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SOME RESULTS ON APPROXIMATIONS USING BOAS TRANSFORM OF WAVELETS

LEENA KATHURIA (1), SHASHANK GOEL(2), NIKHIL KHANNA (3) AND S. K. KAUSHIK (4)

Abstract. The present paper aims to approximate functions belonging to the class $L^2(\mathbb{R})$, using Boas transform of wavelets. We exhibit that Hölder continuity of a function plays a crucial role in the decay of wavelet coefficients and thus assists in approximating it. Further, we give sufficient conditions for uniform approximation of wavelet coefficients of square integrable function.

1. Introduction

Over the past two decades, the theory of wavelets has developed itself as one of the most efficient mathematical instruments for the reach of signal-processing applications, such as compression of data and images, transient detection, reduction of noise, texture analysis, identification of patterns and detection of singularities. In 1984, the notion of wavelet was proposed by Grossmann and Morlet [7]. However, the credit for the construction of multiresolution analysis (MRA), an analysis tool to study a signal from a coarser approximation to a higher resolution approximation, goes to both Mallat [20] and Meyer [21]. Their combined efforts was the reason behind the rise of wavelets to great heights. In wavelet analysis, a varying window called the mother wavelet is considered. Translated and dilated forms of mother wavelet are used to generate other wavelets which creates the foundation of wavelet analysis. A detailed study of the topic can be found in [3, 8].

Wavelets are of fundamental importance in the field of nonlinear approximation theory. The nonlinear approximation has a rich history, dating back to Schmidt's

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work [22] in 1907, and has been studied in various contexts. DeVore [5] published a comprehensive analysis that examines the advantages of nonlinear approximation over linear approximation. This supremacy of nonlinear approximation can be examined in terms of the rate of decay of the approximation error with regard to the number of terms in the approximate representation. In the early 1960s, French mathematician Céa realized that error estimation of finite elements is nothing more than an approximation problem in Sobolev spaces. Approximation of an arbitrary function by wavelet polynomials is a recent advancement in approximation theory. In this context, Islam et al. [11] discussed linear and nonlinear approximation of a function by Haar wavelet in different smoothness spaces. There are several types of wavelets such as Haar wavelet, Mexican-Hat wavelet, Shannon wavelet, Daubechies wavelet, Meyer wavelet and so forth. For further details on wavelets, see [4].

Coifman et al. [2] proposed wavelet packets to ameliorate the poor frequency localization of high frequency wavelet basis and so delivered a more appropriate decomposition of signals (or functions) incorporating both temporal and stationary components. The viability of wavelet packets gives the freedom in determining the suitable basis function for representing a given function. For more details on wavelet packets, one may refer to [8, 17].

Some approximation properties of wavelet bases plays a crucial role in analysis. The analysis of the decay of approximation error on the basis of resolution is an important concern in wavelet theory. In this context, Khanna et al. [16] approximated functions in $L^2(\mathbb{R})$ using Hilbert transform of wavelets. In 2016, Khanna et al. [17] proposed the orthogonal Coifman wavelet packet systems and biorthogonal Coifman wavelet packet systems with the vanishing moments distributed equally between the scaling function and the wavelet packet functions and thereby gave wavelet packet approximation theorem. These systems have good approximation properties with exponential decay. The wavelet packet approximation theorem illustrates the different roles of the vanishing moments of the wavelet packet functions.

The Boas transform of a function $f \in L^2(\mathbb{R})$, denoted by $\mathfrak{B}f$, in terms of principal value integral is defined as

$$\mathfrak{B}f(x) = \frac{1}{\pi} P \int_0^{\infty} \frac{f(x+z) - f(x-z)}{z^2} \sin(z)\, dz = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x+z)}{z^2} \sin(z)\, dz$$

for any x for which the integral exists.

The relationship between the Boas transform (BT) and the Hilbert transform (HT) is given by

$$(\mathfrak{B}f)(x) = (\mathfrak{H}f)(x) - \{\mathfrak{H}f * \mathfrak{g}\}(x), \tag{1.1}$$

where

$$\mathfrak{g}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{1-\cos(x)}{x^2}\right),$$

and $\mathfrak{H}f(\cdot)$ denotes the Hilbert transform of f .

The following equivalence specifies the Hilbert transform on $L^2(\mathbb{R})$,

$$\mathfrak{H}f(x) \overset{\mathfrak{F}}{\longleftrightarrow} -i\ \mathrm{sgn}(\gamma)\hat{f}(\gamma), \tag{1.2}$$

where the signum function $\mathrm{sgn}(\gamma)$ is defined by $\gamma/|\gamma|$, for $\gamma \neq 0$ and takes the value zero for $\gamma = 0$.

Taking Fourier transform on both sides of (1.1), we have

$$\widehat{\mathfrak{B}f}(\gamma) = \widehat{\mathfrak{H}f}(x) - \mathfrak{F}\{\mathfrak{H}f * \mathfrak{g}\}(\gamma), \tag{1.3}$$

where $\hat{f}(\cdot)$ (or $\mathfrak{F}f(\cdot)$) denotes the Fourier transform of f.

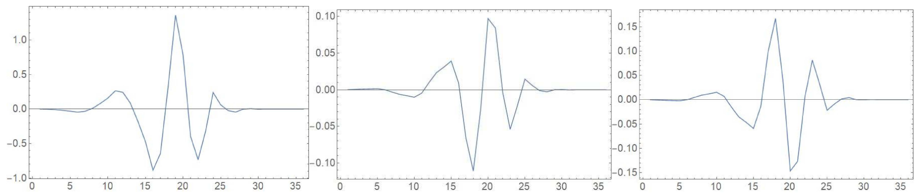
If $\mathfrak{H}f(x) \in L^1(\mathbb{R})$, then using (1.1), (1.2) and (1.3), we get

$$\widehat{\mathfrak{B}f}(\gamma) = -i\,\mathrm{sgn}(\gamma)\hat{f}(\gamma)(1 - \hat{\mathfrak{g}}(\gamma)),$$

where

$$\hat{\mathfrak{g}}(\gamma) = \begin{cases} 0, & \text{if } |\gamma| > 1, \\ 1 - |\gamma|, & \text{if } |\gamma| \leq 1. \end{cases}$$

Boas [1] characterized the Boas transform by analyzing functions whose Fourier transform vanishes on a finite interval. A significant contribution in this field can be seen in the work of Goldberg [6], Heywood [9] and Zaidi [24]. For further details on Boas transforms, one may read [25].



(A) Daubechies wavelet

(B) HT of Daubechies wavelet

(C) BT of Daubechies wavelet

FIGURE 1. Daubechies wavelet and its Hilbert transform and Boas transform

In [15], Khanna et al. studied Boas transform of wavelets and attained various results related to their vanishing moments. In Figure 1, one may observe the transition in the Daubechies wavelet when two different transforms are applied which is actually one of the reason for exploring the properties of Boas transform of wavelets in this work. Recently, Khanna et al. [18] proposed fractional Boas transforms and the associated wavelets which are more efficient than the Boas transforms of wavelets due to an additional degree of freedom in context of fractional order. Recently, parameter (p, q)-Boas transform are introduced in [14], where parameter (p, q)-Boas transform of a signal in linear canonical transform domain is studied. Further, the recent introduction of Fourier-Boas-Like wavelets [12] has improved the inability of original wavelets to study both the symmetries of an asymmetric signal. Very recently, Zothansanga et al. [19] introduced some new generalized wavelets based on the Hartley kernel and Boas transforms.

This paper's intent is to approximate functions lying in the class $L^2(\mathbb{R})$, using Boas transform of wavelets, where vanishing moments play a vital role in approximating smooth functions in $L^2(\mathbb{R})$. We employ Hölder continuity of a function in order to reduce the number of wavelet coefficients, generated by Boas transform of wavelets. Finally, we obtain sufficient conditions for uniform approximation of wavelet coefficients of square integrable function based on modulus of continuity and boundedness of higher derivatives.

2. Main Results

Khanna et al. [15] presented Boas transform of wavelets and delivered various results associated with their vanishing moments. Later, Khanna and Kathuria [13] discussed the convolution of the wavelets introduced in [15] to analyze the Boas transform of convolution of signals. Further, Khanna et al. [16] furnished various results to approximate the functions in the $L^2(\mathbb{R})$ space. In [15], sufficient conditions for the higher order vanishing moments of Boas transform of wavelets are obtained. The result is given below.

Theorem 2.1. Let $\psi, \psi^{(1)}, \hat{\psi} \in L^1(\mathbb{R})$ be such that for some $n \in \mathbb{N}$, $x^{n-1}\psi(x)$, $\gamma^n\hat{\psi}(\gamma) \in L^1(\mathbb{R}), x^n\psi(x) \in L^2(\mathbb{R})$, and $\int_{\mathbb{R}} x^q\, G(x)\, dx = 0$, for $0 \le q \le n$, where $G(x) = \displaystyle\int_{-1}^{1} \left(1 - \frac{1}{|\gamma|}\right) e^{-2\pi i\gamma x}\, \widehat{\psi^{(1)}}(-\gamma)\, d\gamma$. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthogonal system on $\mathbb{R}$, then

$$\int_{\mathbb{R}} x^q\, \mathfrak{B}\psi(x)\, dx = 0, \quad for\ 0 \le q \le n.$$

In [16], the authors proved that wavelet coefficients of a square integrable function decay fast as $j \to +\infty$ subject to the number of vanishing moments and smoothness of f. The result is stated below.

Theorem 2.2. Given $p \in \mathbb{N}$, suppose that the function $f \in L^2(\mathbb{R})$ is C^p on $\mathbb{R}$ and that $f^{(p)} \in L^\infty(\mathbb{R})$. Let $\psi \in L^2(\mathbb{R})$ be a function with compact support such that

$$x^{p-1}\psi(x) \in L^2(\mathbb{R}),$$

and

$$\int_{\mathbb{R}} x^p\, \psi(x)\, dx = 0 \quad 0 \le m \le p-2.$$

Then, there exists a constant $K > 0$ depending on p and $f(x)$ such that for every $j, k \in \mathbb{Z}$, $|\langle f, \mathfrak{H}\psi_{j,k}\rangle| \le K\, 2^{-j(p+\frac{1}{2})}$, where $\mathfrak{H}\psi_{j,k}$ denotes the Hilbert Transform of wavelet $\psi_{j,k}$ given by $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, where $j, k \in \mathbb{Z}$.

Now, we recall from [18] the notion of G-function of order n.

Definition 2.1. Let f be a function such that f, $f^{(1)}$, $\hat{f} \in L^1(\mathbb{R})$.Then, f is said to be G-function of order n if $\int_{\mathbb{R}} x^q\, G(x)\, dx = 0$ for $0 \le q \le n$, where $G(x) = \displaystyle\int_{-1}^{1} \left(1 - \frac{1}{|\gamma|}\right) e^{-2\pi i\gamma x}\, \widehat{\psi^{(1)}}(-\gamma)\, d\gamma$.

Further, recall from [16] that the moment formula for Hilbert transform is given by

$$\mathfrak{H}\{x^s f(x)\} = x^s \mathfrak{H} f(x) - \frac{1}{\pi}\sum_{m=0}^{s-1} x^m \int_{\mathbb{R}} q^{s-1-m} f(q)\, dq, \quad s \geq 0. \tag{2.1}$$

Note that the above formula holds if $x^s f(x) \in L^q(\mathbb{R}), 1 < q < \infty$.

Next result generalizes Theorem 2.2 for Boas transform of wavelets.

Theorem 2.3. Let $f \in L^2(\mathbb{R}) \cap C^p(\mathbb{R})$ $(p \in \mathbb{N})$ be a function such that $f^{(p)} \in L^\infty(\mathbb{R})$. Let ψ be a wavelet with compact support such that

(i) $x^{p-1}\, \psi(x) \in L^2(\mathbb{R})$,
(ii) $\int_{\mathbb{R}} x^l\, \psi(x)\, dx = 0$ for $l = 0, 1, ..., p-1$,
(iii) ψ is a G-function of order p

Then there exists a constant $M > 0$ depending on p and $f(x)$ such that for every $j, k \in \mathbb{Z}$,

$$|\langle f, \mathfrak{B}\psi_{j,k}\rangle| \leq M\, 2^{-j(p+\frac{1}{2})},$$

where $\mathfrak{B}\psi_{j,k}$ denotes the Boas transform of wavelet $\psi_{j,k}(x) = 2^{\frac{j}{2}}\, \psi(2^j x - k), j, k \in \mathbb{Z}$.

Proof. Let ψ be supported in the interval $I = I_{0,0} = [0, b]$ for $b > 0$. Then $\psi_{j,k}(x) = 2^{\frac{j}{2}}\, \psi(2^j x - k)$ will be supported in the interval $I_{j,k} = [2^{-j}k, 2^{-j}(k+b)]$ of length $2^{-j}b$ denoted by $|I_{j,k}|$. Let us denote the center of $I_{j,k}$ by $b_{j,k} = 2^{-(j+1)}b + 2^{-j}k$. Since $f \in C(\mathbb{R})$, it follows that for each $j, k \in \mathbb{Z}$, $f(x)$ can be represented by its Taylor polynomial plus a remainder term. That is

$$\begin{aligned} f(x) &= f(b_{j,k}) + (x - b_{j,k}) f^{(1)}(b_{j,k}) + \dots \\ &+ \frac{1}{(p-1)!}(x - b_{j,k})^{p-1} f^{(p-1)}(b_{j,k}) + R(x), \end{aligned} \tag{2.2}$$

where $R(x) = \frac{1}{p!}(x - b_{j,k})^p f^{(p)}(z)$ for some z between $b_{j,k}$ and x.
Note that

$$\begin{aligned}
\langle f, \mathfrak{B}\psi_{j,k}\rangle &= \int_{\mathbb{R}} f(x)\, \mathfrak{H}\psi_{j,k}(x)\, dx - \int_{\mathbb{R}} f(x)\, (\mathfrak{H}\psi_{j,k} * \mathfrak{g})(x)\, dx \\
&= \int_{\mathbb{R}} f(x)\, \mathfrak{H}\psi_{j,k}(x)\, dx - \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} T_{-x}\, \mathfrak{H}\psi(-t)\, \mathfrak{g}(t)\, dt\, dx \\
&= \int_{\mathbb{R}} f(x)\, \mathfrak{H}\psi_{j,k}(x)\, dx - \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \widehat{\mathfrak{H}T_{-x}\psi}(-\gamma)\, \hat{\mathfrak{g}}(\gamma)\, d\gamma\, dx \\
&= \int_{\mathbb{R}} f(x)\, \mathfrak{H}\psi_{j,k}(x)\, dx + i \int_{\mathbb{R}} f(x) \int_{-1}^{1} sgn(-\gamma)\, \widehat{T_{-x}\psi}(-\gamma)(1 - |\gamma|)\, d\gamma\, dx.
\end{aligned}$$

Since ψ, $\psi^{(1)}$, $\hat{\psi} \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
\langle f, \mathfrak{B}\psi_{j,k}\rangle &= \int_{\mathbb{R}} f(x)\, \mathfrak{H}\psi_{j,k}(x)\, dx - \frac{1}{2\pi} \int_{\mathbb{R}} f(x) \int_{-1}^{1} \left(1 - \frac{1}{|\gamma|}\right) e^{-2\pi i \gamma x}\, \widehat{\psi^{(1)}}(-\gamma)\, d\gamma\, dx \\
&= \int_{\mathbb{R}} f(x)\, (\mathfrak{H}\psi_{j,k}(x) - \frac{1}{2\pi} G(x))\, dx. \qquad (2.3)
\end{aligned}$$

Also, since $\psi(x), x^{p-1}\psi(x) \in L^2(\mathbb{R})$, it follows that $x^l\,\psi(x) \in L^2(\mathbb{R})$ for all $l = 0, 1, \ldots, p-1$.
Thus, using the moment formula for Hilbert transforms and (2.3), we have

$$\begin{aligned}
\langle f, \mathfrak{B}\psi_{j,k}\rangle &= \int_{\mathbb{R}} R(x) \left(\mathfrak{H}\psi_{j,k}(x) - \frac{1}{2\pi} G(x)\right) dx \\
&= \int_{\mathbb{R}} R(x) \mathfrak{B}\psi_{j,k}(x)\, dx \\
&= \frac{1}{p!} f^{(p)}(z) \int_{\mathbb{R}} (x - b_{j,k})^p\, \mathfrak{B}\psi_{j,k}(x)\, dx.
\end{aligned}$$

This gives

$$|\langle f, \mathfrak{B}\psi_{j,k}\rangle| = \frac{|f^{(p)}(z)|}{p!} \left| \int_{\mathbb{R}} (x - b_{j,k})^p\, \mathfrak{B}\psi_{j,k}(x)\, dx \right|.$$

Since $\int_{I_{j,k}} |(x-b_{j,k})^p|^2 \, dx < +\infty$, and using the compact support of $\psi_{j,k}$, it follows that

$$\begin{aligned}
|\langle f, \mathfrak{B}\psi_{j,k}\rangle| &= \frac{\left|f^{(p)}(z)\right|}{p!} \left| \int_{\mathbb{R}} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \mathfrak{F}\{\mathfrak{B}\psi_{j,k}(x)\}(\gamma)\, d\gamma \right| \\
&= \frac{\left|f^{(p)}(z)\right|}{p!} \left| \int_{\mathbb{R}} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \hat{\psi}_{j,k}(\gamma)\, \left(1-\hat{\mathfrak{g}}(\gamma)\right)\, d\gamma \right| \\
&= \frac{\left|f^{(p)}(z)\right|}{p!} \left| \int_{|\gamma|>1} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \hat{\psi}_{j,k}(\gamma)\, \left(1-\hat{\mathfrak{g}}(\gamma)\right)\, d\gamma \right. \\
&\qquad \left. + \int_{|\gamma|\leq 1} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \hat{\psi}_{j,k}(\gamma)\, \left(1-\hat{\mathfrak{g}}(\gamma)\right)\, d\gamma \right| \\
&= \frac{\left|f^{(p)}(z)\right|}{p!} \left| \int_{|\gamma|>1} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \hat{\psi}_{j,k}(\gamma)\, d\gamma \right. \\
&\qquad \left. + \int_{|\gamma|\leq 1} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \hat{\psi}_{j,k}(\gamma)\, |\gamma|\, d\gamma \right| \\
&\leq \frac{\left|f^{(p)}(z)\right|}{p!} \left| \int_{\mathbb{R}} \mathfrak{F}\{(x-b_{j,k})^p\}(\gamma)\, \hat{\psi}_{j,k}(\gamma)\, d\gamma \right| \\
&= \frac{\left|f^{(p)}(z)\right|}{p!} \left| \int_{\mathbb{R}} (x-b_{j,k})^p\, \psi_{j,k}(x)\, dx \right| \\
&\leq \frac{1}{p!} \max_{x\in I_{j,k}} \left|f^{(p)}(x)\right| \int_{I_{j,k}} |(x-b_{j,k})^p\, \psi_{j,k}(x)|\, dx \\
&\leq \frac{1}{p!} \max_{x\in I_{j,k}} \left|f^{(p)}(x)\right| \left[\int_{I_{j,k}} |(x-b_{j,k})^p|^2\, dx\right]^{\frac{1}{2}} \left[\int_{I_{j,k}} |\psi_{j,k}(x)|^2\, dx\right]^{\frac{1}{2}}.
\end{aligned}$$

Since $\psi \in L^2(\mathbb{R})$, there exists a constant $K > 0$ such that

$$\left[\int_{I_{j,k}} |\psi_{j,k}(x)|^2\, dx\right]^{\frac{1}{2}} \leq K.$$

Therefore

$$
\begin{aligned}
|\langle f, \mathfrak{B}\psi_{j,k}\rangle| &\leq \frac{K}{p!} \max_{x\in I_{j,k}} \left|f^{(p)}(x)\right| \left[\int_{I_{j,k}} 2^{-2p(j+1)} b^{2p}\, dx\right]^{\frac{1}{2}} \\
&= \frac{K}{p!} \max_{x\in I_{j,k}} \left|f^{(p)}(x)\right| \, 2^{-p(j+1)} b^{p} \, |I_{j,k}|^{\frac{1}{2}} \\
&= \frac{K}{p!} \|f^{(p)}\|_{\infty} \, 2^{-p} b^{\frac{1}{2}+p} \, 2^{-pj} \, 2^{-\frac{j}{2}} \\
&= M 2^{-j(p+\frac{1}{2})},
\end{aligned}
$$

where $M = \frac{K}{p!}\|f^{(p)}\|_{\infty} \, 2^{-p} b^{\frac{1}{2}+p}$.

Thus, the wavelet coefficients of such a function decays rapidly as $j \to +\infty$. □

An important characteristic of wavelet function is its regularity. Regularity is linked with how many continuous derivatives a function has. Possibly, regularity can be considered as a measure of smoothness. Holschneider and Tchamitchian [10] used the wavelet transform to examine the local regularity of functions in general. The Hölder spaces permit us to define a notion of smoothness or regularity for a function and, in particular, they roughly provide an intermediate level between continuity and differentiability. Khanna et al. [16] obtained sufficient conditions to decrease the wavelet coefficients of a function by employing uniform Hölder continuity. The result is stated below.

Theorem 2.4. Let $f \in L^2(\mathbb{R})$ be n-times continuously differentiable function such that $f^{(n)}$ is Hölder continuous with exponent β for $0 < \beta < 1$ and let $\psi \in L^2(\mathbb{R})$ be a wavelet satisfying the following conditions

$$x^{n+1}\, \psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

$$\int_{\mathbb{R}} x^p \, \psi(x)\, dx = 0 \text{ for } p = 0, 1, \ldots.., n+1.$$

Then $|\langle f, \mathfrak{H}\psi_{j,k}\rangle| \leq C\, 2^{-j(n+\beta+\frac{1}{2})}$.

Following result is a generalized version of Theorem 2.4 in terms of Boas transform of wavelets.

Theorem 2.5. Let $f \in C^p$ be such that $f^{(p)}$ is Hölder continuous with exponent $\delta\,(0 < \delta < 1)$ and let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\psi^{(1)}$, $\hat{\psi} \in L^1(\mathbb{R})$ satisfying the following conditions

(i) $x^{p+1}\,\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

(ii) $\int_{\mathbb{R}} x^m\,\psi(x)\,dx = 0$ for $m = 0, 1, \ldots\ldots, p+1$,

(iii) $x^{p+1}G(x) \in L^1(\mathbb{R})$,where $G(x) = \displaystyle\int_{-1}^{1}\left(1 - \frac{1}{|\gamma|}\right) e^{-2\pi i\gamma x}\,\widehat{\psi^{(1)}}(-\gamma)\,d\gamma.$

Then $|\langle f, \mathfrak{B}\psi_{j,k}\rangle| \le K\,2^{-j(p+\delta+\frac{1}{2})}$.

Proof. Since $f \in C^p(\mathbb{R})$, $f(x)$ can be expanded by using Taylor's expansion about the point b as

$$\begin{aligned} f(x) = f(b) + (x-b)f^{(1)}(b) + \frac{(x-b)^2}{2!}f^{(2)}(b) + \ldots \\ + \frac{(x-b)^{p-1}}{(p-1)!}f^{(p-1)}(b) + R(x), \end{aligned}$$

where

$$R(x) = \frac{1}{(p-1)!}\int_b^x (x-z)^{p-1} f^{(p)}(z)\,dz.$$

This gives

$$f(x) = P^b_{p-1}(x) + R(x),$$

where $P^b_{p-1}(x)$ is a polynomial of degree $(p-1)$ given by

$$P^b_{p-1}(x) = \sum_{r=0}^{p-1} \frac{(x-b)^r}{r!} f^{(r)}(b).$$

Note that

$$\begin{aligned} f(x) - P^b_p(x) &= R(x) - \frac{(x-b)^p}{p!} f^{(p)}(b) \\ &= \frac{1}{(p-1)!}\int_b^x (x-z)^{p-1} f^{(p)}(z)\,dz - \frac{1}{(p-1)!}\int_b^x (x-z)^{p-1} f^{(p)}(b)\,dz \\ &= \frac{1}{(p-1)!}\int_b^x (x-z)^{p-1}\left[f^{(p)}(z) - f^{(p)}(b)\right]\,dz \\ &\le \frac{(x-b)^{p-1}}{(p-1)!}\int_b^x \left[f^{(p)}(z) - f^{(p)}(b)\right]\,dz. \end{aligned}$$

This gives

$$\left|f(x) - P^b_p(x)\right| \le \frac{(x-b)^{p-1}}{(p-1)!}\int_b^x \left|f^{(p)}(z) - f^{(p)}(b)\right|\,dz.$$

Since $f^{(p)}$ is Hölder continuous with exponent $\delta\,(0<\delta<1)$, there exists a constant M such that

$$|f^{(p)}(z)-f^{(p)}(b)|\leq M|z-b|^{\delta}.$$

So, we compute

$$\begin{aligned}|f(x)-P_p^b(x)| &\leq M\,\frac{|x-b|^{p-1}}{(p-1)!}\int_b^x |z-b|^{\delta}\,dz\\ &= M\,\frac{|x-b|^{p-1}}{(p-1)!}\,\frac{(z-b)^{\delta+1}}{\delta+1}\Bigg|_b^x\\ &= \frac{M}{(\delta+1)(p-1)!}\,|x-b|^{p-1}\,|x-b|^{\delta+1}\\ &= M'\,|x-b|^{p+\delta},\end{aligned}$$

where $M'=\frac{M}{(\delta+1)(p-1)!}$.
Since ψ, $\psi^{(1)}$, $\hat{\psi}\in L^1(\mathbb{R})$, and $x^{p+1}\,\psi(x)\in L^2(\mathbb{R})$, we have

$$\langle f,\mathfrak{B}\psi_{j,k}\rangle=\int_{\mathbb{R}}\left[f(x)-P_p^b(x)\right]\,2^{\frac{j}{2}}\,\mathfrak{B}\psi(2^jx-k)\,dx.$$

Taking $b=2^{-j}k$ gives

$$|\langle f,\mathfrak{B}\psi_{j,k}\rangle|\leq K\,2^{-j(p+\delta)}\,2^{\frac{j}{2}}\int_{\mathbb{R}}\left|2^jx-k\right|^{p+\delta}\left|\mathfrak{B}\psi(2^jx-k)\right|dx.$$

Put $2^jx-k=v$. Then

$$\begin{aligned}|\langle f,\mathfrak{B}\psi_{j,k}\rangle| &\leq K\,2^{-j(p+\delta+\frac{1}{2})}\int_{\mathbb{R}}\left|v^{p+\delta}\mathfrak{B}\psi(v)\right|dv\\ &\leq K\,2^{-j(p+\delta+\frac{1}{2})}\left[\int_{\mathbb{R}}\left|v^{p+\delta}\,\mathfrak{H}\psi(v)\right|dv+\int_{\mathbb{R}}\left|v^{p+\delta}(\mathfrak{H}\psi*\mathfrak{g})(v)\right|dv\right]\\ &\leq K\,2^{-j(p+\delta+\frac{1}{2})}\left[\int_{\mathbb{R}}\left|v^{p+\delta}\,\mathfrak{H}\psi(v)\right|dv+\int_{\mathbb{R}}\left|v^{p+\delta}\,G(v)\right|dv\right].\end{aligned}$$

Using $(i),(ii)$ and (iii) and moment formula for the Hilbert Transform, we have

$$|\langle f,\mathfrak{B}\psi_{j,k}\rangle|\leq K\,2^{-j(p+\delta+\frac{1}{2})},$$

where K is a constant independent of j. □

Note that modulus of continuity is a traditional way used for describing smoothness of functions. This idea of modulus of continuity is found in a variety of applications in theory of approximation, functional spaces and different fields of present day analysis. Recall from [23] that modulus of continuity of a function f is given by

$$w_f = sup\{|f(r) - f(s)| \colon |r - s| \leq \delta\} \text{ for each } \delta > 0.$$

In the given result, we give a sufficient condition for uniform approximation of wavelet coefficients of function $f \in L^2(\mathbb{R})$ by means of modulus of continuity and prove that it decays fast as $j \to \infty$.

Theorem 2.6. Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\hat{\psi} \in L^1(\mathbb{R})$ and $\hat{\psi}(0) = 0$. Let Boas transform of wavelet ψ be of compact support and let $\|\mathfrak{H}\psi\|_1$ be bounded. Also, let $|G(x)| < \infty$, where $G(x) = \displaystyle\int_{-1}^{1} \left(1 - \frac{1}{|\gamma|}\right) e^{-2\pi i \gamma x}\, \widehat{\psi^{(1)}}(-\gamma)\, d\gamma$. Then $\|\langle f, \mathfrak{B}\psi_{j,k}\rangle\|_\infty = 2^{\frac{-j}{2}} O(w_f(2^{-j}L))$, where w_f represents the modulus of continuity of a function f.

Proof. We have

$$\begin{aligned}
2^{\frac{j}{2}} |\langle f, \mathfrak{B}\psi_{j,k}\rangle| &= 2^{\frac{j}{2}} \left| \int_{\mathbb{R}} f(x)\, \mathfrak{B}\psi_{j,k}(x)\, dx \right| \\
&= \left| \int_{\mathbb{R}} f\left(2^{-j}(k+x)\right)\, \mathfrak{B}\psi(x)\, dx \right| \\
&= \left| \int_{\mathbb{R}} \left(f\left(2^{-j}(k+x)\right) - f(2^{-j}k)\right) \mathfrak{B}\psi(x)\, dx \right| \\
&\leq \int_a^b \left|f\left(2^{-j}(k+x)\right) - f(2^{-j}k)\right| |\mathfrak{H}\psi(x) - (\mathfrak{H}\psi * \mathfrak{g})(x)|\, dx \\
&\leq w_f\left(2^{-j}(b-a)\right) \left(\int_a^b |\mathfrak{H}\psi(x)|\, dx + \int_a^b |\mathfrak{H}\psi * \mathfrak{g}(x)|\, dx \right) \\
&= w_f\left(2^{-j}(b-a)\right) (b-a) \left(|\mathfrak{H}\psi(x)| + |G(x)|\right) \\
&\leq w_f\left(2^{-j}(b-a)\right) (b-a) \left(\|\mathfrak{H}\psi\|_\infty + K_1\right) \\
&< w_f\left(2^{-j}(b-a)\right) (b-a) \left(K_2 + K_1\right) \\
&= w_f\left(2^{-j}L\right) L\, K\,,
\end{aligned}$$

where $L = b - a$ and $K = K_2 + K_1$. This gives

$$\|\langle f, \mathfrak{B}\psi_{j,k}\rangle\|_\infty = \sup_{k \in \mathbb{Z}} |\langle f, \mathfrak{B}\psi_{j,k}\rangle| = 2^{\frac{-j}{2}} O(w_f(2^{-j}L)).$$

□

Finally, we give a sufficient condition for uniform approximation of wavelet coefficients of function $f \in L^2(\mathbb{R})$ with bounded p^{th} derivative and prove that it decays fast as $j \to \infty$.

Theorem 2.7. Let ψ be such that for some $m \in \mathbb{N}$, $x^{p-2}\psi(x)$, $\gamma^{p-1}\hat{\psi}(\gamma) \in L^1(\mathbb{R})$, $x^{p-1}\psi(x) \in L^2(\mathbb{R})$, and ψ is a G-function of order p with $\|G\|_2 < +\infty$. Let $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ be an orthogonal system on $\mathbb{R}$ such that Boas transform of ψ be of compact support. Then for a function $f \in L^2(\mathbb{R})$ with a bounded p^{th} derivative, we have $\|\langle f, \mathfrak{B}\psi\rangle\|_\infty = 2^{\frac{-j}{2}}\ O(2^{-jp})$.

Proof. We have

$$\begin{aligned}
2^{\frac{j}{2}}\ |\langle f, \mathfrak{B}\psi\rangle| &= 2^{\frac{j}{2}}\ \left|\int_{\mathbb{R}} f(x)\ \mathfrak{B}\psi_{j,k}(x)\ dx\right| \\
&= \left|\int_{\mathbb{R}} f(2^{-j}(k+x))\ \mathfrak{B}\psi(x)\ dx\right| \\
&= \left|\int_{\mathbb{R}} [f(2^{-j}(k+x)) - q(2^{-j}(k+x))]\ \mathfrak{B}\psi(x)\ dx\right|,
\end{aligned}$$

where $q(\cdot)$ is a Taylor polynomial of degree at most $p-1$ which matches f and its first $p-1$ derivatives at $2^{-j}(k+x)$ and let $K = \operatorname{supp}(\mathfrak{B}\psi)$. This yields

$$2^{\frac{j}{2}}\ |\langle f, \mathfrak{B}\psi\rangle| = \left|\int_K R(x)\ \mathfrak{B}\psi(x)\ dx\right|,$$

where $R(x) = \frac{1}{p!}\ (x - 2^{-j}k)\ f^{(p)}(\xi)$ and for each $x \in K$, there exists ξ lying between x and $2^{-j}(k+x)$. This gives

$$\begin{aligned}
2^{\frac{j}{2}}\ |\langle f, \mathfrak{B}\psi\rangle| &\le \left|\int_K R(x)\ (\mathfrak{H}\psi(x) - (\mathfrak{H}\psi * \mathfrak{g})(x))\ dx\right| \\
&\le \left|\int_K R(x)\ \mathfrak{H}\psi(x)\right| dx + \left|\int_K R(x)\ (\mathfrak{H}\psi * \mathfrak{g})(x)\ dx\right| \\
&\le \left|\int_K R(x)\ \mathfrak{H}\psi(x)\right| dx + \left|\int_K R(x)\ G(x)\ dx\right|.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2^{\frac{j}{2}} \ |\langle f, \mathfrak{B}\psi\rangle| &\leq \|R\|_2(\|\mathfrak{H}\psi\|_2 + \|G\|_2) \\ &\leq M_2 \ \sup_{x\in K} |R(x)| \\ &\leq M_2 \ \frac{1}{p!} \ \|f^{(p)}\|_\infty \ 2^{-jp} \\ &= M \ 2^{-jp}, \end{aligned}$$

where $M_2 = \|\mathfrak{H}\psi\|_2 + \|G\|_2$ and $M = M_2 \ \frac{1}{p!} \ \|f^{(p)}\|_\infty$. This gives

$$\sup_{k\in\mathbb{Z}} |\langle f, \mathfrak{B}\psi\rangle| = 2^{\frac{-j}{2}} \ O(2^{-jp}).$$

□

3. Conclusion

In this work, finite energy signals are approximated using Boas transform of wavelets. Certain sufficient conditions are imposed in order to minimize the wavelet coefficients. Finally, uniform approximation of wavelet coefficients of finite energy signals is studied.

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(1) Department of Mathematics, Amity Institute of Applied Sciences, Amity University, Sector 125, Noida 201313 (U.P.), India.

Email address: leenakathuria91@gmail.com

(2) Department of Mathematics, Amity Institute of Applied Sciences, Amity University, Sector 125, Noida 201313 (U.P.), India.

Email address: sgoel54@amity.edu, goel.shashank25@gmail.com

(3) Department of Mathematics, College of Science, Sultan Qaboos University, P. O. Box 36, Al-Khod 123, Muscat, Sultanate of Oman.

Email address: n.khanna@squ.edu.om; nikkhannak232@gmail.com

(4) Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India.

Email address: shikk2003@yahoo.co.in