Jordan Journal of Mathematics and Statistics (JJMS), 16(2), 2023, pp 285 - 307
DOI: https://doi.org/10.47013/16.2.7

ORTHOGONAL CONE METRIC SPACES WITH BANACH ALGEBRAS AND FIXED POINT THEOREMS OF GENERALIZED LIPSCHITZ MAPPINGS

M.H.M. RASHID [(1)], P.P NOUSHIDA[(2)], V. ANIL KUMAR[(3)] AND T. PRASAD [(4)]

Abstract. In this paper, we impose weaker restriction on generalized Lipschitz constants to prove existence and uniqueness of fixed points in the setting of orthogonal cone metric spaces. Our results extend many of previously obtained findings in literature. Furthermore, we provide two example to support and illustrate the usability of the main results.

1. Introduction

The concept of cone metric spaces was first introduced by Huang and Zhang [10]. Observe that cone metric space is a generalization of a metric space in which the underlying space of the metric is replaced by a real Banach space. Then many researchers are established fixed point theorems on cone metric spaces.(see [1],[2],[12], [13],[18],[21]).

Recently, Liu and Xu [14] developed the notion of cone metric spaces over Banach algebras (also known as cone metric spaces over Banach algebras in [14], which replaced Banach spaces as the underlying spaces of cone metric spaces with Banach algebras). They created the notion of cone metric spaces over Banach algebras by replacing the Banach space E with a Banach algebra $\mathfrak{A}$. They used spectral radius to show various fixed point theorems of generalized Lipschitz mappings with weaker and natural constraints on the generalized Lipschitz constant k. It's worth noting that introducing the idea of cone metric spaces with Banach algebras is important

2010 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. Orthogonal cone metric space, fixed point, generalized orthogonal contractions, Lipschitz mapping.

Received: April 26, 2022 Accepted: Sept. 7, 2022 .

since one can show that they're not comparable to metric spaces in terms of the presence of the fixed points of the generalized Lipschitz mappings. In particular, Liu and Xu demonstrated that using the methods in the literature, the primary conclusions established in [14] could not be reduced to a consequence of similar results in metric spaces. This is beneficial to the field of cone metric spaces research.

Gordji et al [5], on the other hand, defined the idea of orthogonal sets and orthogonal metric spaces in 2017. In certain research works, theorems in this topic have been extended. (see [3],[15], [16]).

Orthogonal completeness, a new notion of orthogonal cone metric space, was presented lately, and orthogonal continuity is described in [6].
Orthogonal contractive mappings were suggested by the authors of [20] and several fixed point theorems for them were shown. They demonstrated their results in orthogonal bounded complete metric spaces using the idea of τ-distances. In contrast, they defined the idea of generalized orthogonal sets in [19] by expanding orthogonal sets. they also present the generalization of $\perp_F$-contractions known as $\perp_{\psi F}$-contractions. Some fixed point theorems are demonstrated for these contractions.

This new set of concepts is accompanied by examples. An example of orthogonal complete cone metric space, which is not complete cone metric space, is also presented. On orthogonal cone metric spaces, fixed point theorems and their corollaries are also established.

The existence and uniqueness of the fixed point for generalized Lipschitz mappings in the context of orthogonal cone metric spaces over Banach algebras are obtained in this study. The methodologies and approaches employed in this work are not the same as those used in previous research. We'll also demonstrate the existence and uniqueness of solutions to the initial value problem.
Throughout this paper $\mathbb{F}$ will denote either the real field, $\mathbb{R}$, or the complex field $\mathbb{C}$.

2. Preliminaries

An algebra over a field $\mathbb{F}$ is a vector space $\mathfrak{A}$ over $\mathbb{F}$ that also has a multiplication defined on it that makes $\mathfrak{A}$ into a ring such that if $\alpha \in \mathbb{F}$ and $a, b \in \mathfrak{A}$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$ (see [4]).

Definition 2.1. [4] A Banach algebra is an algebra $\mathfrak{A}$ over $\mathbb{F}$ that has a norm $\|.\|$ relative to which $\mathfrak{A}$ is a Banach space and such that for all $a, b \in \mathfrak{A}$,

$$\|ab\| \leq \|a\| \|b\| .$$

If $\mathfrak{A}$ has an identity, e, then it is assumed that $\|e\| = 1$. That is, $\mathfrak{A}$ is called a Banach algebra (with unit) if:

(i) $\mathfrak{A}$ is a Banach space;

(ii) There is a multiplication $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ that has the following properties:

$$(xy)z = x(yz) =, \quad (x+y)z = xz+yz, \quad x(y+z) = xy+xz, \quad \alpha(xy) = (\alpha x)y = x(\alpha y)$$

for all $x, y, z \in \mathfrak{A}$ and $\alpha \in \mathbb{F}$. Moreover, there is a unit element e: $ex = xe = x$ for all $x \in \mathfrak{A}$;

(iii) $\|e\| = 1$;

(iv) $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in \mathfrak{A}$.

An element x in a Banach algebra $\mathfrak{A}$ is said to be invertible if there exists $y \in \mathfrak{A}$ such that $xy = yx = e$, the inverse of x is denoted by x^{-1}. Let $\mathfrak{A}$ be a Banach algebra and $x \in \mathfrak{A}$, then the spectrum of x is given by $\sigma(x){:} = \{\lambda : x - \lambda e \text{ is singular}\}$, the spectral radius r of x is defined as $r(x) = sup\{|\lambda| : \lambda \in \sigma(x)\} = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}}$. Let $\mathfrak{A}$ be a Banach algebra with identity e, and $x \in \mathfrak{A}$. If the spectral radius $r(x)$ of x is less than 1, then $e-x$ is invertible and $(e-x)^{-1} = \sum_{i=1}^{\infty} x^i$ (see [17] for more details).

Remark 1. *From* [4]*, we see that the spectral radius $r(x)$ of x satisfies $r(x) \leq \|x\|$ for all $x \in \mathfrak{A}$, where $\mathfrak{A}$ is a Banach algebra with a unit e.*

In the following we always suppose $\mathfrak{A}$ is a Banach algebra with a unit, P is a cone in $\mathfrak{A}$ with $\text{int}(P) \neq \emptyset$ and $\preceq$ is partial ordering with respect to P.

Definition 2.2. [9] Let $\mathfrak{A}$ be a Banach algebra and P a subset of $\mathfrak{A}$. P is called a cone if and only if

(i) P is closed, nonempty, $P \neq \{0_{\mathfrak{A}}\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = 0_{\mathfrak{A}}$.

Given a cone $P \subset \mathfrak{A}$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ while $x \prec\prec y$ will stand for $y - x \in \operatorname{int}(P)$, $\operatorname{int}(P)$ denotes the interior of P and if $\operatorname{int}(P) \neq \emptyset$, then P is called a solid cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathfrak{A}$, $0_{\mathfrak{A}} \preceq x \preceq y$ implies $\|x\|_{\mathfrak{A}} \leq K \|y\|_{\mathfrak{A}}$. The least positive number K satisfying above is called the normal constant of P (see [9]).

Definition 2.3. [7] Let X be a nonempty set and $\perp \subseteq X \times X$ be a binary operation. If $\perp$ satisfies the following condition:

$$\exists x_0 \in X; (\forall y \in X, y \perp x_0) \vee (\forall y \in X, x_0 \perp y), \tag{2.1}$$

it is called an orthogonal set (shortly O-set). And $(X, \perp)$ is called O-set. And the element x_0 is called an orthogonal element.

Example 2.1. [8] *Let $X = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set.*

By the following example, we can see that x_0 is not necessarily unique.

Example 2.2. [8] *Let $X = [0, \infty)$, we define $x \perp y$ if $xy \in \{x, y\}$ then by setting $x_0 = 0$ or $x_0 = 1$, $(X, \perp)$ is an O-set.*

Definition 2.4. [7] Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$.

Definition 2.5. [7] Let $(X, \perp)$ be an orthogonal set. A sequence (x_n) in X is called an orthogonal sequence (O-sequence) if

$$x_n \perp x_{n+1} \ \vee \ x_{n+1} \perp x_n, \forall n \in \mathbb{N}. \tag{2.2}$$

Similarly, a Cauchy sequence $\{x_n\}$ is said to be an orthogonally Cauchy sequence (shortly O-Cauchy sequence) if

$$x_n \perp x_m \vee x_m \perp x_n, \forall n, m \in \mathbb{N} \text{ such that } n \geq m. \tag{2.3}$$

Definition 2.6. [5, 7] A mapping $d : X \times X \to \mathfrak{A}$ is called a cone metric on the orthogonal set (X, d), if the following conditions are satisfied:

(OC1) $0_{\mathfrak{A}} \leq d(x, y)$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,

(OC2) $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,

(OC3) $d(x, y) = d(y, x)$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,

(OC4) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$ such that $x \perp y$, $y \perp z$ and $x \perp z$.

Then the ordered triple $(X, \perp, d)$ is called an orthogonal cone metric space.

Example 2.3. *Let $\mathfrak{A} = \mathbb{R}^2$, $P = \{(x, y) \in \mathfrak{A} : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{Z}$ and $d : X \times X \to \mathfrak{A}$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where α is a nonnegative constant. Assume that binary relation $\perp$ on $X = \mathbb{Z}$ as Example 2.1, then $(X, \perp, d)$ is orthogonal cone metric space.*

Let X be an orthogonal set and $d : X \times X \to \mathfrak{A}$ be a mapping. For every $x \in X$ we define the set

$$\mathcal{O}(X, d, x) = \{(x_n) \subset X : \lim_{n\to\infty} d(x_n, x) = 0 \text{ and } x_n \perp x, n \in \mathbb{N}\}. \tag{2.4}$$

Definition 2.7. [11] Let $(X, \perp, d)$ be an orthogonal cone metric space. A sequence (x_n) in X is said to be

(i) an orthogonal convergent (in short O-convergent) to x if and only if $(x_n) \in \mathcal{O}(X, d, x)$,

(ii) an orthogonal Cauchy (in short O-Cauchy) if and only $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $x_n \perp x_m$ or $x_m \perp x_n$, $\forall n, m \in \mathbb{N}$.

Remark 2. *In orthogonal cone metric space $(X, \perp, d)$, an orthogonal convergent sequence may be not an orthogonal Cauchy.*

Definition 2.8. [11] An orthogonal cone metric space $(X, \perp, d)$ is said to be an orthogonal complete (O-complete)if every orthogonal Cauchy sequence converges in X.

Remark 3. *It is easy to see that every complete cone metric space is orthogonal cone complete but the converse is not true. For this remark, see the following examples.*

Example 2.4. [11] *Let $X = \mathbb{Q}$ and $P = \{x \in \mathbb{Q} : x \geq 0\}$. Define $x \perp y$ if and only if $x = 0$ or $y = 0$. Then $(X, \perp)$ is an orthogonal set. It is clear that $\mathbb{Q}$ is not a complete cone metric space with respect to the Euclidean metric, but it is orthogonal cone complete. If (x_n) is any orthogonal Cauchy sequence in $\mathbb{Q}$, then there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k} = 0$ for all $k \in \mathbb{N}$. Then (x_{n_k}) converges to $0 \in X$. We know that every Cauchy sequence with a convergent subsequence is convergent, so (x_n) is convergent.*

Example 2.5. [11] *Let $X = [0, 1)$ and define the orthogonal relation on X by*

$$x \perp y \Longleftrightarrow x \leq y \leq \frac{1}{4} \vee x = 0.$$

Then $(X, \perp)$ is an orthogonal set. We have X is not a complete metric space with respect to the Euclidean metric but it is orthogonal metric. Consider (x_n) is an orthogonal Cauchy sequence in X. Then there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k} = 0$ for all $k \in \mathbb{N}$, or there exists a monotone subsequence (x_{n_k}) of (x_n) for which $x_{n_k} \leq \frac{1}{4}$. We see that (x_{n_k}) converges to a point $[0, \frac{1}{4}] \subseteq X$. We know that every Cauchy sequence with a convergent subsequence is convergent, so (x_n) is convergent in X.

Definition 2.9. [11] Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be an orthogonal continuous (O-continuous or $\perp$-continuous) at a point x_0 in X if for each orthogonal sequence (x_n) in X converging to x_0 such that $f(x_n) \to f(x_0)$. Also f is said to be orthogonal continuous on X if f is orthogonal continuous at each point in X.

Remark 4. *It is easy to see every continuous mapping is orthogonal continuous. The following examples show the converse is not true in general.*

Example 2.6. [11] *Let $X = \mathbb{R}$. Define the orthogonality relation on X by $x \perp y$ if and only if $x = 0$ or $y \neq 0$ in $\mathbb{Q}$. Then $(X, \perp)$ is an orthogonal set. Define a function*

$f : X \to X$ by

$$f(x) = \begin{cases} 2, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then f is an orthogonal continuous but is not continuous on $\mathbb{Q}$.

Example 2.7. [11] *Let $X = \mathbb{R}$. Define $x \perp y$ if $x, y \in (q + \frac{1}{7}, q + \frac{2}{7})$ for some $q \in \mathbb{Z}$ or $x = 0$. Then $(X, \perp)$ is an orthogonal set. Define a function $f : X \to X$ by $f(x) = [x]$. Then f is an orthogonal continuous on X. Because for an orthogonal sequence (x_n) in X converging to $x \in X$, then we have*

> **Case-I:** *If $x_n = 0$ for all n, then $x = 0$ and $f(x_n) = 0 = f(x)$.*
>
> **Case-II:** *If $x_{n_0} \neq 0$ for some n_0, then there exists $k \in \mathbb{Z}$ such that $x_n \in (k + \frac{1}{7}, k + \frac{2}{7})$ for all $n \geq n_0$. Then $x \in [q + \frac{1}{7}, q + \frac{2}{7}]$ and $f(x_n) = k = f(x)$. It follows that f is orthogonal continuous on X but it is not continuous on X.*

Definition 2.10. [11] Let $(X, \perp, d)$ be an orthogonal metric space and $0 < K < 1$. A mapping $T : X \to X$ is called an orthogonal contraction (O-contraction or $\perp$-contraction) with Lipschitz constant K, if for all $x, y \in X$ with $x \perp y$ then $d(Tx, Ty) \leq Kd(x, y)$.

Remark 5. *It is clear that every contraction is orthogonal contraction but the converse is not true.*

Example 2.8. *Let $X = [0, 20)$ and d be the Euclidean metric on X. Define $x \perp y$ if $xy \leq x \vee y$. Let $F : X \to X$ be a map defined by*

$$F(x) = \begin{cases} \frac{x}{8}, & \text{if } x \leq 8; \\ 0, & \text{if } x > 4. \end{cases}$$

Let $x \perp y$ and $x \leq y$ then we have

> **Case I:** *If $x = 0$ and $y \leq 8$ then $F(x) = 0$ and $F(y) = \frac{y}{8}$.*
>
> **Case II:** *If $x \leq 8$ and $y > 8$ then $F(x) = \frac{x}{8}$ and $F(y) = 0$.*
>
> **Case III:** *If $x \leq 8$ and $y \leq 8$ then $F(x) = \frac{x}{8}$ and $F(y) = \frac{y}{8}$.*
>
> **Case IV:** *If $x > 8$ and $y \leq 8$ then $F(x) = 0$ and $F(y) = \frac{y}{8}$.*

Therefore we have $|F(x)-F(y)| \leq \frac{1}{8}|x-y|$*, and hence,* F *is an orthogonal contraction. But* F *is not a contraction, because for each* $K < 1$ *then* $|F(9) - F(8)| = 1 > K = K|9 - 8|$.

Example 2.9. [11] *Let* $X = [0, 1)$ *and* d *be the Euclidean metric on* X*. Define* $x \perp y$ *if* $xy \in \{x, y\}$ *for all* $x, y \in X$*. Let* $F : X \to X$ *be a mapping defined by*

$$F(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \mathbb{Q} \cap X; \\ 0, & \text{if } x \in \mathbb{Q}^c \cap X. \end{cases}$$

Then F *is an orthogonal contraction on* X *but it is not a contraction.*

Definition 2.11. Let $(X, \perp, d)$ be an orthogonal cone metric space. A mapping $T : X \to X$ is said to be an orthogonal preserving (O-preserving or $\perp$-preserving)if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in X$.

Definition 2.12. [11] Let $(X, \perp, d)$ be an orthogonal cone metric space. A mapping $T : X \to X$ is said to be a weakly orthogonal preserving (weakly O-preserving or weakly $\perp$-preserving)if $x \perp y$ implies $Tx \perp Ty$ or $Ty \perp Tx$ for all $x, y \in X$.

Example 2.10. [11] *Let* X *be the set of all peoples in the world. We define* $x \perp y$ *if* x *can give blood to* y*. According to the following table, if* x_0 *is a person such that his/her blood type is* O^-*, then we have* $x_0 \perp y$ *for all* $y \in X$*. Then* $(X, \perp)$ *is an orthogonal set. In the following, we see that in this orthogonal set* x_0 *is not unique.*

Type	*You can give blood to*	*You can receive blood from*
A^+	A^+, AB^+	A^+, A^-, O^+, O^-
O^+	O^+, A^+, B^+, AB^+	O^+, O^-
B^+	B^+, AB^+	B^+, B^-, O^+, O^-
AB^+	AB^+	*Everyone*
A^-	A^+, A^-, AB^+, AB^-	A^-, O^-
O^-	*Everyone*	O^-
B^-	B^+, B^-, AB^+, AB^-	B^-, O^-
AB^-	AB^+, AB^-	AB^-, B^-, O^-, A^-

Remark 6. *We have every orthogonal preserving mapping is weakly preserving, but the converse is not true.*

For this let $(x, \perp)$ be an orthogonal set defined in the Example 2.10. Let O_1 in X be a person with blood type O^-; P_1 be a person with blood type A^+. Define a mapping $F : X \to X$ by

$$F(x) = \begin{cases} P_1, & \text{if } x = O_1; \\ O_1, & \text{if } x \in X \setminus \{O_1\}. \end{cases}$$

Let $O_2 \in X \setminus \{O_1\}$ be a person with blood type O^-. Then we get $O_1 \perp O_2$ but we do not have $F(O_1) \perp F(O_2)$. Therefore F is not an orthogonal preserving but it is weakly orthogonal preserving.

Finally, let us recall the concept of generalized Lipschitz mapping defining on the cone metric spaces over Banach algebras, which is introduced in [14].

Definition 2.13. [14] Let (X, d) be a cone metric space over a Banach algebra $\mathfrak{A}$. A mapping $T : X \to X$ is called a generalized Lipschitz mapping if there exists a vector $k \in P$ with $r(k) < 1$ and for all $x, y \in X$, one has

$$d(Tx, Ty) \preceq kd(x, y).$$

Remark 7. *In Definition 2.13, we only suppose the spectral radius of k is less than 1, while $\|k\| < 1$ is not assumed. Generally speaking, it is meaningful since by Remark 1, the condition $r(k) < 1$ is weaker than that $\|x\| < 1$.*

Remark 8. *If $r(k) < 1$, then $\|k^n\| \to 0 (n \to \infty)$.*

To arrive at our primary conclusions, it is crucial to first understand the facts regarding spectral radius.

Lemma 2.1. [22] *Let $\mathfrak{A}$ be a Banach algebra and let x, y be vectors in $\mathfrak{A}$. If x and y commute, then the following hold:*

(i) $r(xy) \leq r(x)r(y)$;

(ii) $r(x + y) \leq r(x) + r(y)$;

(iii) $|r(x) - r(y)| \leq r(x - y)$.

Lemma 2.2. [22] *Let $\mathfrak{A}$ be a Banach algebra and let (x_n) be a sequence in $\mathfrak{A}$. Suppose that (x_n) converges to x in $\mathfrak{A}$ and that x_n and x commute for all n, then we have $r(x_n) \to r(x)$ as $n \to \infty$.*

Lemma 2.3. [22] *Let $\mathfrak{A}$ be a Banach algebra and let k be a vector in $\mathfrak{A}$. If $0 \leq r(k) < 1$, then we have*

$$r((e-k)^{-1}) \leq (1-r(k))^{-1}.$$

3. Examples

In this section we give some examples for orthogonal cone metric spaces with Banach algebras.

Example 3.1. [5] *Let $\mathfrak{A} = \mathbb{R}^2$, with $\| (x_1, x_2) \| = |x_1| + |x_2|$. The multiplication is defined by*

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_1 + x_1y_2).$$

Then $\mathfrak{A}$ is a Banach algebra with unit $(1, 0)$.

Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, then P is a normal cone with normal constant $M = 1$.

Let $X = \mathbb{R}^2$, the metric d be defined by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, \alpha|x_2 - y_2|),\ \alpha \in \mathbb{R}, \alpha > 0.$$

Then (X, d) is a complete cone metric space with Banach algebra.

Now consider the binary relation on X by,

$$x \perp y \text{ if and only if } \langle x, y\rangle = 0.$$

Then by setting $x_0 = (0, 0)$, we get $x_0 \perp (x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$, and so X is an orthogonal set, $(X, \perp, d)$ is an O-complete cone metric space with Banach algebra.

Example 3.2. *Let $\mathfrak{A} = \mathbb{R}^2$, with*

$$\| (x_1, x_2) \| = \max_{i=1,2} |x_i|.$$

The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2).$$

Then $\mathfrak{A}$ is a Banach algebra with unit $(1, 1)$.

Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, then P is a normal cone with normal constant $M = 1$.

(1) *Let $X = [0, 1) \times [0, 1) \subseteq \mathbb{R}^2$, the metric d be defined by*

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|),$$

then (X, d) is a non complete cone metric space with Banach algebra.

Now consider the binary relation defined on X by

$$x \perp y \text{ if and only if } x.y \in \{x, y\}.$$

Then $(0,0) \perp (x_1, x_2)$ for all $(x_1, x_2) \in X$, so $(X, \perp, d)$ is an O-complete cone metric space with Banach algebra.

(2) *Consider $\mathbb{Z}$ with the binary relation*

$$m \perp n \text{ if there is } k \in \mathbb{Z} \text{ with } m = kn.$$

Then if we define the metric $d : \mathbb{Z} \times \mathbb{Z} \to \mathfrak{A}$ by

$$d(x, y) = (|x - y|, |x - y|),$$

then $(\mathbb{Z}, \perp, d)$ is a O-complete cone metric space with Banach algebra.

Example 3.3. *Let $\mathfrak{A} = \mathbf{C}[0, 1]$, be the space of all real valued continuous functions on $[0, 1]$, with pointwise multiplication and*

$$\| x \| = \sup_{t \in [0,1]} |x(t)|,$$

then it is known that $\mathfrak{A}$ is a commutative unital Banach algebra with this norm and $P = \{x : x(t) \geq 0\}$, is a normal cone in $\mathfrak{A}$ with normal constant 1. Let $X = \{f \in \mathbf{C}[0, 1] : f(x) \geq 1\}$, consider the binary relation on X by

$$x \perp y \text{ if and only if } x(t).y(t) \geq x(t) \text{or } y(t).$$

Then $(X, \perp)$ is an orthogonal set. Define a metric $d : X \times X \to \mathfrak{A}$ by

$$d(f, g) = |f(t) - g(t)|.$$

Then $(X, \perp, d)$ is an orthogonal cone metric space with Banach algebra.

Example 3.4. *Let $\mathfrak{A} = M_n(\mathbb{R}) = \{a = (a_{ij})_{n \times n} | a_{ij} \in \mathbb{R}, \forall 1 \leq i, j \leq n\}$ be the algebra of all n-square real matrices, and define the norm*

$$\|a\| = \sum_{1 \leq i,j \leq n} |a_{ij}|.$$

Then $\mathfrak{A}$ is a real Banach algebra with the unit e the identity matrix.
Let $P = \{a \in \mathfrak{A} | a_{ij} \geq 0, \forall 1 \leq i, j \leq n\}$. Then $P \subset \mathfrak{A}$ is a normal cone with normal constant $M = 1$. Now we define a binary relation $\perp$ on $\mathfrak{A}$ as $a \perp b$ if $a - b \geq 0$. Clearly

for all $a \in P$, $a \perp 0$.
Let $X = M_n(\mathbb{R})$, *and define the metric* $d : X \times X \to \mathfrak{A}$ *by*

$$d(x, y) = d((x_{ij})_{n\times n}, (y_{ij})_{n\times n}) = (|x_{ij} - y_{ij}|)_{n\times n} \in \mathfrak{A}.$$

Then $(X, \perp, d)$ *is a cone metric space over Banach algebra* $\mathfrak{A}$ *with normality.*

4. Main Results

The following results, which extend the theorems in cone metric spaces [9], to orthogonal cone metric spaces with Banach algebras, are proved in this section. Other fixed point theorems of generalized Lipschitz mappings in the case of an orthogonal cone metric space over Banach algebra will be presented in the following sections.

Definition 4.1. Let $(X, \perp, d)$ be an orthogonal cone metric space over a Banach algebra $\mathfrak{A}$. A mapping $T : X \to X$ is called a generalized Lipschitz mapping if there exists a vector $k \in P$ with $r(k) < 1$ and for all $x, y \in X$ with $x \perp y$, one has

$$d(Tx, Ty) \preceq kd(x, y).$$

Theorem 4.1. *Let* $(X, \perp, d)$ *be an O-complete cone metric space with Banach algebra* $\mathfrak{A}$ *and* P *is a normal cone in* $\mathfrak{A}$ *with normal constant* M, *let* $f : X \to X$ *be a mapping which is* $\perp$*-preserving,* $\perp$*-continuous and satisfies the generalized orthogonal contraction condition,*

$$d(fx, fy) \preceq ad(x, y), \text{ if } x \perp y,$$

where $a \in P$ *with* $r(a) < 1$. *Then* f *has a unique fixed point in* X.

Proof. Since X is an orthogonal set, there exists x_0 in X such that

$$x \perp x_0 \text{ or } x_0 \perp x \text{ for all } x \in X.$$

So we have $x_0 \perp f(x_0)$ or $f(x_0) \perp x_0$. Let

$$x_1 := f(x_0), x_2 := f(x_1) = f^2(x_0), ..., x_{n+1} := f(x_n) = f^n(x_0), \text{ for } n \geq 1.$$

Since f is $\perp$-preserving we have $\{x_n\}$ is an O-sequence and

$$d(x_{n+1}, x_n) \preceq ad(x_n, x_{n-1}) \preceq ... \preceq a^n d(x_1, x_0)$$

So for $n > m$,

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &\preceq (a^{n-1} + a^{n-2} + ... + a^m) d(x_0, x_1) \\ &\preceq (\sum_{i=0}^{\infty} a^i) a^m d(x_1, x_0) \\ &= (e - a)^{-1} a^m d(x_1, x_0). \end{aligned}$$

Since P is a normal cone with normal constant M, and $\| a^n \| \to 0$ as $n \to \infty$, we have

$$\| d(x_n, x_m) \| \to 0 \text{ as } n \to \infty.$$

Hence the O-sequence $\{x_n\}$ is a Cauchy O-sequence and by the O-completeness of X, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now since f is $\perp$-continuous $f(x_n) \to f(x)$ as $n \to \infty$, and

$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x.$$

Therefore x is a fixed point of f in X.

Now we prove the uniqueness of the fixed point. Let $y \in X$ be another fixed point of f. Then we have $f^n(y) = y$ for all $n \in \mathbb{N}$, and X is an O-set implies,

$$x_0 \perp y \text{ or } y \perp x_0.$$

As f is $\perp$-preserving, we have

$$f^n(x_0) \perp f^n(y) \text{ or } f^n(y) \perp f^n(x_0), \text{ for all } n \in \mathbb{N}, \text{ and}$$

$$f^n(x_0) \perp f^n(x) \text{ or } f^n(x) \perp f^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Since f is a $\perp -$ contraction

$$\begin{aligned} d(x, y) &= d(f^n(x), f^n(y)) \\ &\preceq d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(y)) \\ &\preceq a^n [d(x_0, x) + d(y, x_0)]. \end{aligned}$$

As P is a normal cone with normal constant M and $r(a) < 1$, we have

$$\| d(x, y) \| \leq M(\| a^n \| [\| d(x, x_0) \| + \| d(x_0, y) \|]) \to 0,$$

and $x = y$ as $n \to \infty$. Hence the fixed point of f is unique. □

Example 4.1. *Let* $\mathfrak{A} = \mathbb{R}^2$, *with* $\| (x_1, x_2) \| = \frac{|x_1|+|x_2|}{2}$. *The multiplication is defined by*

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2).$$

Then $\mathfrak{A}$ *is a Banach algebra with unit* $(1, 1)$.

Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, *then* P *is a normal cone with normal constant* $M = 1$.

Let $X = \{(x, 0) : 0 \leq x \leq 1, x \in \mathbb{R}\}$. *Define a binary relation on* X *by*

$$(x, 0) \perp (y, 0) \text{ if } xy \leq min\{x, y\},$$

setting $x_0 = (0, 0)$ *we get* $(X, \perp)$ *is an orthogonal set.*

Define $d : X \times X \to \mathfrak{A}$ *by*

$$d((x, 0), (y, 0)) = (|x - y|, \alpha|x - y|), \alpha \in \mathbb{R}, \alpha > 0.$$

Then $(X, \perp, d)$ *is an O-complete cone metric space with Banach algebra.*

Consider the map $f : X \to X$ *defined by*

$$f((x, 0)) = (arctan(x + \tfrac{1}{2}), 0),$$

then f *is* $\perp$*-preserving,* $\perp$*-continuous and satisfies the generalized orthogonal contractive condition*

$$d(f((x, 0)), f((y, 0))) \preceq (\tfrac{4}{5}, \tfrac{4}{5})d((x, 0), (y, 0))$$

where $k = (\frac{4}{5}, \frac{4}{5}) \in P$,*with* $r(k) < 1$ *and* $\| k \| > 1$. *Hence we can apply the Theorem 4.1 to guarantee the existence of a fixed point for* f *in* X.

Theorem 4.2. *Let* $(X, \perp, d)$ *be an O-complete cone metric space with Banach algebra* $\mathfrak{A}$ *and* P *is a normal cone in* $\mathfrak{A}$ *with normal constant* M, *let* $f : X \to X$ *be* $\perp$*-preserving,* $\perp$*-continuous satisfying the following generalized orthogonal contraction type condition*

$$d(fx, fy) \preceq a[d(x, fx) + d(y, fy)],$$

where $x, y \in X$ *with* $x \perp y$ *and* $a \in P$ *with* $r(a) < \frac{1}{2}$. *Then* f *has a fixed point in* X.

Proof. As in Theorem 4.1 we get an O-sequence by

$$x_0, x_1 := f(x_0), x_2 := f(x_1) = f^2(x_0), .., x_n := f(x_{n-1}) = f^n(x_0), \text{ and}$$

$$\begin{aligned} d(x_{n+1}, x_n) &= d(fx_n, fx_{n-1}) \\ &\preceq a[d(x_n, fx_n) + d(x_{n-1}, fx_{n-1})] = a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ d(x_{n+1}, x_n) &\preceq (e-a)^{-1} a d(x_n, x_{n-1}). \end{aligned}$$

Take $h = a(e-a)^{-1}$, then we prove that $r(h) < 1$. Since $0 \leq r(a) < \frac{1}{2} < 1$,

$$(e-a)^{-1} = e + \sum_{i=0}^{\infty} a^i = e + \lim_{n\to\infty} \sum_{i=0}^{n} a^i$$

Then by lemma(2.2) and lemma(2.3), we have

$$\begin{aligned} r((e-a)^{-1}) &\leq r(e) + r(\lim_{n\to\infty} \sum_{i=0}^{n} a^i) \\ &\leq r(e) + \lim_{n\to\infty} \sum_{i=0}^{n} r(a)^i \\ &= 1 + \sum_{i=0}^{\infty} r(a)^i = (1 - r(a))^{-1}. \end{aligned}$$

In fact, a and $(e-a)^{-1}$ commutes and $r(a) < \frac{1}{2}$ gives

$$\begin{aligned} r(a(e-a)^{-1}) &\leq r(a) r((e-a)^{-1}) \\ &\leq \frac{r(a)}{1 - r(a)} < 1. \end{aligned}$$

So we have $d(x_{n+1}, x_n) \preceq h d(x_n, x_{n-1})$, where $r(h) < 1$, then by the proof of Theorem 4.1, we can see that the O-sequence $\{x_n\}$ is Cauchy. since X is O-complete there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. Next, we prove that x is a fixed point of f. Consider

$$\begin{aligned} d(fx, x) &\preceq d(fx, fx_n) + d(fx_n, x) \\ &\preceq k[d(fx, x) + d(fx_n, x_n)] + d(x_{n+1}, x) \\ &\preceq (e-k)^{-1}[k d(x_n, x) + (e+k) d(x_{n+1}, x)]. \end{aligned}$$

Since P is a normal cone with normal constant M, in the light of the above inequality,

$$d(fx, x) \leq M \parallel (e-k)^{-1} \parallel [\parallel k \parallel (\parallel d(x, x_n) \parallel + \parallel e+k \parallel \parallel d(x, x_{n+1}) \parallel)] \to 0 \text{ as } n \to \infty.$$

Hence x is a fixed point of f. □

Next by modifying the generalized contractive condition of the mapping, we prove the following fixed point result.

Theorem 4.3. *Let $(X, \perp, d)$ be an O-complete cone metric space with Banach algebra $\mathfrak{A}$ and P is a normal cone in $\mathfrak{A}$ with normal constant M, let $f : X \to X$ be $\perp$-preserving, $\perp$-continuous and satisfying the following generalized contraction type condition*

$$d(fx, fy) \preceq ad(x, y) + b[d(x, fx) + d(y, fy)] + c[d(y, fx) + d(x, fy)],$$

where $x, y \in X$ with $x \perp y$ and a, b, c are commuting elements in P with $r(a) + 2[r(b) + r(c)] < 1$. Then f has a fixed point in X.

Proof. As in Theorem 4.1, we get an O-sequence:

$$x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), .., x_n = f(x_{n-1}) = f^n(x_0), \text{ and}$$

$$\begin{aligned} d(x_{n+1}, x_n) &= d(fx_n, fx_{n-1}) \\ &\preceq ad(x_n, x_{n-1}) + b[d(x_n, fx_n) + d(x_{n-1}, fx_{n-1})] + c[d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})] \\ &= ad(x_n, x_{n-1}) + b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + c[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\preceq (e - b - c)^{-1}(a + b + c)d(x_n, x_{n-1}). \end{aligned}$$

Take $h = (e - b - c)^{-1}(a + b + c)$, then we prove that $r(h)$ is less than 1. By Lemma 3.1 and Lemma 3.2 , we have

$$\begin{aligned} r((e - b - c)^{-1}(a + b + c)) &\leq r(a + b + c)r((e - b - c)^{-1}) \\ &\leq \frac{r(a + b + c)}{1 - r(b + c)} \\ &\leq \frac{r(a) + r(b) + r(c)}{1 - r(b) - r(c)} < 1. \end{aligned}$$

Thus we have $d(x_{n+1}, x_n) \preceq hd(x_n, x_{n-1})$ with $r(h) < 1$, then by the proof of theorem 4.1, we can see that the O-sequence $\{x_n\}$ is Cauchy. Since X is O-complete there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. Next, we prove that x is a fixed point of f, consider

$$\begin{aligned} d(fx,x) &\preceq \lim_{n\to\infty} d(fx,fx_n)+d(fx_n,x) \\ &\preceq \lim_{n\to\infty}[ad(x,x_n)+bd[d(fx,x)+d(fx_n,x_n)]+c[d(fx,x_n)+d(fx_n,x)]+d(x_{n+1},x)] \\ &\preceq (b+c)d(fx,x), as\ n\to\infty. \end{aligned}$$

Observe that $b+c\in P$ with $r(b+c)<1$, implies $d(fx,x)=0_{\mathfrak{A}}$ and hence x is a fixed point of f. □

Here we prove fixed point result for mapping satisfying generalized Hardy Rogers contraction type conditions in O-complete cone metric spaces.

Theorem 4.4. *Let $(X,\perp,d)$ be an O-complete cone metric space over Banach algebra with normal cone P, $f:X\to X$ be $\perp$- preserving, $\perp$-continuous mapping and satisfying the following condition:*

$$d(fx,fy)\preceq a_1d(x,y)+a_2d(x,fx)+a_3d(y,fy)+a_4d(y,fx)+a_5d(x,fy),$$

where $x,y\in X$ with $x\perp y$ and a_1,a_2,a_3,a_4,a_5 are commuting elements in P with $r(a_1)+r(a_5)+r(a_3)+2r(a_2)+2r(a_4)<1$. *Then f has a fixed point in X.*

Proof. As in Theorem 4.1, we get an O-sequence:

$$x_0,x_1=f(x_0),x_2=f(x_1)=f^2(x_0),..,x_n=f(x_{n-1})=f^n(x_0),\text{ and}$$

$$\begin{aligned} d(x_{n+1},x_n) &= d(fx_n,fx_{n-1}) \\ &\preceq a_1d(x_n,x_{n-1})+a_2d(x_n,fx_n)+a_3d(x_{n-1},fx_{n-1}) \\ &+ a_4d(x_{n-1},fx_n)+a_5d(x_n,fx_{n-1}) \\ &= a_1d(x_n,x_{n-1})+a_2d(x_n,x_{n+1})+a_3d(x_{n-1},x_n)] \\ &+ a_4d(x_{n-1},x_{n+1})+a_5d(x_n,x_n) \\ &\preceq (e-a_2-a_4)^{-1}(a_1+a_2+a_3+a_4)d(x_n,x_{n-1}). \end{aligned}$$

Take $h = a(e - a_2 - a_4)^{-1}(a_1 + a_2 + a_3 + a_4)$, then we prove that $r(h)$ is less than 1. We have

$$\begin{aligned} r(e - a_2 - a_4)^{-1}(a_1 + a_2 + a_3 + a_4) &\leq r((e - a_2 - a_4)^{-1})r(a_1 + a_2 + a_3 + a_4) \\ &\leq \frac{r(a_1 + a_2 + a_3 + a_4)}{1 - r(a_2 - a_4)} \\ &\leq \frac{r(a_1) + r(a_2) + r(a_3) + r(a_4)}{1 - r(a_2) - r(a_4)} < 1. \end{aligned}$$

Thus we have $d(x_{n+1}, x_n) \preceq hd(x_n, x_{n-1})$ with $r(h) < 1$, then by the proof of Theorem 4.1, we can see that the O-sequence $\{x_n\}$ is Cauchy. Since X is O-complete there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. Next, we prove that x is a fixed point of f, consider

$$\begin{aligned} d(fx, x) &\preceq \lim_{n\to\infty} d(fx, fx_n) + d(fx_n, x) \\ &\preceq \lim_{n\to\infty} [a_1 d(x, x_n) + a_2 d(fx, x) + a_3 d(fx_n, x_n) \\ &+ a_4 d(fx, x_n) + a_5 d(fx_n, x) + d(x_{n+1}, x)] \\ &\preceq (a_2 + a_4)d(fx, x), as\ n \to \infty. \end{aligned}$$

Observe that $a_2 + a_4 \in P$ with $r(a_2 + a_4) < 1$, implies $d(fx, x) = 0_{\mathfrak{A}}$ and hence x is a fixed point of f. □

Definition 4.2. Let $(X, \perp, d)$ be an orthogonal cone metric space over the Banach algebra $\mathfrak{A}$ with cone P. We say $F : X \to X$ is α-admissible if there is $\alpha : X \times X \to P$ such that

$$\alpha(x, y) \succeq e \text{ implies } \alpha(Fx, Fy) \succeq e, \text{ for } x, y \in X.$$

Next we prove a fixed point result for mappings satisfying α-admissible contraction condition in O-complete cone metric space over Banach algebra.

Theorem 4.5. *Let $(X, \perp, d)$ be an O-complete cone metric space over Banach algebra $\mathfrak{A}$ with normal cone P, $f : X \to X$ be $\perp$- preserving, $\perp$-continuous, α-admissible mapping and satisfying the following conditions:*

(i) $\alpha(x, y)d(fx, fy) \preceq ad(x, y)$, *if* $x \perp y$, *where* $a \in P$ *with* $r(a) < 1$;

(ii) *There exists* $x_0 \in X$ *such that* $x_0 \perp fx_0$ *and* $\alpha(x_0, fx_0) \succeq e$.

Then f *has a fixed point in* X.

Proof. By condition (ii) there exists $x_0 \in X$ such that $x_0 \perp fx_0$ and $\alpha(x_0, fx_0) \succeq e$. Take $x_n = f^n(x_0)$, since f is $\perp$-preserving we have $\{x_n\}$ is an orthogonal sequence and the α-admissibility of f gives;

$$\alpha(x_n, x_{n+1}) \succeq e, \; for \, n \geq 0.$$

Also $\{x_n\}$ satisfies:

$$\begin{aligned} d(x_1, x_2) &\preceq \alpha(x_0, fx_0)d(x_1, x_2) = \alpha(x_0, fx_0)d(fx_0, fx_1) \\ &\preceq ad(x_0, x_1). \end{aligned}$$

Similarly

$$\begin{aligned} d(x_2, x_3) &\preceq \alpha(x_0, fx_0)d(x_2, x_3) = \alpha(x_0, fx_0)d(f(fx_0), f(f^2(x_0)) \\ &\preceq ad(fx_0, fx_1) = ad(x_1, x_2) \\ &\preceq a^2 d(x_0, x_1). \end{aligned}$$

Then by continuing this process we see that

$$d(x_n, x_{n+1}) \preceq a^n d(x_0, x_1).$$

Now for $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m+1}, x_m) \\ &\preceq (a^n + a^{n+1} + ... + a^{m-1})d(x_0, x_1) \\ &\preceq a^n((e-a)^{-1})d(x_0, x_1). \end{aligned}$$

Take $h = a^n(e-a)^{-1}$, then $r(a^n(e-a)^{-1}) < 1$, hence $\{x_n\}$ is a Cauchy sequence. Then by the O-completeness of X, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now since f is $\perp$-continuous we have x is a fixed point of f. □

5. Application

In this section, we give a simple application of Theorem 4.1.

Let $\mathfrak{A} = \mathbf{C}[0,1]$, be the space of all real valued continuous functions on $[0,1]$, with pointwise multiplication and

$$\| x \| = \sup_{t\in[0,1]} |x(t)|,$$

then $\mathfrak{A}$ is a commutative unital Banach algebra with this norm and
$P = \{x : x(t) \geq 0\}$, is a normal cone in $\mathfrak{A}$ with normal constant 1.

Now consider the initial value problem:

$$\begin{cases} y' = g(t, y(t)); & t \in I = [0,1] \\ y(0) = y_0, & y_0 \geq 1, \end{cases} \tag{5.1}$$

where $g : I \times \mathfrak{A} \to \mathfrak{A}$ be continuous and satisfies the following

(1) $g(s,x) \in P$ for all $x \in P$ and $s \in I$.

(2) g satisfies:

$$|g(s,x) - g(s,y)| \preceq f(s)|x(s) - y(s)|$$

where $f(s) = \frac{1+s^3}{3}$, $s \in I$ and $x, y \in P$ with $xy - x \in P$ or $xy - y \in P$. Note that f need not be a generalized Lipschitz mapping under the given conditions.

Here we prove the existence of a solution for the above initial value problem by applying theorem 4.1.

Let $X = \{x \in \mathbf{C}[0,1] : x(t) > 0, \forall t \in I\}$. Then define $d : X \times X \to \mathfrak{A}$ by $d(x,y) = \sup_{t\in I} |x(t) - y(t)|e^t$ for all $x, y \in X$. Then (X, d) is a cone metric space over the Banach algebra $\mathfrak{A}$. Now consider the orthogonality relation on X by

$$x \perp y \text{ if and only if } x(t)y(t) \geq x(t) \text{ or } y(t),$$

then $(X, \perp, d)$ is an O-complete cone metric space with Banach algebra.

Note that the initial value problem 5.1 is equivalent to the integral equation

$$x(t) = y_0 + \int_0^t g(s, x(s))\, ds.$$

Define a map $G : X \to X$ by

$$(Gx)(t) = y_0 + \int_0^t g(s, x(s))\, ds.$$

Note that the fixed points of G are the solutions of equation 5.1. To prove this we need the following steps;

(1) G is $\perp$-preserving: Let $x, y \in X$ with $x \perp y$, then

$$(Gx)(t) = y_0 + \int_0^t g(s, x(s))\, ds \geq 1.$$

Thus $(Gx)(t)(Gy)(t) \geq (Gy)(t)$, and hence $Gx \perp Gy$.

(2) G is a generalized $\perp$-contraction: Let $x, y \in X$ with $x \perp y$, then

$$\begin{aligned} |(Gx)(t) - (Gy)(t)| &\leq \int_0^t |g(s, x(s)) - g(s, y(s))|\, ds \\ &\leq \int_0^t |f(s)||x(s) - y(s)|\, ds \\ &= F(t)|x(t) - y(t)|, \end{aligned}$$

where $F(t) = \int_0^t |f(s)|\, ds$, for $t \in I$, then

$$\begin{aligned} d(Gx, Gy) &= \sup_{t\in I} |(Gx)(t) - (Gy)(t)|e^t \\ &\preceq \sup_{t\in I} F(t)|x(t) - y(t)|e^t \preceq kd(x, y). \end{aligned}$$

where $k \in P$ with $r(k) < 1$, then G is a generalized orthogonal contraction.

(3) G is $\perp$-continuous: Let $\{x_n\}$ be an O-sequence in X, and let $x_n \to x$ in X. As $x_n(t) \geq 1$, we have $x(t) \geq 1$. Hence $x_n \perp x$, and as in step.2, we get

$$d((Gx_n)(t) - (Gx)(t)) \preceq kd(x_n, x),$$

then as $r(k) < 1$, we have $Gx_n \to Gx$ as $n \to \infty$.

So G satisfies all the conditions in Theorem 4.1, hence the existence and uniqueness of solution to the initial value problem has been guaranteed by Theorem 4.1.

Acknowledgement

The authors take this opportunity to thank the editor and the referees for their very helpful comments on the submitted version of the paper. The second author would like to thank Kerala State Council for Science Technology and Environment (KSCSTE) for the financial support during the preparation of this paper.

Availability of data and materials: no data are available for this paper because it is a theoretical study.

References

[1] M. Abbas, and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *Journal of Mathematical Analysis and Applications*, 341(1)(2008), 416–420.

[2] M. Abbas, and B.E. Rhoades, Fixed and periodic point results in cone metric spaces, *Applied Mathematics Letters*, 22(4)(2009), 511–515.

[3] H. Baghani, M.E. Gordji, and M. Ramezani, Orthogonal sets: The axiom of choice and proof of a fixed point theorem, *Journal of Fixed Point Theory and Applications*, 18(3)(2016), 465–477.

[4] Conway, J. B.(1990), *A course in Functional analysis*, 2nd edt., Springer-Verlag, New York, London, Paris.

[5] N. B. Gungor, and M. Surmelioglu, Some fixed point theorems for contractive mappings on ordered orthogonal cone metric spaces, *Journal of Universal Mathematics,* 3(1)(2020), 83–93.

[6] N.B. Gungor, and D. Turkoglu, Fixed Point Results for Extensions of Orthogonal Contraction on Orthogonal Cone Metric Space, *Sahand Communications in Mathematical Analysis*, 17(4)(2020), 95–107.

[7] M. Eshaghi Gordji, M. Ramezani, Manuel De La Sen, and Yeol Je Cho, On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory* , 18(2)(2017), 569–578.

[8] M. Eshaghi Gordji, and H. Habibi, Fixed point theory in generalized orthogonal metric space, *Journal of Linear and Topological Algebra (JLTA)*, 6(3)(2017), 251–260.

[9] H. Liu, and S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, *Fixed Point Theory Appl 2013*, 320(2013). https://doi.org/10.1186/1687-1812-2013-320.

[10] Long-Guang Huang, and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications*, 332(2)(2007), 1468–1476.

[11] S.A. Mohiuddine, and T. Acar(2018), *Advances in Summability and Approximation Theory*, Springer Singapore.

[12] D. Ilic, and V. Rakocevic, Common fixed points for maps on cone metric space, *Journal of Mathematical Analysis and Applications*, 341(2)(2008), 876–882.

[13] S. Janković, Z. Kadelburg, and S. Radenović, On cone metric spaces: a survey, *Nonlinear Analysis: Theory, Methods and Applications*, 74(7)(2011), 2591–2601.

[14] H. Liu, and S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, *Fixed Point Theory Applications 2013*, 320(2013).

[15] M. Ramezani, Orthogonal metric space and convex contractions, *Int. J. Nonlinear Anal. Appl*, 6(2)(2015), 127–132.

[16] M. Ramezani, and H. Baghani, Contractive gauge functions in strongly orthogonal metric spaces, *International Journal of Nonlinear Analysis and Applications (IJNAA)*, 8(2)(2017), 23–28.

[17] G.F. Simmons(1963), *Introduction to Topology and Modern Analysis*, New York McGraw-Hill.

[18] D.Turkoglu, and M. Abuloha, Cone metric spaces and fixed point theorems in diametrically contractive mappings, *Acta mathematica sinica, English series*, 26(3)(2010), 489–496.

[19] Y. Touail, and D. El-Moutawakil, $\perp_{\psi F}$-contractions and some fixed point results on generalized orthogonal sets, *Rendiconti del Circolo Matematico di Palermo Series 2*, 70(2021), 1459–1472. https://doi.org/10.1007/s12215-020-00569-4.

[20] Y. Touail, and D. El-Moutawakil, Fixed point theorems on orthogonal complete metric spaces with an application, *International Journal of Nonlinear Analysis and Applications*, 12(2)(2021), 1801–1809.

[21] P. Vetro, Common fixed points in cone metric spaces, *Rendiconti del circolo Matematico di Palermo*, 56(3)(2007), 464–468.

[22] S. Xu, and S. Radenović, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, *Fixed Point Theory Appl 2014*, 102(2014). https://doi.org/10.1186/1687-1812-2014-102.

(1) Department of Mathematics, Faculty of Science P.O. Box(7), Mu'tah university, Al-Karak-Jordan.

Email address: malik_okasha@yahoo.com

(2,3,4) Department of Mathematics, University of Calicut, Calicut University P.O., Kerala, 673635.

Email address: noushidanvar@gmail.com, anil@uoc.ac.in, prasadt@uoc.ac.in