x \succeq 0_E$, $\alpha x \succeq 0_E$.*

Moreover, if E is equipped with a norm $\|.\|$, then E is called normed ordered space.

Definition 1.2. [21]. *Let E be a real normed space, E^+ be a non-empty closed and convex subset of E, and 0_E be a zero element in E. Then E^+ is called a positive cone if it satisfies*

(1) *$x \in E^+$ and $a \geq 0$ imply $ax \in E^+$;*
(2) *$x \in E^+$ and $-x \in E^+$ imply $x = 0_E$.*

Definition 1.3. [23] *Let E be a real normed space and E^+ a positive cone in E. We say $\preceq$ is a partial ordering relation on E if*

$$x, y \in E,\ x \preceq y \text{ if and only if } y - x \in E^+.$$

Clearly,

$$x \in E^+ \text{ if and only if } 0_E \preceq x.$$

Definition 1.4. [28] *Let E be a real normed space and E^+ a positive cone in E. Then E^+ is called:*

(1) *a solid cone if $intE^+ \neq \varnothing$;*
(2) *a normal cone if there exists an $K > 0$ such that*

$$0_E \preceq x \preceq y \quad \text{imply} \quad \|x\| \leq K\|y\|, \quad \text{for all } x, y \in E.$$

The least positive number satisfying the above is called the normal constant of E^+.

Definition 1.5. [17] *The cone E^+ is called regular if every increasing sequence which is bounded from above is convergent.*

That is, if $\{y_n\}_{n\geq 1}$ is sequence such that

$$y_1 \preceq y_2 \preceq ... \preceq y_n \preceq ... \preceq x, \text{ for some } x \in E^+,$$

then there is $y \in E^+$ such that

$$\|y_n - y\| \to 0 \ \ (n \to \infty).$$

Equivalently, the cone E^+ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

The following definition of an E-metric space defined in [6].

Definition 1.6. [6]. *Let X be a non-empty set and let E be an ordered space over the real scalars. An ordered E-metric on X is an E-valued function $d^E : X \times X \to E$ such that for all x, y and $z \in X$, we have*

(1) $0_E \preceq d^E(x, y)$, $d^E(x, y) = 0_E$ *if and only if* $x = y$;
(2) $d^E(x, y) = d^E(y, x)$;
(3) $d^E(x, y) \preceq d^E(x, z) + d^E(z, y)$.

Then the pair d^E (X, d) is called $E-$metric space.

Example 1.7. [7] *Let* $E = \mathbb{R}^2$, $E^+ = \{(x,y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ *and* $d : X \times X \to E$ *be defined by* $d^E(x,y) = (|x-y|, \alpha|x-y|)$, *where* $\alpha \geq 0$ *is a constant. Then* (X, d^E) *is an* E*-cone metric space.*

2. Partially E-Cone Metric Space

Let E be an ordered normed space ordered by the positive cone E^+, we shall denote by 0_E the zero of E. We say that

$$\text{The closed unit ball of } E \text{ is } B = \{x \in E : \|x\| \leq 1\},$$

and that

$$\text{the positive part of } B \text{ is } B_+ = B \cap E^+.$$

The point $x_0 \in E^+$ is called a semi-interior point of E^+ if there exists a real number $\lambda > 0$ such that

$$x_0 - \lambda B_+ \subseteq E^+.$$

Here and thereafter, denote by $(E^+)^{\oslash}$ the set of all semi-interior points of E^+.

Any interior point of E^+ is a semi-interior point. However, the converse is not true as shown by the following (Example 2.5 in [15])

Example 2.1. *Let* $X_n = \mathbb{R}^2$ *ordered point wise and endowed with norm* $\|.\|_n$, *where*

$$||(x,y)||_n = \begin{cases} |x| + |y| & \text{if } xy \geq 0 \\ max\{|x|,|y|\} - \frac{n-1}{n} min\{|x|,|y|\}, & \text{if } xy < 0 \end{cases}$$

It is easy to show, that the unit ball of X_n *is the polygon* D_n *with vertices,* $(-n,n), (-1,0), (0,-1), (n,-n), (1,0), (0,1)$. *Let*

$$E = \left\{ \begin{array}{c} \mathbf{x} = (x_n)_{n\in N}, x_n = (x_n^1, x_n^2) \in X_n \\ and\ \|x_n\|_n \leq m_x, m_x > 0 \ depends\ on\ x \end{array} \right\}.$$

Suppose that E *is ordered by the use of* $E^+ = \{\mathbf{y} = (y_n) \in E : y_n \in \mathbb{R}^2_+ \text{ for any } n\}$ *and normed by* $\|y\|_\infty = \sup_{n\in N} \|y_n\|_n$.

Let $X = E^+ - E^+$ *be the subspace of* E *generated by* E^+ *ordered by* $X^+ = E^+$.

Now if $\mathbf{1} = (y_n) \in X, y_n = (1,1)$ *for every* n, *then* $\mathbf{1}$ *cannot be an interior point of* X^+. *In fact, if for any positive integer* k, *let* $\mathbf{x} = (x_n)$ *of* X *with* $x_m = (-2,2)$ *and* $x_n = (0,0)$ *for any* $n \neq m$. *It is easy to show that* $\|x\|_\infty = \frac{2}{m}$ *and* $\mathbf{1} + \mathbf{x} \notin X^+$.

Therefore, $\mathbf{1} + \lambda B_+ \nsubseteq X^+$*, for any* $\lambda > 0$*. Hence,* $\mathbf{1}$ *is cannot be an interior point of the space* X^+*. Similarly one can show that* $int(X^+) = \phi$ *and the point* $\mathbf{1} = (y_n)$ *is a semi interior point of* X^+*.*

Now, let E be a normed space ordered by its positive cone E^+. For $x, y \in E^+$, $x \lll y$ if and only if $y - x \in (E^+)^{\oslash}$.

It is clear that

$$x \in \left(E^+\right)^{\oslash} \text{ if and only if } 0_E \lll x.$$

We shall give some topological properties relevant to semi-interior points in E-metric spaces.

Proposition 2.2. [21]. *If* $x, y \in E$*. Then* $y \lll x$ *implies* $y \preceq x$*.*

Proposition 2.3. *Let* $x, y, z \in E$*. If* $0 \preceq z$ *and* $x \preceq y - z$*, then* $x \preceq y$*.*

Proof. Let $x, y, z \in E$ and $0 \preceq z$, $x \preceq y - z$, then

$$0 \preceq z, y - z - x \in E^+.$$

Noting that E^+ is a positive cone, it follows that

$$y - x = (y - z - x) + z \in E^+,$$

Thus, $y - x \in E^+$, that is $x \preceq y$. □

Proposition 2.4. *Let* $x, y, z \in E$*. Then* $0 \preceq z$*,* $x \lll y - z$ *implies* $x \lll y$*.*

Proof. Let $x, y, z \in E$ and $x \lll y - z$, then

$$y - z - x \in \left(E^+\right)^{\oslash}.$$

Hence, there exists $\lambda > 0$ such that

$$y - z - x - \lambda B_+ \subseteq E^+.$$

Noting that E^+ is a positive cone and $z - x \in E^+$, it follows that

$$y - x - \lambda B_+ = (y - z - \lambda B_+) + (z - x) \subseteq E^+.$$

Thus, $y - x \in (E^+)^{\oslash}$, that is $x \lll y$. □

We now state the following definition of partially E-cone metric space.

Definition 2.5. *Let $X \neq \phi$ and E be an ordered space over the real scalars ordered by its positive cone with the assumption that $(E^+)^{\oslash}$ is non-empty. A partially E-cone metric on X is a function $p^E : X \times X \to E^+$ such that for all $x, y, z \in X$;*

- **(p$_1$):** $0_E \preceq p^E(x,x) \preceq p^E(x,y)$,
- **(p$_2$):** $x = y \Longleftrightarrow p^E(x,x) = p^E(x,y) = p^E(x,y)$,
- **(p$_3$):** $p^E(x,y) = p^E(y,x)$,
- **(p$_4$):** $p^E(x,y) \preceq p^E(x,z) + p^E(x,y) - p^E(z,z)$.

A partial E-cone metric space is a pair (X, p^E) such that X is non-empty set and p^E is a partially E-cone metric on the set X.

It is obvious that, if $p^E(x,y) = 0_E$, then from (p$_1$) and (p$_2$), $x = y$. But if $x = y$, $p^E(x,y)$ may not be equal to 0_E.

Example 2.6. *Let $E = \mathbb{R}^2$, $E^+ = \{(y,z) \in E : y, z \geq 0\}$, $X = \mathbb{R}^+$ and $p^E : X \times X \to E^+$ defined by*

$$p^E(y,z) = (\max\{y,z\}, a\max\{y,z\}), \text{ where } a \geq 0 \text{ is a constant.}$$

Then (X, p^E) is a partially E-cone metric space.

Now we define the e-convergence and the e-Cauchy convergence criteria in the ordered normed space E, with non-solid cone E^+.

Definition 2.7. *Let E be a ordered normed space with the assumption that $(E^+)^{\oslash}$ is non-empty and (X, p^E) be a partially E-cone metric. Let (x_n) be a sequence in X and $x \in X$. Then*

(i) *A sequence (x_n) is said to be e-converges to x if for every $0_E \lll e$, there exists a natural number n_0 such that*

$$p^E(x_n, x) \lll e, \quad \text{for all } n \geq n_0.$$

In this case, we write $\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{e} x$.

(ii) *A sequence (x_n) is said to be e-Cauchy sequence if for every $0_E \lll e$, there exists a natural number n_0 such that*

$$p^E(x_n, x_m) \lll e, \quad \text{for all } n, m \geq n_0.$$

(iii) $\left(X, p^E\right)$ *is e-complete if every e-Cauchy sequence is e-convergent.*

Theorem 2.8. *Let* $\left(X, p^E\right)$ *be a partially E-cone metric space and* $\{x_n\}$ *a sequence in X satisfying*

$$p^E\left(x_n, x_{n+1}\right) \preceq \lambda p^E\left(x_{n-1}, x_n\right) \quad (n = 1, 2, ...),$$

where $0 \leq \lambda < 1$ *is a constant. Then* $\{x_n\}$ *is an e-Cauchy sequence in X.*

Proof. Suppose that x_n is a contractive sequence in X. Then for some real number $\lambda \in [0, 1)$, we have

$$p^E\left(x_n, x_{n+1}\right) \preceq \lambda p^E\left(x_{n-1}, x_n\right) \preceq \lambda^2 p^E\left(x_{n-2}, x_{n-1}\right) \preceq ... \preceq \lambda^n p^E\left(x_0, x_1\right).$$

For any $n, m \in \mathbb{N}$ using Proposition 2.3, we have

$$\begin{aligned} p^E\left(x_m, x_n\right) &\preceq p^E\left(x_m, x_{m-1}\right) + p^E\left(x_{m-1}, x_{m-2}\right) + ... + p^E\left(x_{n+1}, x_n\right) \\ &\quad - \sum_{r=1}^{m-n-1} p^E\left(x_{m-r}, x_{m-r}\right) \\ &\preceq p^E\left(x_m, x_{m-1}\right) + p^E\left(x_{m-1}, x_{m-2}\right) + ... + p^E\left(x_{n+1}, x_n\right) \\ &\preceq \left(\lambda^{m-1} + \lambda^{m-2} + ... + \lambda^n\right) p^E\left(x_0, x_1\right) \\ &\preceq \lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E\left(x_1, x_0\right). \end{aligned}$$

Let $0_E \lll e$ be given, choose $\rho > 0$ such that $e - \rho B_+ \subseteq E^+$ and a natural number k_1 such that $\lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E\left(x_1, x_0\right) \in \frac{\rho}{2} B_+$ for any $m, n \geq k_1$. Therefore,

$$e - \lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E\left(x_1, x_0\right) - \frac{\rho}{2} B_+ \subseteq e - \rho B_+ \subseteq E^+.$$

Hence,

$$p^E\left(x_m, x_n\right) \preceq \lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E\left(x_1, x_0\right) \lll e, \qquad \text{for all } n, m \geq k_1,$$

which implies (x_n) is an e-Cauchy sequence in X. □

Theorem 2.9. *Let* $\left(X, p^E\right)$ *be a partially E-cone metric space with closed positive cone* E^+ *such that* $\left(E^+\right)^{\oslash} \neq \emptyset$*. If* $(E, \|.\|)$ *is an e-complete space and* $\{x_n\}, \{y_n\}$ *are e-Cauchy sequences in X, then* E^+ *is not normal cone provided that* $\left\{p^E\left(x_n, y_n\right)\right\}$ *is not e-convergent in E.*

Proof. Assume that $\{p^E(x_n, y_n)\}$ is not e-convergent in E and E^+ is a normal cone with the normal constant K. As $\{x_n\}$ and $\{y_n\}$ are e-Cauchy sequences, for $\varepsilon \geqslant 0$ and $e \in (E^+)^{\oslash}$ with $\|e\| < \frac{2\varepsilon}{2K+1}$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$p^E(x_n, x_m) \lll \frac{e}{4}, \text{ for all } n, m > n_1, \tag{2.1}$$

$$p^E(y_n, y_m) \lll \frac{e}{4}, \text{ for all } n, m > n_2. \tag{2.2}$$

Let $n = \max\{n_1, n_2\}$, then for all $n, m > n$,

$$\begin{aligned} p^E(x_n, y_n) &\preceq p^E(x_n, x_m) + p^E(x_m, y_m) + p^E(y_m, y_n) \\ &\quad -p^E(x_m, x_m) - p^E(y_m, y_m) &(2.3)\\ &\lll p^E(x_m, y_m) + \frac{e}{2}, &(2.4) \end{aligned}$$

$$\begin{aligned} p^E(x_m, y_m) &\preceq p^E(x_m, x_n) + p^E(x_n, y_n) + p^E(y_n, y_m) \\ &\quad -p^E(x_n, x_n) - p^E(y_n, y_n) &(2.5)\\ &\lll p^E(x_n, y_n) + \frac{e}{2}. &(2.6) \end{aligned}$$

Combing (2.4) and (2.6), we have

$$\begin{aligned} 0_E &\lll p^E(x_m, y_m) + \frac{e}{2} - p^E(x_n, y_n) \\ &\lll p^E(x_n, y_n) + \frac{e}{2} + \frac{e}{2} - p^E(x_n, y_n) = e. &(2.7) \end{aligned}$$

By applying(2.7), we establish

$$0_E \lll p^E(x_m, y_m) + \frac{e}{2} - p^E(x_n, y_n) \lll e. \tag{2.8}$$

Since E^+ is a normal cone, then it may be verified from (2.8) that

$$\left\| p^E(x_m, y_m) + \frac{e}{2} - p^E(x_n, y_n) \right\| \leq K\|e\|. \tag{2.9}$$

Hence, using (2.9), we obtain

$$\begin{aligned} \left\| p^E(x_m, y_m) - p^E(x_n, y_n) \right\| &\leq \left\| p^E(x_m, y_m) + \frac{e}{2} - p^E(x_n, y_n) \right\| + \left\| \frac{e}{2} \right\| \\ &\leq \left(K + \frac{1}{2} \right) \|e\| < \varepsilon, \end{aligned}$$

which means that $\{p^E(x_n, y_n)\}$ is an e-Cauchy in E. Since $(E, \|.\|)$ is an e-complete, then $\{p^E(x_n, y_n)\}$ is $e-$convergent. This leads to a contradiction with the hypothesis. $\square$

3. Fixed Point Theorems In Partially E-Cone Metric Space

Now we present a generalization of Theorem 1 of [28] as follows:

Theorem 3.1. *Let (X, p^E) be an $e-$complete partially E-cone metric space ordered by its closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. If $T : X \to X$ is a mapping satisfying*

$$p^E(Tx, Ty) \preceq \lambda p^E(x, y), \quad \text{for all } x, y \in X \text{ and some } \lambda \in [0, 1),$$

then T has a unique fixed point in X, and for each $x \in X$, the sequence $(T^n x)_{n\geq 0}$ converges to the fixed point of T.

Proof. For any $x_0 \in X$, consider the iterative sequence

$$x_{n+1} = Tx_n = T^n x_0$$

with $x_n \neq x_{n+1}$ for $n \in \mathbb{N}$. Using Theorem 2.8 we get (x_n) is an e-Cauchy sequence. But X is e-complete so there exists some $x \in X$ such that $x_n \overset{e}{\to} x$. For a given $0_{E,} \lll e$, choose $k \in N$, such that $p^E(x, x_n) \lll \frac{e}{2}$ for all $n \geq k$.

$$\begin{aligned} p^E(x, Tx) &\preceq p^E(x, x_n) + p^E(x_n, Tx) - p^E(x_n, x_n) \\ &\preceq p^E(x, x_n) + p^E(x_n, Tx) \\ &\preceq p^E(x, x_n) + \lambda p^E(x, x_{n-1}) \\ &\preceq p^E(x, x_n) + p^E(x, x_{n-1}) \\ &\lll \frac{e}{2} + \frac{e}{2} = e. \end{aligned}$$

Since $p^E(x, Tx) \lll e$ for any $0_{E,} \lll e$, therefore $e - p^E(x, Tx) \in E^+$ which implies $-p^E(x, Tx) \in E^+$. But $p^E(x, Tx) \in E^+$. Therefore, $p^E(x, Tx) = 0_E$ and hence, $x = Tx$.

To prove that the fixed point x is unique, let $y \in X$ be such that $x \neq y = Ty$, then

$$p^E(x, y) = p^E(Tx, Ty) \preceq \lambda p^E(x, y),$$

which implies $p^E(x,y) = 0_E$. This proves the theorem. □

Corollary 3.2. *Let* (X, p^E) *be an e-complete partially E-cone metric space with closed positive cone* E^+ *such that* $(E^+)^{\oslash} \neq \emptyset$. *For* $0_E \lll e$ *and* $x_0 \in X$, *set* $\mathcal{B}(x_0, e) = \{y \in X : p^E(x_0, y) \lll e\}$. *If* $T : X \to X$ *is a mapping such that*

$$p^E(Tx, Ty) \preceq \lambda p^E(x,y),$$

for all $x, y \in \mathcal{B}(x_0, e)$, *where* $\lambda \in [0,1)$ *is a constant and* $p^E(x_0, Tx_0) \lll (1-\lambda)e$, *then* T *has a unique fixed point in* $\mathcal{B}(x_0, e)$.

Proof. First we show that $\mathcal{B}(x_0, e)$ as an e-complete space. Let $\{x_n\}$ be an e-Cauchy sequence in $\mathcal{B}(x_0, e)$, then $\{x_n\}$ is also e-Cauchy sequence in the given e-complete space X, therefore there exists some $x \in X$ such that $x_n \overset{e}{\to} x$. as $n \to \infty$.

Now we have

$$\begin{aligned} p^E(x, x_0) &\preceq p^E(x, x_n) + p^E(x_n, x_0) - p^E(x_n, x_n). \\ &\preceq p^E(x, x_n) + p^E(x_n, x_0) \\ &\lll e. \end{aligned}$$

Thus, $x \in \mathcal{B}(x_0, e)$.

To complete the proof, we have to show that T is a self mapping on $\mathcal{B}(x_0, e)$. Let $z \in \mathcal{B}(x_0, e)$. Then

$$\begin{aligned} p^E(x, Tz) &\preceq p^E(x_0, Tx_0) + p^E(Tx_0, Tz) - p^E(Tx_0, Tx_0) \\ &\preceq p^E(x_0, Tx_0) + p^E(Tx_0, Tz) \\ &\lll (1-\lambda)e + \lambda e = e. \end{aligned}$$

Using Theorem 3.1, we conclude that T has a unique fixed point in $\mathcal{B}(x_0, e)$. □

Corollary 3.3. *Let* (X, p^E) *be an e-complete partially E-cone metric space ordered by its closed positive cone* E^+ *such that* $(E^+)^{\oslash} \neq \emptyset$. *If for some* $n \in N$, *the mapping* $T : X \to X$ *satisfies*

$$p^E(T^n x, T^n y) \preceq \lambda p^E(x,y), \tag{3.1}$$

for all $x, y \in X$, *where* $\lambda \in [0,1)$ *is a constant, then* T *has a unique fixed point in* X.

Proof. Let $W = T^n$. Then from (3.1), we get

$$p^E(Wx, Wy) \preceq \lambda p^E(x, y), \text{ for all } x, y \in X.$$

So by Theorem 3.1, W has a unique fixed point x_0. But

$$T^n(Tx_0) = T(T^n x_0) = Tx_0.$$

So Tx_0 is also a fixed point of $W = T^n$. Hence $Tx_0 = x_0$ and x_0 is a fixed point of T. Since the fixed of T is also fixed point of T^n, the fixed point of T is unique. □

Next we generalize Theorem 2 in [28] and Theorem 2.6 in [34] as follows:

Theorem 3.4. *Let (X, p^E) be an e-complete partially E-metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. Let $T : X \to X$ be mapping satisfying*

$$p^E(Tx, Ty) \preceq \lambda \left[p^E(Tx, x) + p^E(Ty, y)\right]$$

for all $x, y \in X$ and some $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point in X, and for any $x \in X$, the sequence $(T^n x)_{n \geq 0}$ e-converges to the fixed point of X.

Proof. For any $x_0 \in X$, consider the interactive sequence (x_n) such that

$$x_{n+1} = Tx_n \text{ with } x_n \neq x_{n+1} \text{ for } n \in \mathbb{N}.$$

Then,

$$\begin{aligned} p^E(x_{n+1}, x_n) &= p^E(Tx_n, Tx_{n-1}) \preceq \lambda\left(p^E(Tx_n, x_n) + p^E(Tx_{n-1}, x_{n-1})\right) \\ &\preceq \lambda\left(p^E(x_{n+1}, x_n) + p^E(x_n, x_{n-1})\right). \end{aligned}$$

So,

$$\begin{aligned} p^E(x_{n+1}, x_n) &\preceq \frac{\lambda}{1-\lambda} p^E(x_n, x_{n-1}) = \eta p^E(x_n, x_{n-1}) \\ &\preceq \eta^n p^E(x_1, x_0), \quad \text{where } \eta = \frac{\lambda}{1-\lambda} \in [0, 1). \end{aligned}$$

Now for $n > m$, using the same argument in Theorem 2.8, we obtain

$$p^E(x_n, x_m) \preceq \eta^m \left(\frac{1-\eta^{n-m}}{1-\eta}\right) p^E(x_1, x_0),$$

which implies that (x_n) is an e-Cauchy sequence, as X is e-complete, there exists $x \in X$ such that $x_n \xrightarrow{e} x$. For a given $0_{E,} \lll e$, choose $k \in \mathbb{N}$, such that $p^E(x_{n+1}, x_n) \lll \frac{e(1-\lambda)}{2\lambda}$, and $p^E(x_{n+1}, Tx) \lll \frac{e(1-\lambda)}{2}$ for all $n \geq k$. Then,

$$\begin{aligned} p^E(Tx, x) &\preceq p^E(Tx_n, Tx) + p^E(Tx_n, x) - p^E(x_{n+1}, x_{n+1}) \\ &\preceq p^E(Tx_n, Tx) + p^E(Tx_n, x) \\ &\preceq \lambda\left[p^E(Tx_n, x_n) + p^E(Tx, x)\right] + p^E(x_{n+1}, Tx) \\ &\preceq \frac{1}{1-\lambda}\left[\lambda p^E(x_{n+1}, x_n) + p^E(x_{n+1}, Tx)\right] \\ &\lll e, \quad \text{for all } n \succeq k. \end{aligned}$$

Since, $p^E(x, Tx) \lll e$, therefore, $e - p^E(x, Tx) \in E^+$, which implies $-p^E(x, Tx) \in E^+$. But $p^E(x, Tx) \in E^+$. Hence $p^E(x, Tx) = 0_E$, and $x = Tx$.

To prove uniqueness, let $y \in X$ be such that $x \neq y = Ty$, then

$$p^E(x, y) = p^E(Tx, Ty) \preceq \lambda\left[p^E(Tx, x) + p^E(Ty, y)\right] = 0_E,$$

which implies $p^E(x, y) = 0_E$. This proves the theorem. □

Now we present the generalized versions of the Theorem 3 in [28].

Theorem 3.5. *Let $\left(X, p^E\right)$ be an e-complete partially E-cone metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. Let $T : X \to X$ be mapping satisfying*

$$p^E(Tx, Ty) \preceq \lambda\left[p^E(Tx, y) + p^E(Ty, x)\right]$$

for all $x, y \in X$ and some $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point in X, and for each $x \in X$, the sequence $(T^n x)_{n\geq 0}$ e-converges to the fixed point of T.

Proof. For any $x_0 \in X$, consider the sequence (x_n) such that

$$x_{n+1} = Tx_n \text{ with } x_n \neq x_{n+1} \text{ for } n \in \mathbb{N}.$$

Then,

$$\begin{aligned} p^E(x_{n+1},x_n) &= p^E(Tx_n,Tx_{n-1}) \\ &\preceq \lambda\left(p^E(Tx_n,x_{n-1})+p^E(Tx_{n-1},x_n)\right) \\ &= \lambda\left(p^E(x_{n+1},x_{n-1})+p^E(x_n,x_n)\right) \\ &\preceq \lambda\left(p^E(x_{n+1},x_n)+p^E(x_n,x_{n-1})-p^E(x_n,x_n)+p^E(x_n,x_n)\right) \\ p^E(x_{n+1},x_n) &\preceq \frac{\lambda}{1-\lambda}p^E(x_n,x_{n-1}) \\ &\preceq \left(\frac{\lambda}{1-\lambda}\right)^n p^E(x_1,x_0). \end{aligned}$$

For $\delta = \frac{\lambda}{1-\lambda} \in [0,1)$, following a similar argument in Theorem 3.4, it is easy to see T has a fixed point in X, and for each $x \in X$, the iterative sequence $(T^n x)_{n\geq 0}$ converges to the fixed point of T.

To prove uniqueness, let $x, y \in X$ be two fixed points of T such that $x \neq y$. Then,

$$\begin{aligned} p^E(x,y) &= p^E(Tx,Ty) \preceq \lambda\left(p^E(Tx,y)+p^E(Ty,x)\right) \\ &\preceq \lambda\begin{pmatrix} p^E(Tx,x)+p^E(x,y)-p^E(x,x) \\ +p^E(Ty,y)+p^E(y,x)-p^E(y,y) \end{pmatrix} \\ &\preceq 2\lambda p^E(x,y), \quad \text{for } 2\lambda \in [0,1), \end{aligned}$$

which implies $p^E(x,y) = 0_E$. This proves the theorem. □

Following Reich type contraction mapping [33], we will prove another fixed point theorem in partially E-cone metric space.

Theorem 3.6. *Let $\left(X, p^E\right)$ be an e-complete partially E-cone metric space ordered by its closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. If $T : X \to X$ is a mapping satisfying*

$$p^E(Tx,Ty) \preceq \alpha_1 p^E(Tx,x) + \alpha_2 p^E(Ty,y) + \alpha_3 p^E(x,y),$$

for all $x, y \in X$, where $0 \leq \alpha_1+\alpha_2+\alpha_3 < 1$ and $\alpha_1, \alpha_2, \alpha_3 \geq 0$, then T has a unique fixed point in X, and for each $x \in X$, the sequence $(T^n x)_{n\geq 0}$ e-converges to the unique fixed point of T.

Proof. Choose $x_0 \in X$. Define (x_n) as

$$x_{n+1} = Tx_n = T^{n+1}x_0.$$

Then,

$$\begin{aligned} p^E(x_{n+1}, x_n) &= p^E(Tx_n, Tx_{n-1}) \\ &\preceq \alpha_1 p^E(Tx_n, x_n) + \alpha_2 p^E(Tx_{n-1}, x_{n-1}) + \alpha_3 p^E(x_n, x_{n-1}) \\ &\preceq \alpha_1 p^E(x_{n+1}, x_n) + \alpha_2 p^E(x_n, x_{n-1}) + \alpha_3 p^E(x_n, x_{n-1}), \end{aligned}$$

which implies that

$$p^E(x_{n+1}, x_n) \preceq \frac{\alpha_2 + \alpha_3}{1 - \alpha_1} p^E(x_n, x_{n-1}) = \gamma p^E(x_n, x_{n-1}),$$

where $\gamma = \frac{\alpha_2+\alpha_3}{1-\alpha_1} < 1$.

For $n > m$,

$$\begin{aligned} p^E(x_m, x_n) &\preceq p^E(x_m, x_{m+1}) + p^E(x_{m+1}, x_{m+2}) + ... + p^E(x_{n-1}, x_n) \\ &\quad - \sum_{r=1}^{n-m-1} p^E(x_{m+r}, x_{m+r}) \\ &\preceq p^E(x_m, x_{m+1}) + p^E(x_{m+1}, x_{m+2}) + ... + p^E(x_{n-1}, x_n) \\ &\preceq \left(\gamma^m + \gamma^{m+1} + ... + \gamma^{n+m-1}\right) p^E(x_1, x_0) \\ &\preceq \gamma^m \left(1 + \gamma + \gamma^2 + ... + \gamma^{n-m-1}\right) p^E(x_1, x_0) \\ &\preceq \gamma^m \left(\frac{1 - \gamma^{n-m}}{1 - \gamma}\right) p^E(x_1, x_0). \end{aligned}$$

Let $e \ggg 0$ be given, choose $\rho > 0$ such that $e - \rho B_+ \subseteq E^+$ and a natural number $k_1 \in \mathbb{N}$ such that $\gamma^m \left(\frac{1-\gamma^{n-m}}{1-\gamma}\right) p^E(x_1, x_0) \in \frac{\rho}{2} B_+$ for any $m, n \geq k_1$. Therefore,

$$e - \gamma^m \left(\frac{1 - \gamma^{n-m}}{1 - \gamma}\right) p^E(x_1, x_0) - \frac{\rho}{2} B_+ \subseteq e - \rho B_+ \subseteq E^+, \quad \text{for all } n, m \geq k_1.$$

Thus,

$$p^E(x_m, x_n) \preceq \gamma^m \left(\frac{1 - \gamma^{n-m}}{1 - \gamma}\right) p^E(x_1, x_0) \lll e, \quad \text{for all } n, m \geq k_1,$$

which implies that (x_n) is an e-Cauchy sequence, since X is e-complete so there exists some $x \in X$ such that $x_n \xrightarrow{e} x$.

For a given $e \ggg 0_E$, choose $k_2 \in \mathbb{N}$, such that $p^E(x_{n+1}, x_n) \lll \frac{(1-\alpha_2)e}{3\alpha_1}$, $p^E(x_n, x) \lll \frac{(1-\alpha_2)e}{3\alpha_3}$ and $p^E(x_{n+1}, x) \lll \frac{e}{3}$ for all $n \geq k_2$. Then,

$$\begin{aligned} p^E(Tx, x) &\preceq p^E(Tx_n, Tx) + p^E(Tx_n, x) - p^E(Tx_n, Tx_n) \\ &\preceq p^E(Tx_n, Tx) + p^E(Tx_n, x) \\ &\preceq \alpha_1 p^E(Tx_n, x_n) + \alpha_2 p^E(Tx, x) + \alpha_3 p^E(x_n, x) + p^E(x_{n+1}, x) \\ &\preceq \alpha_1 p^E(x_{n+1}, x_n) + \alpha_2 p^E(Tx, x) + \alpha_3 p^E(x_n, x) + p^E(x_{n+1}, x). \end{aligned}$$

Hence,

$$\begin{aligned} p^E(Tx, x) &\preceq \frac{1}{1-\alpha_2}\left(\alpha_1 p^E(x_{n+1}, x_n) + \alpha_3 p^E(x_n, x) + p^E(x_{n+1}, x)\right) \\ &\lll \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = e, \text{ for all } n \geq k_2. \end{aligned}$$

Thus, $p^E(x, Tx) \lll e$ for any $e \ggg 0_E$. Therefore $e - p^E(x, Tx) \in E^+$ which implies $-p^E(x, Tx) \in E^+$. Since $p^E(x, Tx) \in E^+$, it follows that $p^E(Tx, x) = 0_E$ and hence x is a fixed point of T.

To prove uniqueness, let y be another fixed point of T such that $x \neq y = Ty$ and $0 \leq \alpha_1 + \alpha_2 + \alpha_3 < 1$. Then,

$$\begin{aligned} p^E(x, y) &= p^E(Tx, Ty) \\ &\preceq \alpha_1 p^E(Tx, x) + \alpha_2 p^E(Ty, y) + \alpha_3 p^E(Tx, y) \\ &= \alpha_3 p^E(x, y), \end{aligned}$$

which implies that $p^E(x, y) = 0_E$ and hence $x = y$. □

Theorem 3.7. *Let (X, p^E) be an e-complete partially E-cone metric space ordered by its closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. If $T : X \to X$ is a mapping satisfying*

$$p^E(Tx, Ty) \preceq \lambda \max\left\{p^E(x, y), p^E(x, Tx), p^E(y, Ty)\right\} \tag{3.2}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$, then, T has a unique fixed point $x \in X$ and $p^E(Tx, x) = 0_E$.

Proof. For the existence of fixed point, let $x_0 \in X$ be arbitrary and define a sequence (x_n) by

$$x_{n+1} = Tx_n \text{ for all } n \geq 0.$$

Now, for any n we obtain from (3.2) that

$$\begin{aligned} p^E(x_{n+1}, x_n) &= p^E(Tx_n, Tx_{n-1}) \\ &\preceq \lambda \max\left\{p^E(x_n, x_{n-1}), p^E(x_n, Tx_n), p^E(x_{n-1}, Tx_{n-1})\right\} \\ &= \lambda \max\left\{p^E(x_n, x_{n-1}), p^E(x_n, x_{n+1}), p^E(x_{n-1}, x_n)\right\} \\ &= \lambda \max\left\{p^E(x_n, x_{n-1}), p^E(x_n, x_{n+1})\right\}. \end{aligned}$$

If $\max\left\{p^E(x_n, x_{n-1}), p^E(x_n, x_{n+1})\right\} = p^E(x_n, x_{n+1})$, then we obtain

$$p^E(x_{n+1}, x_n) \preceq \lambda p^E(x_n, x_{n+1}) \preceq p^E(x_{n+1}, x_n),$$

which is a contradiction. Therefore, we must have

$$\max\left\{p^E(x_n, x_{n-1}), p^E(x_n, x_{n+1})\right\} = p^E(x_n, x_{n-1}).$$

Consequently,

$$p^E(x_{n+1}, x_n) \preceq \lambda p^E(x_n, x_{n-1}).$$

Following the argument in Theorem 3.1 It is easy to see that T has a fixed point in X, and for each $x \in X$, the sequence $(T^n x)_{n\geq 0}$ e-converges to the fixed point of T.

To prove uniqueness of the fixed point, let $x, y \in X$ be two fixed points of T such that $x \neq y$. Then,

$$\begin{aligned} p^E(x, y) &= p^E(Tx, Ty) \preceq \lambda \max\left\{p^E(x, y), p^E(x, Tx), p^E(y, Ty)\right\} \\ &= \lambda \max\left\{p^E(x, y), p^E(x, x), p^E(y, y)\right\} \\ &= \lambda p^E(x, y). \end{aligned}$$

This is also a contradiction. Therefore, we must have $p^E(x, y) = 0_E$, that is, $x = y$. This proves the theorem. □

4. Conclusion

Some additional properties of partially E-cone metric space have been established in this paper. We have generalized some more fixed theorems due to Kannan, Chatterjea and Reich in partially E-cone metric space with non solid and non-normal cones. However, these results have vast potential in solving various nonlinear problems in functional analysis, integral and differential equations, computer science and many other fields.

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