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EXISTENCE RESULT FOR A MODEL COUPLING A QUASI-LINEAR PARABOLIC EQUATION AND A LINEAR HYPERBOLIC SYSTEM

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Abstract. We study a coupled Fluid - Structure system describing the motion of an elastic solid interacting with an incompressible viscous fluid in two dimensions. The behavior of the solid is described by the Lamé system of linear elasticity and the fluid obeys the incompressible stokes equations. The quasi-linear nature of the considered Stokes equation is characterized by the nonlinear dependence of the stress tensor on the gradient of the fluid velocity; this encompasses the case of Newtonian as well as non-Newtonian fluids. At the Fluid Solid interface, natural conditions are imposed, continuity of the velocities and of the Cauchy stress forces. The fluid and the solid are coupled through these conditions. By this interaction, the fluid deforms the boundary of the solid which in turn influences the fluid motion. We prove the existence of globally-in-time solution for the problem coupling the linear Lamé system and the quasi-linear Stokes equation. To achieve this, we interpret the solution as the fixed point of some non-linear operator T associated to the global problem. Then we construct, using a regularization procedure, a sequence $(T^\epsilon)_\epsilon$ of auxiliary compact operators that approximate T. Next we establish, using a combination of Banach and Schaeffer fixed point theorems, the existence of fixed points to every operator T^ϵ, these auxiliary fixed point are actually solution of auxiliary problems. Finally we prove that these fixed points converge to the fixed point of T.

1. Introduction

The mathematical and numerical analysis of Fluid—Structure interaction problems have been an important area of research in the recent years, (cf. [5], [9]) and the

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references therein. In the sequel, the fluid-structure interaction problem is denoted by FSI. Let us precisely describe the problem we are interested in. We study a system modeling the interaction between a fluid flow obeying the incompressible Stokes equations and an elastic solid obeying the Lamé system of linear elasticity.

Setting of the main problem. We address the issue of existence of solution for the coupled system which reads:

$$
\begin{cases}
\partial_t v_f - \mathrm{div} a(t, \nabla v_f) - \mu \Delta v_f + \nabla \pi = F \text{ in } \Omega_f^T, & (1.1)\\
\mathrm{div}\, v_f = 0 \text{ in } \Omega_f^T, \quad v_f = 0 \quad \text{on} \quad (0,T) \times (\partial\Omega_f - \Sigma), & (1.2)\\
v_f(0,.) = v_f^0 \quad \text{in} \quad \Omega_f, & (1.3)\\
\displaystyle\int_0^t v_f(r)\mathrm{d}r = u_s(t), \quad (S(v_f,\pi) + a(t,\nabla v_f)) \cdot \overrightarrow{n} = \sigma(u_s) \cdot \overrightarrow{n} \quad \text{on} \quad \Sigma^T, & (1.4)\\
\partial_{tt} u_s - \mathrm{div}\, \sigma(u_s) = 0 \text{ in } \Omega_s^T, & (1.5)\\
u_s = 0 \quad \text{on } (0,T) \times (\partial\Omega_s - \Sigma), & (1.6)\\
u_s(0,.) = 0, \quad \partial_t u_s(0,.) = 0 \text{ in } \Omega_s. & (1.7)
\end{cases}
$$

Let us interpret the symbols and the notations in system (1.1)-(1.7). We let $T > 0$ to be a positive real number. We have denoted $\omega^T := (0,T) \times \omega$. The function u_s denotes the displacement of the solid structure, the function v_f denotes the velocity of the fluid and π denotes its pressure. System (1.1)-(1.7) is formed out of:

- The parabolic Stokes equation with a quasi-linear diffusion term that describes the motion of a fluid inside a fluid domain $\Omega_f \subset \mathbb{R}^2$, cf. equations (1.1)-(1.3).
- The coupling condition, cf. equation (1.4).
- The second order linear hyperbolic Lamé system that describes the deformation, due to interaction with the fluid, of a structure occupying a solid domain $\Omega_s \subset \mathbb{R}^2$, cf. equations (1.5)-(1.7).

We assume that $\Omega := \Omega_s \cup \Omega_f(t) \subset \mathbb{R}^2$, for all $t \in (0,T)$, that is the global domain Ω doesn't vary in time during the interaction. Moreover, the boundaries $\partial\Omega_f$ resp. $\partial\Omega_s$ of the fluid resp. the structure domains are both assumed to meet the minimal C^2-regularity.

The solid domain Ω_s and the fluid domain $\Omega_f(t)$ share a common part of their respective boundaries, this contact interface is denoted Σ i.e. $\partial\Omega_s \cap \partial\Omega_f(t) = \Sigma(t)$, the variation in time of Σ isn't taken into account, cf. the conclusive discussion at the end of this paper. We assume furthermore that Σ is connected. The coupling condition, relating the fluid and solid problems, is prescribed on $(0,T)\times\Sigma$. It consists of imposing, on the contact interface Σ, the two equalities:

$$\begin{aligned} \int_0^t v_f(r,x)\mathrm{d}r &= u_s(t,x) \\ (S(v_f,\pi) + a(t,\nabla v_f))\cdot\overrightarrow{n} &= \sigma(u_s)\cdot\overrightarrow{n}, \quad (t,x)\in\Sigma^T, \end{aligned} \tag{1.8}$$

where $\overrightarrow{n}$ denotes the exterior unit normal defined at each point of Σ. The term $S(v_f,\pi)$ denotes the Cauchy stress tensor:

$$S(v_f,\pi) := -\pi\mathbb{I} + 2\nu\varepsilon(v_f), \tag{1.9}$$

where ν denotes the fluid viscosity and the function $a : [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$ such that $a := (a_1, a_2)$ satisfy (3.2) and other assumptions that will be precised in section 3. On the other hand, $\sigma(u_s)\cdot\overrightarrow{n}$ denotes the normal component of the stress tensor:

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda\mathrm{Tr}\varepsilon(u)\mathrm{Id}, \quad \text{with} \quad \varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right), \tag{1.10}$$

the symbols μ and λ in (1.10) denotes the Lamé coefficients characteristic of the solid medium. Unlike what is usually done regarding the coupling condition, we imposed the equality of the displacements functions, u_s and u_f, on the contact interface instead of the equality of the velocities.

The condition (1.8) is completed by an homogeneous Dirichlet data (1.2), (1.6) on the remaining part of the boundaries of the domain and by initial time conditions (1.3) and (1.7). The fluid is assumed to be divergence-free, this translate the incompressibility of the fluid. We emphasize that, except for the fluid initial condition, (1.3), the restriction to homogeneous data for both Stokes and Lamé problems are adopted only for the sake of simplicity of presentation, one can refer to [5] and the references therein for the case where non homogeneous data are considered but with restrictive assumption on the geometry of the domain, namely the flatness of the contact interface.

Application examples. The two equations involved in the coupled problem (1.1)-(1.7) are the time-dependent Lamé system and the unsteady Stokes equations. It is well known that the Lamé equation models the elastic behavior of materials by considering their characteristic Lamé coefficients. The Lamé coefficients reflect the nature of the material, this flexibility in the choice of these coefficients allows this model to be applied to various situations. On the other hand, the Stokes or Navier-Stokes equations model the flow of a varied number of fluids by considering the linear or non-linear dependence of the viscosity on the strain, which makes it possible to include the study of the Newtonian case or non-Newtonian fluid as in the present study.

The coupling problem has proven very efficient to model which part important fluid-structure interaction phenomena that arise in many practical situations. This is the case, for example, of a dam (concrete dam) which is subjected to water load. In this case, the fluid is Newtonian (water). Another example is that of a high-speed train; HST, subjected to the aerodynamic forces of the wind or an airplane that is subjected to aerodynamic forces (the fluid in this case represents the air). Other important situations can be considered, such as the circulation of blood in a vein, the fluid in this case is non-Newtonian. One can also cite the example of pipelines such as those of gas which exert a constraint on the parishes of the pipelines.

Motivation behind this study. Let's take a closer look at some examples that reflect the importance of the presently addressed study. The case when the structure represents a dam occupying the solid domain Ω_s that has, say, a trapezoidal shape. This dam is subjected to a hydraulic load on its downstream face. The fluid in this case represents water. Let us imagine that the whole of the structure and the hydraulic load are subjected to a seismic excitation, this is interpreted by non-homogeneous boundary conditions for both of the structure (displacement) and the fluid (velocity).

During earthquake exposure (or in the case of ordinary filling of the dam), the fluid and the structure interact. This interaction takes the form of a physical exchange of data. This transmission concerns on the one hand the velocity and on the other hand the constraints. This transmission occurs at the level of the contact interface, this interface is the part of the boundary which is shared by the structure and the fluid. The existence of a solution allows us to proceed to the numerical study, but not only.

Indeed, the idea used in the proof of existence of a solution to the coupled problem is mainly based on the fixed point theory. This idea can be successfully adopted for numerical purposes in that it suggests the method to be used in numerical simulation. Once these numerical data are in hand, we can move on to a very important step, that of adapting the geometry of the solid domain (dam). Indeed, the dam is made of a reinforced concrete structure whose characteristics obey well-established standards, it cannot support a hydraulic load beyond a certain critical threshold. So if the data show that the standards are exceeded, it will be necessary to think of introducing a reinforcement in order to guarantee the equilibrium. We can clearly see that the issue of existence, in addition to being important in itself, it constitutes the support of a numerical and optimization study that is both interesting and applicable.

Indeed, starting from these numerical results, we can decide on the relevance of the model as well as the mechanical and dynamic characteristics of the materials to be adopted. One of the goals of this kind of study is to seek and reach the state of equilibrium during the interaction by adapting the characteristics of the materials, which consists of the Lamé coefficients (for reinforced concrete, type of steel that will be used for the manufacture of the airplane wings, the curvature of the shell, the aircraft fuselage, etc.) These coefficients must be well adapted in order to avoid cracks and overflows which can lead to disasters in the case of the hydraulic dam or an airplane.)

The other type of coefficient to be adapted are the viscosity. For example that of drugs or any other physiological fluids which are naturally dilating or pseudo-plastic. These coefficients should be well suited for medical treatment in case of abnormalities); or for medical reasons, in the case of blood and/or drugs which interact with the the walls of blood vessels and tissues in order to avoid life-threatening or physiological dysfunctions in general.

It should also be noted that the fluid in the example which has just been presented is water, therefore the stress tensor involved in the equation which governs its flow is a linear function of the strain. This being so, one can think of modeling an interaction between a viscous fluid with a solid structure of another type, for example the blood which interacts with the walls of the vessels in human body. Since blood is a pseudo-plastic fluid, a quasi-linear term should be introduced into the equation describing its flow. This is the subject of the stokes equations which is part of the coupled problem (1.1)-(1.7). One should note that what has just been stated for the case of the dam with hydraulic loading is perfectly transposable for other situations of fluid-structure interaction.

Relevance of the non-linearity in the Fluid equation. The new ingredient in the considered Stokes equation is the introduction of the quasi-linear term a. The principal and widely used prototype for such a term is given by the $p-$laplacian:

$$a(t, \nabla v) = |\nabla v|^{p-2} \nabla v.$$

The p-Laplacian operator arises in various fields, such as non-Newtonian fluids, non-linear diffusion problems, filtration of fluids in porous medium. The quantity p is a characteristic of the medium. Three cases can be distinguished. The first case corresponds to $p > 2$: the fluid is called dilatant (like thick suspensions of particles in a liquid). The second case corresponds to $p \in (1, 2)$: the fluid is called pseudo-plastic (like Ice and blood). The last case corresponds to $p = 2$: the fluid is Newtonian. The present study encompasses all the Newtonian, non Newtonian fluid as well as Generalized Newtonian fluid.

In the case $p = 2$, which represents Newtonian fluids, we recover the Stokes equation that describes the flow of an incompressible Newtonian fluid, that is to say whose equation is of the form

$$v_t - \text{div} T(v, \pi) = 0$$

where

$$T(v, \pi) = \nu \varepsilon_{jk}(v) - \delta_{jk} \pi$$

is the usual Cauchy stress tensor and where ε_{jk} is the strain tensor given by (1.10) and $\nu > 0$ denote the fluid viscosity. That is, the stress tensor T depends linearly on the symmetric part of the gradient (i.e. the strain tensor ε). Thus we have in this case $\text{div} T(v, \pi) = \Delta v - \nabla \pi$, cf. [6]. This yields the linear Stokes equation. However, in the case of a Non-Newtonian fluid we have

$$T(u,\pi) = \nu(|\nabla v|^{p-2})\varepsilon_{jk} - \delta_{jk}\pi$$

i.e. the viscosity ν is a function of $|\nabla v|$. We can interpret this in the one-dimensional case, that is to say in the case where the flow of the fluid is uni-directional, say in the $x-$direction, by saying that the deformations in this case reduces to $\varepsilon(v) = \partial_x v$. So the stress tensor depends non-linearly on the strain via the nonlinear dependence of the viscosity on the strain. Such quasi-linear Stokes Problems in one dimension occur in some physical models such as non-Newtonian fluids and chemical reactions, see e.g. [2], [6].

In dimensions two and above, it is possible to consider the nonlinear dependence of the viscosity, and therefore also that of the stress tensor, on the strain tensor in the following way:

$$a(t,\varepsilon(v)) = |\varepsilon(v)|^{p-2}\varepsilon(v)$$

where $\varepsilon(u)$ is the second order strain tensor. Such an operator is monotone, this characteristic is very important because it will be the key ingredient to prove the existence of a solution of the coupled problem (1.1)-(1.7) along with an appropriate energy estimate. The term $a(t,\varepsilon(v))\cdot\overrightarrow{n} = |\varepsilon(v)|^{p-2}\varepsilon(v)\cdot\overrightarrow{n}$ represents the normal component of the tensor of the constraints whose value on the boundary constitutes Neumann data.

The main result of the present work is stated in the following theorem:

Theorem 1.1. *$\forall T > 0$, $\forall F \in L^2(0,T;H^{-1}(\Omega_f))$ and $\forall v_f^0 \in L^2(\Omega_f)$, the coupled problem (1.1)-(1.7) admits at least one solution:*

$$\begin{aligned}(v_f,\pi,u_s) \in & L^2\left(0,T,H^1(\Omega_f)\right)\cap H^1(0,T;H^{-1}(\Omega_f)\times L^2\left(0,T;L^2(\Omega_f)/\mathbb{R}\right)\\ &\times\left(H^1((0,T)\times\Omega_s)\cap H^2(0,T;H^{-1}(\Omega_s))\right).\end{aligned} \tag{1.11}$$

The focus is on establishing existence of globally-in-time solution to the coupled problem (1.1)-(1.7) which is analogous to problem [5, Problem 2.10, p.556]. The interest in the regularity of the solutions is secondary in our current considerations.

Novelty of the study. Our contribution to the analysis of FSI type problems can be summarized by mean of two main ideas. The first one is to have included a non-linear (or quasi-linear) term in the Stokes equation. This has the advantage, as pointed out above, of encompassing non-Newtonian fluids (with a viscosity that depends no linearly on the strain). To our knowledge, the present work is the first to undertake the study of a coupled problem with a nonlinear term in the equation of the fluid part, i.e., which considers a non-Newtonian fluid interacting with a solid structure. Moreover, we prove the globally-in-time existence of at least one solution to the coupled problem while dealing with this non-linearity. To achieve that, we view the solution of (1.1)-(1.7) as a fixed point of some non-linear operator T and we use a regularization method in order to apply fixed point theory. The key ingredient in order to apply such a theory is the fact that the operator defined by the quasi-linear term is monotone. This allows us to use the theory of monotone operator cf. [7], this is the second new original feature of the present work.

Organization of the paper. In the second section we establish a well-posedness result for the Dirichlet problem associated to the Lamé operator along with an inverse estimate for the solid displacement. In the third section we derive an energy estimate for the quasi-linear Stokes system and we introduce the operator $T : \mathcal{X} \to \mathcal{X}$ whose fixed point is a solution of problem (1.1)-(1.7), the space $\mathcal{X}$ is given by (2.6). Next we apply the regularization method to construct a sequence of auxiliary compact operators $(T^\epsilon)_\epsilon$ such that $T = \lim\limits_{\epsilon\to 0} T^\epsilon$, then we establish the boundedness and compactness of T^ϵ using the preceding estimates. In the fourth section, we prove the existence of a fixed point u^ϵ to T^ϵ by combining Banach and Schaeffer fixed point theorems. Finally we conclude by showing that the fixed points u^ϵ converge to a fixed point u^0 of T.

2. Inverse estimate for the Lamé system.

Throughout this section, we let $\Omega_s \subset \mathbb{R}^2$ to be a bounded planar domain with boundary $\partial\Omega_s \in C^2$. We consider the auxiliary Dirichlet problem (2.2) associated to the time dependent second order Lamé operator $\mathcal{H}$ given by:

$$\mathcal{H}u := \partial_{tt}u - \mathrm{div}\sigma(u). \tag{2.1}$$

We prescribe a non-homogeneous Dirichlet condition on $(0,T)\times\Sigma$ and a homogeneous Dirichlet condition on the remaining part of the boundary:

$$\begin{cases} \partial_{tt}u_s - \mathrm{div}\sigma(u_s) = 0 \quad \text{in } (0,T)\times\Omega_s, \\ u_s = u_s^d \quad \text{on } (0,T)\times\Sigma, \\ u_s = 0 \quad \text{on } (0,T)\times(\partial\Omega_s - \Sigma)\,, \end{cases} \tag{2.2}$$

where $T>0$. Moreover, the following initial-time conditions are prescribed:

$$u_s(0) = 0, \quad \partial_t u_s(0) = 0. \tag{2.3}$$

The Dirichlet data u_s^d in (2.2) is assumed to be compatible with (2.3). Let us define the space:

$$\mathcal{D}_s := \{v \in C^\infty(\overline{\Omega_s}),\ \mathrm{supp}v \cap (\partial\Omega_s - \mathring{\Sigma}) = \emptyset\},$$

where $\mathring{\Sigma}$ denotes the topological interior of Σ. We set:

$$\mathcal{U} := \overline{\mathcal{D}_s}^{H^1} \tag{2.4}$$

to be the completion of $\mathcal{D}_s$ with respect to the $H^1(\Omega_s)-$norm. Consider the space:

$$\{\gamma_\Sigma(v):\ v\in\mathcal{U}\} \equiv H_0^{\frac{1}{2}}(\Sigma), \tag{2.5}$$

where $\gamma_\Sigma(v)$ denotes the trace, on the boundary Σ, of the function v. The space $H_0^{\frac{1}{2}}(\Sigma)$ is defined as the completion of $C_0^\infty(\Sigma)$ with respect to the $H^{\frac{1}{2}}(\Sigma)-$norm. Denote $\mathcal{X}$ to be the space:

$$\mathcal{X} := H^{\frac{1}{2}}\left(0,T;L^2(\Sigma)\right) \cap L^2\left(0,T;\ H_0^{\frac{1}{2}}(\Sigma)\right) \subset H^{\frac{1}{2}}((0,T)\times\Sigma), \tag{2.6}$$

and denote $[\mathcal{X}]^*$ its topological dual space. Let us define the operator T_1 by:

$$\begin{aligned} T_1 : \mathcal{X} &\to L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right) \\ u_s^d &\mapsto g_s = T_1(u_s^d), \end{aligned} \tag{2.7}$$

where $H^{-\frac{1}{2}}(\Sigma)$ denotes the dual of $H_0^{\frac{1}{2}}(\Sigma)$. The operator T_1 associates to every Dirichlet data $u_s^d\in\mathcal{X}$ on the solid part of the contact interface, $(0,T)\times\Sigma$, the uniquely determined Neumann data $g_s := \sigma(u_s)\cdot\overrightarrow{n}$ corresponding to the solution u_s of the Dirichlet problem (2.2)-(2.3). In the sequel we will denote equally by u_s the Dirichlet data u_s^d. The main result of this section is given in the following proposition:

Proposition 2.1. *The operator T_1, given by (2.7), is well defined and bounded i.e. there exists a constant $C_s > 0$ such that:*

$$||\sigma(u_s)\cdot\overrightarrow{n}||_{L^2\left(0,T;\ H^{-\frac{1}{2}}(\Sigma)\right)} \leq C_s||u_s||_{\mathcal{X}} \tag{2.8}$$

for all $u_s \in \mathcal{X}$, where $\mathcal{X}$ is defined by (2.6).

Before passing on to the proof of Proposition 2.1, we give a remark stating a lifting property within the context of the Bochner space $L^2(0,T;\mathcal{U})$:

Remark 1. *Consider the map γ_Σ defined by:*

$$\gamma_\Sigma : L^2(0,T;\mathcal{U}) \to L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)$$

$$v \mapsto \gamma_\Sigma v,$$

this map associates to every $v\in L^2(0,T;\mathcal{U})$ its trace on $(0,T)\times\Sigma$. We claim that γ_Σ is onto. Indeed, let $v \in L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)$. The function can be extended by zero to the rest of the boundary. This results in a function belonging to $L^2\left(0,T;H^{\frac{1}{2}}(\partial\Omega_s)\right)$ which we still denote by v. By using the lifting property, one can easily find a family of functions $(0,T)\ni t\mapsto \tilde{v}(t,.)\in H^1(\Omega_s)$ such that $\gamma_{\partial\Omega_s}\tilde{v}(t,.) = v(t,.)$, $\forall t\in(0,T)$, and such that one also has: $||\tilde{v}(t,.)||_{H^1(\Omega_s)} \leq ||v(t,.)||_{H^{\frac{1}{2}}(\Omega_s)}$, $\forall t \in (0,T)$. The existence of a $\tilde{v}$ satisfying such estimate can be established, for instance, within the Banach space $\{w(t,.)\in H^1(\Omega_s):\ \int_{\Omega_s} w(t,x)\Delta_x\phi(x)\mathrm{d}x = 0, \forall\phi\in C_c^\infty(\Omega_s), \forall t\in(0,T)\}$. Thus the function $\tilde{v}$ satisfy:

$$||\tilde{v}||_{L^2(0,T;\mathcal{U})} \leq ||v||_{L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)},$$

and moreover, by combining [3, Theorem 1, p.518] *and* [2, Theorem 2.1, p.731], *we have $\tilde{v}(t,.)\in\mathcal{U}$, $\forall t\in(0,T)$; we thus conclude the surjectivity of γ_Σ.*

Let us state a lemma about a useful existence and regularity result:

Lemma 2.1. *Consider the problem*

$$\begin{cases} \partial_{tt}\phi - \mathrm{div}\sigma(\phi) = F \quad in\ (0,T)\times\Omega_s, \\ \phi = 0 \quad on\ (0,T)\times\partial\Omega_s, \\ \phi(0,.) = 0, \quad \partial_t\phi(0,.) = 0, \end{cases} \tag{2.9}$$

we claim that for every $F \in L^2(0,T;H^{-1}(\Omega_s))$, problem (2.9) admits a unique solution $\phi \in L^2(0,T;H^1(\Omega_s))$ and moreover $\sigma(\phi)\cdot\overrightarrow{n} \in L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega_s))$.

Proof. One can, by density, find a sequence $(F_n)_n$ of elements in $L^2(0,T;L^2(\Omega_s)) \subset L^1(0,T;L^2(\Omega_s))$ such that:

$$||F - F_n||_{L^2(0,T;H^{-1}(\Omega_s))} \to 0. \tag{2.10}$$

Next one can use [10, Theorem 2.1, p.151] to show the existence of a unique solution $\phi_n \in L^2(0,T;H^1(\Omega_s)) =: \Lambda$ to problem (2.9) with F_n as a right hand side instead of F. Starting from the weak formulation of problem (2.9), one writes:

$$< \partial_t\phi_n, \partial_t\psi > + < \varepsilon(\phi_n), \varepsilon(\psi) >_{L^2(0,T;L^2(\Omega_s))} = < F_n, \psi >_{\Lambda^*,\Lambda} \tag{2.11}$$

for all $\psi \in L^2(0,T;H^1_0(\Omega))$. Choosing $\psi = \phi_n$ and using (2.10) we infer easily that

$$||\partial_t\phi_n||_{L^2(0,T;L^2(\Omega_s))} + ||\varepsilon(\phi_n)||_{L^2(0,T;L^2(\Omega_s))} < C \tag{2.12}$$

for some $C > 0$, and deduce the existence of a unique solution $\lim_n \phi_n := \phi \in \Lambda$ to problem (2.9). $\square$

Now we pass into the proof of Proposition 2.1:

Proof. Define the subspace

$$\mathcal{A} := \left\{ \begin{array}{l} v \in H^1((0,T)\times\Omega_s) \cap H^2(0,T;H^{-1}(\Omega_s)) : \\ \int_0^T \int_{\Omega_s} v\mathcal{H}(\phi)\mathrm{d}x\mathrm{d}t = 0, \forall\phi \in C_c^\infty((0,T)\times\Omega_s), \\ ||v(0)||_{L^2(\Omega_s)} = ||\partial_t v(0)||_{H^{-1}(\Omega_s)} = 0, \end{array} \right\} \tag{2.13}$$

where $\mathcal{H}$ is defined in (2.1). Recall that: $H^1(0,T;L^2(\Omega_s)) \cap L^2(0,T;H^1(\Omega_s)) = H^1((0,T)\times\Omega_s)$. One sees that $\mathcal{A}$ is a Banach space when endowed with the norm:

$$||v||_{\mathcal{A}} := ||v||_{H^1((0,T)\times\Omega_s)} + ||\partial_{tt}v||_{L^2(0,T;H^{-1}(\Omega_s))},$$

moreover $\mathcal{A}$ is reflexive. The idea of the proof consists at writing T_1 as a composition $T_1 = N \circ \gamma_0^{-1}$ of two linear operators and then establishing their boundedness. The rest of the proof is divided into two main steps:

Step 1: first inverse estimate. Consider the trace operator:

$$\gamma_0 : \mathcal{A} \to H^{\frac{1}{2}}((0,T)\times\partial\Omega_s)$$

$$u_s \mapsto \gamma_0 u_s,$$

this operator associates to every function $u_s \in \mathcal{A}$ its trace on $(0,T)\times\partial\Omega_s$, the space $\mathcal{A}$ is given by (2.13). The linear operator γ_0 is clearly one-to-one, we claim that it is also onto. Indeed, let $u_s \in H^{\frac{1}{2}}((0,T)\times\partial\Omega_s)$. we show the existence of $U_s \in \mathcal{A}$ such that $u_s = \gamma_0 U_s$. To do this, it suffices to prove that the problem:

$$\begin{cases} \partial_{tt}U_s - \mathrm{div}\sigma(U_s) = 0 \quad \text{in } (0,T)\times\Omega_s, \\ U_s = u_s \quad \text{on } (0,T)\times\partial\Omega_s, \\ U_s(0,.) = 0, \quad \partial_t U_s(0,.) = 0, \end{cases} \tag{2.14}$$

admits a solution $U_s \in \mathcal{A}$. Let $u_s^n \in C^\infty((0,T)\times\partial\Omega_s)$ be such that:

$$||u_s^n - u_s||_{H^{\frac{1}{2}}((0,T)\times\partial\Omega_s)} \to 0; \tag{2.15}$$

such functions do exist by a density argument. Denote U_s^n to be the solution of problem (2.14) corresponding to u_s^n as a Dirichlet data, according to [10, Theorem 2.1, p.151], this problem admits a unique solution $U_s^n \in \mathcal{A}$. The first equation of (2.14) yields us:

$$< U_s^n, \partial_{tt}\phi > - < U_s^n, \mathrm{div}\sigma(\phi) >=< U_s^n, \sigma(\phi)\cdot\overrightarrow{n} >, \tag{2.16}$$

for all $\phi \in L^2(0,T;H^1(\Omega_s))$ that is solution of problem (2.9). Since, by Lemma 2.1, $\sigma(\phi)\cdot\overrightarrow{n} \in L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega_s))$ then using the convergence (2.15) we have:

$$< U_s^n, \mathcal{H}(\phi) >=< U_s^n, \sigma(\phi)\cdot\overrightarrow{n} >\to< u_s, \sigma(\phi)\cdot\overrightarrow{n} >, \tag{2.17}$$

for every $\phi \in L^2(0,T;H^1(\Omega_s))$ solution of problem (2.9) i.e.

$$< U_s^n, F > \quad \text{converges} \quad \forall F \in L^2(0,T;H^{-1}(\Omega_s)). \tag{2.18}$$

Given that $L^2(0,T;H^{-1}(\Omega_s))$ is reflexive, then (2.18) implies:

$$\exists C > 0, \quad ||U_s^n||_{L^2(0,T;H^1(\Omega_s))} \leq C, \quad \forall n, \tag{2.19}$$

and thus $(U_s^n)_n$ converges weakly to $\lim_{n\to\infty} U_s^n =: U_s \in L^2(0,T;H^1(\Omega_s))$. Furthermore, we have:

$$< \partial_{tt} U_s^n, \phi > = < U_s^n, \partial_{tt}\phi >; \tag{2.20}$$

letting $n \to \infty$, we obtain

$$< \partial_{tt} U_s, \phi > = < U_s, \partial_{tt}\phi >, \quad < \partial_t U_s, \partial_t \phi > = < U_s, \partial_{tt}\phi >, \tag{2.21}$$

for all $\phi \in \mathcal{A}$, which implies that $U_s \in \mathcal{A}$.

Following the continuity argument stated in [10, Theorem 2.3, p.153], we see that $||\partial_t U_s^n||_{L^2(0,T;H^{-1}(\Omega_s))}$ is uniformly bounded. Consequently, by combining this last fact and (2.19) we deduce, by invoking the Aubin-Lions lemma and the continuity of the trace operator, that $||u_s^n - U_s||_{L^2(0,T;L^2(\Sigma))} \leq ||U_s^n - U_s||_{L^2(0,T;H^{\frac{1}{2}}(\Omega_s))} \to 0$, this show that $\gamma_0(\lim U_s^n) = \gamma_0(U_s) = u_s$, which concludes the surjectivity of γ_0. We thus infer that the operator γ_0 is an isomorphism. Since the domain and codomain of the operator γ_0 are Banach spaces and since γ_0 is bounded, then by applying the Banach isomorphism theorem, we deduce that the inverse operator γ_0^{-1} is bounded i.e. $\exists C_{\gamma_0^{-1}} > 0$ such that:

$$||u_s||_{\mathcal{A}} \leq C_{\gamma_0^{-1}} ||u_s||_{H^{\frac{1}{2}}((0,T)\times\Sigma)}, \tag{2.22}$$

for all $u_s \in \mathcal{X} \subset H^{\frac{1}{2}}((0,T)\times\Sigma)$.

Step 2: second inverse estimate. Consider the following operator:

$$N : \mathcal{A} \to L^2(0,T;H^{-\frac{1}{2}}(\Sigma))$$

$$u_s \mapsto N(u_s) = g_s,$$

where $\mathcal{A}$ is defined by (2.13). The operator N associates to every displacement $u_s \in \mathcal{A}$ the corresponding Neumann data, $g_s := \sigma(u_s)\cdot\overrightarrow{n}$, on the boundary $(0,T)\times\Sigma$. We claim that $N(\mathcal{A}) \subset L^2(0,T;H^{-\frac{1}{2}}(\Sigma))$. Indeed, using the density of smooth functions in the space $\left(\mathcal{X}, ||\ ||_{H^{\frac{1}{2}}((0,T)\times\Sigma)}\right)$, we can construct with the aid of estimate (2.22) a sequence $(u_s^n)_n$ of elements in $C^\infty((0,T)\times\Omega) \cap \mathcal{A}$ such that:

$$||u_s^n - u_s||_{\mathcal{A}} \to 0 \text{ as } n \to \infty. \tag{2.23}$$

Let $\phi \in L^2(0,T;\mathcal{U})$, where $\mathcal{U}$ is given by (2.4). We integrate by part the first equation in (2.2) against the test function $\phi(t,.) \in \mathcal{U}$ to obtain:

$$\begin{aligned}
&\int_0^T < \partial_{tt} u_s^n, \phi >_{[1]^*,[1]} \mathrm{d}t + \int_0^T \int_{\Omega_s} \varepsilon(u_s^n)\varepsilon(\phi) \, \mathrm{d}x\mathrm{d}t \\
&= \int_0^T < g_s^n, \phi >_{[-\frac{1}{2}],[\frac{1}{2}],\Sigma} \mathrm{d}t,
\end{aligned} \tag{2.24}$$

for every $\phi \in L^2(0,T;\mathcal{U})$, where $g_s^n := \sigma(u_s^n) \cdot \overrightarrow{n} \in C^\infty((0,T) \times \Sigma)$, the bracket $< .,. >_{[\beta]^*,\beta,E}$ denotes the duality pairing between $[H^\beta(E)]^*$ and $H^\beta(E)$. The tensor ε is defined by (1.10). Using (2.23), the expression (2.24), the claim stated in Remark 1 and (2.5), we deduce that $\forall \phi \in L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)$:

$$\left(|\int_0^T < g_s^n, \phi >_{-\frac{1}{2},\frac{1}{2},\Sigma} \mathrm{d}t| \right)_n \text{ is a Cauchy sequence,} \tag{2.25}$$

the completeness of $\mathbb{R}$ yields $\sup_n |\int_0^T < g_s^n, \phi >_{-\frac{1}{2},\frac{1}{2},\Sigma} \mathrm{d}t| < \infty$ for every $\phi \in L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)$. Using the uniform boundedness principle we infer that

$$\sup_n ||g_s^n||_{\mathcal{L}\left(L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right),\mathbb{R}\right)} < \infty, \tag{2.26}$$

given the completeness and the separability of $L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)$, we infer using the fundamental theorem of weak* convergence and estimate (2.26) that the sequence $(g_s^{\alpha(n)})_n$ converges weakly* to some $g_s \in L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$, for some subsequence $(\alpha(n))_n$. Actually one can easily remark, using (2.25), that the whole sequence converges to g_s. We infer that the operator N is well defined. The reader should notice that we have only proved: $N(\mathcal{A}) \subset L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$.

Moreover, the operator N is bounded. Indeed, considering (2.5), we easily infer from the above arguments that the operator N sends every weakly convergent sequence in $\mathcal{A}$ into a weakly* convergent sequence in $L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$. But given the reflexivness of the space $L^2\left(0,T;H_0^{\frac{1}{2}}(\Sigma)\right)$, the weak* convergence and the weak convergence agree. This shows that N is sequentially weakly continuous. Since N is linear, we deduce that it is bounded i.e. $\exists C_N > 0$ such that:

$$||g_s||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)} \leq C_N ||u_s||_{\mathcal{A}}, \tag{2.27}$$

for every $u_s \in \mathcal{A}$.

Finally, by combining (2.22) and (2.27), we infer that $\exists C_s > 0$ such that:

$$||\sigma(u_s) \cdot \overrightarrow{n}||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)} \leq C_s ||u_s||_{H^{\frac{1}{2}}((0,T)\times\partial\Omega_s)}, \tag{2.28}$$

for every $u_s \in H^{\frac{1}{2}}((0,T)\times\Sigma)$. Estimate (2.28) holds for every $u_s \in \mathcal{X} \subset H^{\frac{1}{2}}((0,T)\times\Sigma)$, thus we conclude immediately estimate (2.8). □

3. Estimates for the quasi-linear Stokes problem

Problem setting for the fluid part. Assume $\Omega_f \subset \mathbb{R}^2$ to be a sufficiently smooth domain, say with boundary $\partial\Omega_f = \Sigma \cup (\partial\Omega_f - \Sigma) \in C^2$. We consider the unsteady Stokes operator with a quasi-linear diffusion term appearing in the first equation of (3.1). This operator is endowed, cf. system (3.1), with mixed boundary conditions. We prescribe a non-homogeneous Neumann condition on the contact interface, $(0,T)\times\Sigma$, via the Cauchy stress tensor, and prescribe a homogeneous Dirichlet condition on the remaining part of the boundary. The fluid is assumed to be divergence-free. Let $F \in L^2(0,T;H^{-1}(\Omega_f))$, $g_f \in L^2(0,T;H^{-\frac{1}{2}}(\Sigma))$ and $v_f^0 \in L^2(\Omega_f)$. The fluid part of the coupled problem reads:

$$\begin{cases} \partial_t v_f - \text{div } a(t,\nabla v_f) - \mu\Delta v_f + \nabla\pi = F \quad \text{in } (0,T)\times\Omega_f, \\ \text{div} v_f = 0 \quad \text{in } (0,T)\times\Omega_f, \\ (a(t,\nabla v_f) + S(v_f,\pi)) \cdot \overrightarrow{n} = g_f \quad \text{on } (0,T)\times\Sigma, \\ v_f = 0 \quad \text{on } (0,T)\times(\partial\Omega_f - \Sigma), \\ v_f(0,.) = v_f^0 \quad \text{in } \Omega_f, \end{cases} \tag{3.1}$$

where v_f is the unknown fluid velocity vector, π denotes the unknown pressure and $S(v_f,\pi)$ denotes the Cauchy stress tensor given by (1.9). Let the vector function $a := (a_1,a_2)$ be such that the functions $a_j : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$, with $j = 1,2$, satisfy the assumptions stated in [7, Example 6.A, p.139] in the two dimensional case. One easily sees that these assumptions imply the hypothesis of [7, Proposition 5.1, p.129]. Actually we assume a stronger condition than [7, Condition 6.6.c, p.139], that is: $\exists c_m > 0$ such that:

$$(a(t,\xi) - a(t,\eta))(\xi - \eta) \geq c_m|\xi - \eta|_2^2, \quad \forall\xi,\eta \in \mathbb{R}^2, \forall t \in [0,T]. \tag{3.2}$$

and

$$a(t,0)=0, \forall t>0 \tag{3.3}$$

Consider the space:

$$\mathcal{D}_f := \{v \in C^\infty(\overline{\Omega_f}), \quad \text{div } v=0, \quad \text{supp } v \cap (\partial\Omega_f - \Sigma) = \emptyset\}. \tag{3.4}$$

Denote $\mathcal{V}$ to be the closure of $\mathcal{D}_f$ with respect to the $H^1(\Omega_f)-$norm i.e.

$$\mathcal{V} := \overline{\mathcal{D}_f}^{H^1}, \tag{3.5}$$

the closed subspace $\mathcal{V}$ is endowed with the H^1-norm and thus it is a Hilbert space.

Well-posedness and energy estimate.

Proposition 3.1. *$\forall F \in L^2(0,T;H^{-1}(\Omega_f))$, $\forall v_f^0 \in L^2(\Omega_s)$ and $\forall g_f \in L^2(0,T;H^{-\frac{1}{2}}(\Sigma))$, there exists a unique solution v_f to problem (3.1) such that:*

$$v_f \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;H^{-1}(\Omega_f)); \tag{3.6}$$

moreover, one has the following energy estimate:

$$\begin{aligned} &||\partial_t v_f||_{L^2(0,T;H^{-1}(\Omega_f))} + ||v_f||_{L^2(0,T;\mathcal{V})} \\ &\le C_f(||\,(S(v_f,\pi)+a(t,\nabla v_f))\cdot \overrightarrow{n}||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)} \\ &+ ||F||_{L^2(0,T;H^{-1}(\Omega_f))} + ||v_f^0||_{L^2(\Omega_s)}). \end{aligned} \tag{3.7}$$

The well-posedness result stated in Proposition 3.1 is rather classic. An equivalent weak formulation of (3.1) can be derived by integrating the first equation in (3.1), against $\phi \in L^2(0,T;\mathcal{V})$, to obtain, cf. [11, Problem 3.15, p.371]:

$$(\partial_t v_f,\phi) + \mathcal{A}(v_f)(\phi) - < (a(t,\nabla v_f)+S(v_f,\pi))\cdot\overrightarrow{n},\phi >_{[\frac{1}{2}]^*,[\frac{1}{2}],\Sigma} = < F,\phi >, \tag{3.8}$$

for a.e. time $0 \le t \le T$, where

$$\mathcal{A}(v_f)(\phi) := \int_{\Omega_f} a(t,\nabla v_f)\nabla\phi \mathrm{d}x + \int_{\Omega_f} \varepsilon(v_f)\varepsilon(\phi)\mathrm{d}x. \tag{3.9}$$

We emphasize that, in order to obtain (3.8), we used the fact $\text{div}(\nabla v_f + \nabla^t v_f) = 0$ which holds since $\text{div} v_f = 0$, consequently one has $\text{div} S(v_f,\pi) = \Delta v_f - \nabla\pi$ in this case, cf. [6, Problem 1.1, p.237-240]. Given the assumption (3.2), the operator $\mathcal{A}$ is strongly monotone. Problem (3.8) rewrites:

Find $v_f \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;H^{-1}(\Omega_f))$ s.t.:

$$
\begin{aligned}
&-\int_0^T (v_f, \partial_t \phi)\mathrm{d}t + \int_0^T \mathcal{A}(v_f)(t)(\phi)(t)\mathrm{d}t \\
&= \int_0^T < (a(t,\nabla v_f) + S(v_f,\pi)) \cdot \overrightarrow{n}, \phi(t) >_{-\frac{1}{2},\frac{1}{2},\Sigma} \mathrm{d}t + \int_0^T < F, \phi > \mathrm{d}t,
\end{aligned} \tag{3.10}
$$

$\forall \phi \in L^2(0,T;\mathcal{V})$ with $\partial_t \phi \in L^2(0,T;H^{-1}(\Omega_f))$ and $\phi(T) = 0$. Problem (3.10) fits in the class of quasi-linear parabolic problems. One deals with such a problem using classical arguments, see for instance [7, Porposition 5.1, p.129] and [7, Example 6.A, p.139]. Applying these last results we infer that problem (3.10) admits a unique solution $v_f \in L^2(0,T;\mathcal{V})$ for every $g_f \in L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$. Regarding the existence issue, also cf. [9, Theorem 1.1, p.225]. Furthermore, one can derive the energy estimate (3.7) by choosing as test function $\phi = v_f \in L^2(0,T;\mathcal{V})$ in (3.10) and using the assumptions (3.2) and (3.3).

Remark 2. *Let $v_f \in L^2\left(0,T;H^{\frac{1}{2}}(\Sigma)\right) \underset{continuous}{\hookrightarrow} L^2(0,T;L^2(\Sigma))$. It is a classical fact that v_f can be arbitrarily approximated by an element $v_f^n \in C_0^\infty\left(0,T;H^{\frac{1}{2}}(\Sigma)\right)$ w.r.t. the norm of the space $L^2\left(0,T;H^{\frac{1}{2}}(\Sigma)\right)$. On another hand, by applying the Poincaré inequality in the time variable with v_n, we can easily show that:*

$$
\int_0^t ||u_f^n(r)||^2_{L^2(\Sigma)}\mathrm{md}r \le C_p \int_0^t ||v_f^n(r)||^2_{L^2(\Sigma)}\mathrm{d}r + ||u_f^n(0)||_{L^2(\Sigma)}. \tag{3.11}
$$

On the other hand:

$$
\begin{aligned}
&\int_0^T |u_f^n(t)|^2_{H^{\frac{1}{2}}(\Sigma)}\mathrm{d}t := \int_\Sigma \int_\Sigma \int_0^T \frac{|u_f^n(t,x) - u_f^n(t,y)|^2}{|x-y|^2}\mathrm{d}t\mathrm{d}x\mathrm{d}y \\
&\le C_p \int_\Sigma \int_\Sigma \int_0^T \frac{|v_f^n(t,x) - v_f^n(t,y)|^2}{|x-y|^2}\mathrm{d}t\mathrm{d}x\mathrm{d}y + T|u_f^n(0)|^2_{H^{\frac{1}{2}}(\Sigma)}.
\end{aligned} \tag{3.12}
$$

Combining estimates (3.11), (3.12) and using $u_f(0,x) = 0$ for $x \in \Sigma$, then letting $n \to \infty$:

$$
\int_0^T ||u_f(r)||^2_{H^{\frac{1}{2}}(\Sigma)}\mathrm{d}r \le C_p \int_0^T ||v_f(r)||^2_{H^{\frac{1}{2}}(\Sigma)}\mathrm{d}r. \tag{3.13}
$$

The same conclusion holds in case of fractional Sobolev spaces $H^s(\Sigma)$ with $s \in \mathbb{R}_+^*$.

We infer from Remark 2, that the fluid displacement u_f satisfies:

$$
u_f \in L^2\left(0,T;H^{\frac{1}{2}}(\Sigma)\right) \cap H^1\left(0,T;L^2(\Sigma)\right) \subset H^{\frac{1}{2}}((0,T)\times\Sigma). \tag{3.14}
$$

The idea. Let us explain the main idea of this section. Define T_2 to be the operator:

$$
\begin{aligned}
T_2 : L^2\left(0, T : H^{-\frac{1}{2}}(\Sigma)\right) &\rightarrow \mathcal{X} \subset H^{\frac{1}{2}}((0,T)\times\Sigma) \\
g_f &\mapsto T_2(g_f) = u_f,
\end{aligned} \tag{3.15}
$$

this operator associates to every Neumann data, on the fluid part of the contact interface, the displacement u_f corresponding to the velocity v_f which is a solution of (3.1). It is easily seen, by combing (3.7), (3.13) and by applying (3.14), that:

$$
\begin{aligned}
&||T_2(g_f^1) - T_2(g_f^2)||_{H^{\frac{1}{2}}((0,T)\times\Sigma)} \\
&= ||u_f^1 - u_f^2||_{H^{\frac{1}{2}}(0,T;L^2(\Sigma))} + ||u_f^1 - u_f^2||_{L^2(0,T;H^{\frac{1}{2}}(\Sigma))} \\
&\leq C_p||v_f^1 - v_f^2||_{L^2(0,T;H^1(\Omega_f))} \\
&\leq C_f||g_f^1 - g_f^2||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)}
\end{aligned} \tag{3.16}
$$

for every $g_f^1, g_f^2 \in L^2\left(0, T; H^{-\frac{1}{2}}(\Sigma)\right)$, and thus the operator T_2 is continuous. Let T_1 and T_2 be defined respectively by (2.7) and (3.15). Define the operator:

$$
\begin{aligned}
T : \mathcal{X} &\rightarrow \mathcal{X} \subset H^{\frac{1}{2}}((0,T)\times\Sigma) \\
u_s &\mapsto T(u_s) := T_2 \circ T_1(u_s) = u_f,
\end{aligned} \tag{3.17}
$$

We remark that the global solution of the coupled problem (1.1)-(1.7) is a fixed point of T, then to show existence of a solution to (1.1)-(1.7) it suffices to prove existence of a fixed point of the of the operator T. To be able to use fixed point theory we need some compactness. However, T sends solid displacements from : $\mathcal{X} \subset H^{\frac{1}{2}}((0,T)\times\Sigma)$ into no more spatial-regular fluid displacements, that is into: $\mathcal{X} \subset H^{\frac{1}{2}}((0,T)\times\Sigma)$. In order to recover some compactness we need to consider a sequence of auxiliary operators T_2^ϵ. To achieve that, we proceed into a regularization of the Stokes problem (3.10) i.e. to define a sequence of problems depending on a small real parameter, $\epsilon > 0$, in such a way that the new operator $T_2^\epsilon \circ T_1$ sends $H^{\frac{1}{2}}((0,T)\times\Sigma)$ into a more regular space in the spatial variable, this will ensure the needed compactness. The (solution of the) original problem will be recovered by letting $\epsilon \rightarrow 0$.

Regularized problem. Consider the space:

$$
\mathcal{W} := \overline{\mathcal{D}_f}^{H^2} \tag{3.18}
$$

to be the completion of $\mathcal{D}_f$, defined by (3.4), with respect to the Sobolev H^2-norm. It is indeed a Hilbert spaces. We denote by $(.,.)_{H^2}$ its canonical inner product, and by $(.,.)_{H^2_{sn}}$ the part of $(.,.)_{H^2}$ that involves only the second derivatives. We denote $\mathcal{W}^*$ its dual. Given $g_f \in L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$, consider the regularized problem:

Find $v_f^\epsilon \in L^2(0,T;\mathcal{W})$ such that:

$$\begin{aligned}&\int_0^T (\partial_t v_f^\epsilon,\phi)_{L^2}\mathrm{d}t + \mathcal{A}_\epsilon(v_f^\epsilon)(\phi)\\ &= \int_0^T < F(t),\phi(t) >_{-1,1,\Omega_f}\,\mathrm{d}t \int_0^T < g_f(t),\phi(t) >_{-\frac{1}{2},\frac{1}{2},\Sigma}\,\mathrm{d}t,\end{aligned} \tag{3.19}$$

for all $\phi \in L^2(0,T;\mathcal{W})$, where

$$\mathcal{A}_\epsilon(v_f^\epsilon)(\phi) := \int_0^T \mathcal{A}(v_f^\epsilon)(\phi)\mathrm{d}t + \epsilon\int_0^T\int_{\Omega_f}(v_f^\epsilon,\phi)_{H^2_{sn}}\,\mathrm{d}x\mathrm{d}t,$$

and where $\mathcal{A}$ is given by (3.9). It is easily seen that the quasi-linear elliptic operator $\mathcal{A}_\epsilon : \mathcal{W}\to\mathcal{W}^*$ satisfy the assumptions of [7, Proposition 5.1, p.129] and that of [7, Theorem 5.1, p.128]. Then for every $g_f \in L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$, problem (3.19) admits a unique solution:

$$v_f^\epsilon \in L^2(0,T;\mathcal{W}) \tag{3.20}$$

for all $\epsilon > 0$, where $\mathcal{W}$ is given by (3.18). An energy estimate can be derived by using the strong monotony of the operator $\mathcal{A}_\epsilon$. We thus obtain the estimate:

$$||v_f^{1,\epsilon} - v_f^{2,\epsilon}||_{L^2(0,T;\mathcal{W})} \le C_f||g_f^1 - g_f^2||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)}. \tag{3.21}$$

Estimate (3.21) is obtained by choosing as test function $\phi = v_f^{1,\epsilon} - v_f^{2,\epsilon}$, in (3.19), corresponding to g_f^1, g_f^2. Using the continuity of the trace operator

$$\gamma_0 : H^2(\Omega_f) \to H^{\frac{3}{2}}(\Sigma),$$

we infer from (3.21):

$$\begin{aligned}&||v_f^{1,\epsilon} - v_f^{2,\epsilon}||_{L^2\left(0,T;H^{\frac{3}{2}}(\Sigma)\right)}\\ &\le C_f||g_f^1 - g_f^2||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)}\end{aligned} \tag{3.22}$$

Remark 3. *The same fact that was stated in Remark 2 holds with u_f^ϵ and v_f^ϵ in the space $L^2\left(0,T;H^{\frac{3}{2}}(\Sigma)\right)$. Thus we obtain by combining (3.13) corresponding to the space $H^{\frac{3}{2}}(\Sigma)$ and estimate (3.22):*

$$
\begin{aligned}
&||u_f^{\epsilon,1}-u_f^{\epsilon,2}||_{L^2(0,T;H^1(\Sigma))}+||u_f^{\epsilon,1}-u_f^{\epsilon,2}||_{H^1(0,T;L^2(\Sigma))}\\
&\leq C_f||g_f^1-g_f^2||_{L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)},
\end{aligned}
\tag{3.23}
$$

for every $g_f^1, g_f^2 \in L^2\left(0,T:H^{-\frac{1}{2}}(\Sigma)\right)$.

The auxiliary operator T^ϵ. First define $\mathcal{Z}$ to be the space:

$$
\mathcal{Z}:=H^1\left(0,T;L^2(\Sigma)\right)\cap L^2\left(0,T;H_0^1(\Sigma)\right)\subset H^1((0,T)\times\Sigma),
\tag{3.24}
$$

it is endowed with the standard $H^1((0,T)\times\Sigma)-$norm. Define the operator T_2^ϵ by:

$$
\begin{aligned}
T_2^\epsilon: L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)&\rightarrow\mathcal{Z}\subset\mathcal{X}\\
g_f&\mapsto T_2^\epsilon(g_f)=u_f^\epsilon,
\end{aligned}
\tag{3.25}
$$

i.e. it associates to every $g_f\in L^2\left(0,T;H^{-\frac{1}{2}}(\Sigma)\right)$, the trace on $(0,T)\times\Sigma$ of the fluid displacement u_f^ϵ corresponding to the fluid velocity v_f^ϵ which is the solution of the regularized problem (3.19) with g_f as a Neumann data. Estimate (3.23) translates the boundedness of T_2^ϵ. Let us introduce the auxiliary operator $T^\epsilon:=T_2^\epsilon\circ T_1$, that is:

$$
\begin{aligned}
T^\epsilon:\mathcal{X}&\rightarrow\mathcal{Z}\subset\mathcal{X}\\
u_s&\mapsto u_f^\epsilon=T_2^\epsilon\circ T_1(u_s),
\end{aligned}
\tag{3.26}
$$

where $\mathcal{X}$ and $\mathcal{Z}$ are respectively given by (2.6) and (3.24), moreover the operators T_1 and T_2^ϵ are respectively defined by (2.7) and (3.25). The operator T^ϵ associates to every solid displacement $u_s\in\mathcal{X}$, the fluid displacement $u_f^\epsilon\in\mathcal{X}$ defined on the fluid part of the contact interface $(0,T)\times\Sigma$.

The auxiliary coupled problem is formed out of the Lamé solid problem (2.2) and the quasi-linear Stokes regularized problem (3.19). By combining the estimates (2.8) and (3.23) on one hand, and using the coupling condition (1.8) on the other hand, we infer the estimate:

$$
\forall\epsilon>0,\quad ||T^\epsilon(u_s^1)-T^\epsilon(u_s^2)||_{\mathcal{X}}\leq C_sC_f||u_s^1-u_s^2||_{\mathcal{X}}\ ,
\tag{3.27}
$$

for all $u_s^1, u_s^2 \in \mathcal{X}$ i.e. the non-linear operator T^ϵ is Lipschitz for every $\epsilon > 0$. One should notice that the constant $C_s C_f$ doesn't depend on ϵ.

4. Existence of solution for the main coupled problem

4.1. **Existence result for the auxiliary coupled problem.** Let T^ϵ be as defined by (3.26) and $\mathcal{X}$ be as defined by (2.6). We propose to prove the following proposition:

Proposition 4.1. *For every $\epsilon > 0$, the operator T^ϵ admits a fixed point $u^\epsilon \in \mathcal{X}$.*

The idea for proving Proposition 4.1 is to combine the Banach and the Schaefer fixed point theorems. Let us recall Schaefer's theorem, cf. [8, Theorem 4.3.2, p.29]:

Theorem 4.1. *(Schaefer). Let $\mathcal{K}$ be a Banach space. Let $T : \mathcal{K} \to \mathcal{K}$ be a continuous and compact mapping. Assume that the set*

$$\{v \in \mathcal{K} :\ v = \rho T v \text{ for some } \rho,\ 0 \le \rho \le 1\}$$

is bounded, then T admits a fixed point in $\mathcal{K}$.

We are going now to prove Proposition 4.1:

Proof. We deal at first with a contraction mapping. Fix $\rho \in \mathbb{R}$ such that $\rho C_s C_f < 1$. The coefficient ρ doesn't depend on ϵ. Consider the operator ρT^ϵ defined by:

$$\rho T^\epsilon : \mathcal{X} \to \mathcal{X} \cap \mathcal{Z} \subset \mathcal{X}$$
$$u_s \mapsto u_f^\epsilon = \rho T_2^\epsilon \circ T_1(u_s),$$

where $\mathcal{Z}$ is defined by (3.24). Applying estimate (3.27), we have:

$$||\rho T^\epsilon(u_s^1) - \rho T^\epsilon(u_s^2)||_{\mathcal{X}} \le \rho C_s C_f ||u_s^1 - u_s^2||_{\mathcal{X}}, \tag{4.1}$$

for all $u_s^1, u_s^2 \in \mathcal{X}$. Let us note the following four facts:

- *Continuity of the operators T^ϵ.* Estimate (3.27) means that T^ϵ is Lipschitz and thus continuous. Moreover, estimate (4.1) implies that ρT^ϵ is a contraction mapping for every $\epsilon > 0$. Consequently, since $\mathcal{X}$ is a Banach space, we deduce, by applying the Banach fixed point theorem, that ρT^ϵ admits a unique $[0, T]-$globally defined fixed point $u_\rho^\epsilon \in \mathcal{X}$.

- *Boundedness condition.* We deduce from the preceding point that, $\forall \epsilon > 0$, the set $\{u \in \mathcal{X},\ u = \rho T^\epsilon u\ \}$ reduces to the unique fixed point u^ϵ_ρ, and thus it is bounded.
- *Stability of the operators T^ϵ.* According to definition (3.26), we have $T^\epsilon(\mathcal{X}) \subset \mathcal{X}$, thus T^ϵ is stable for every $\epsilon > 0$.
- *Compactness of the operator T^ϵ.* Let $\epsilon > 0$. By combining estimates (2.8), (3.23) and the boundedness of $||T^\epsilon(0)||_{\mathcal{Z}}$, we remark that the operator T^ϵ sends every bounded subset $E \subset \mathcal{X}$ into an H^1-bounded subset $T^\epsilon(E) \subset H^1((0,T)\times\Sigma)$. Using the compact embedding $H^1((0,T)\times\Sigma) \underset{compact}{\hookrightarrow} H^{\frac{1}{2}}((0,T)\times\Sigma)$, we deduce that $T^\epsilon(E)$ is compact in $\mathcal{X}$, thus T^ϵ is compact $\forall\epsilon > 0$.

Thus we checked for T^ϵ the sufficient conditions of Theorem 4.1; this proves Proposition 4.1. □

4.2. **Existence of a solution to the coupled problem (1.1)-(1.7).** We now establish the existence of a solution for the global coupled problem which, as pointed out above, amounts at establishing the existence of a fixed point of the operator T given by (3.17). According to Proposition 4.1, $\forall\epsilon > 0$, there exists $u^\epsilon \in \mathcal{X}$ such that

$$T^\epsilon u^\epsilon = u^\epsilon, \tag{4.2}$$

where $\mathcal{X}$ is given by (2.6). Since $(u^\epsilon)_{\epsilon>0}$ is an uncountable family of vectors belonging to the separable normed vector space $(\mathcal{X}, ||\ ||_{\mathcal{X}})$, then it must have a limit point with respect to the topology induced by the norm of $\mathcal{X}$ i.e. there exists $u^0 \in \mathcal{X}$ such that:

$$||u^\epsilon - u^0||_{\mathcal{X}} \to 0, \tag{4.3}$$

as $\epsilon \to 0$. To be able to pass to the limit $\epsilon \to 0$ in (4.2), we need Lemma 4.1:

Lemma 4.1. *Let T^ϵ and T be respectively as defined by (3.26) and (3.17), then:*

$$T^\epsilon u \underset{strongly\ in\ \mathcal{X}}{\to} Tu, \quad \forall u \in \mathcal{X}. \tag{4.4}$$

Proof. Fix $u \in \mathcal{X}$ and pose $g_f = T_1 u$, where T_1 is given by (2.7). Choose in (3.19) $\phi = u^\epsilon_f$ where $u^\epsilon_f = T^\epsilon_2(g_f)$ and where T^ϵ_2 is given by (3.25). Using (3.23), we infer immediately that $T^\epsilon u \underset{weakly\ in\ \mathcal{X}}{\rightharpoonup} U_f$ with $U_f \in \mathcal{X}$, let us show that $U_f = Tu$.

Denote v_f^ϵ to be the velocity $v_f^\epsilon := \partial_t u_f^\epsilon$. It is clear, by using (3.21) with $g_f^2 = 0$ and $g_f^1 = g_f$, that $\left(||v_f^\epsilon||_{L^2(0,T;H^1(\Omega_f))}\right)_\epsilon$ is uniformly bounded, thus:

$$v_f^\epsilon \underset{weakly}{\rightharpoonup} V_f \quad \text{in } L^2(0,T;H^1(\Omega_f)). \tag{4.5}$$

Moreover, by combining (3.21) and (4.5) we infer that both $\epsilon \int_0^T |v_f^\epsilon(t)|^2_{H^2_{sn}(\Omega_f)} \mathrm{d}t := \epsilon \int_0^T (v_f^\epsilon, v_f^\epsilon)_{H^2_{sn}} \mathrm{d}t$ and $||\partial_t v_f^\epsilon||_{L^2(0,T;H^{-1}(\Omega_f))}$ are also uniformly bounded with respect to $\epsilon > 0$, consequently:

$$\begin{aligned} \partial_t v_f^\epsilon &\underset{weakly}{\rightharpoonup} \partial_t V_f \quad \text{in } L^2(0,T;H^{-1}(\Omega_f)), \\ \epsilon \partial^2_{x_i x_j} v_f^\epsilon &\underset{weakly}{\rightharpoonup} 0 \quad \text{in } L^2(0,T;L^2(\Omega_f)). \end{aligned} \tag{4.6}$$

Combining the weak problems (3.10), (3.19) and applying the last convergence in (4.6) on one hand, and by considering the auxiliary function u^n corresponding to problem (3.10) with f^n an dg^n as right hand sides such that $||f^n - f|| \to 0$ and $||g^n - g|| \to 0$, we deduce that:

$$||\nabla v_f^\epsilon - \nabla v_f||_{L^2(0,T;L^2(\Omega_f))} \to 0, \tag{4.7}$$

it follows that $V_f = v_f$. Given that a is Lipschitz in the second variable, then (4.7) yields:

$$a(t, \nabla v_f^\epsilon) \to a(t, \nabla V_f) \quad \text{converges strongly.} \tag{4.8}$$

Finally, letting $\epsilon \to 0$ in (3.19) by mean of combining (4.5), (4.6) and (4.8), one infers that V_f is the fluid velocity field corresponding to the solid displacement $u \in \mathcal{X}$. Consequently we have $\partial_t u = V_f$ and $U_f = Tu$, this yields convergence (4.4). $\square$

Proof of Theorem 1.1. Now we are ready to present a proof of the main result:

Proof. Let u^0 be such as defined by (4.3), we have:

$$\begin{aligned} &||T^\epsilon u^\epsilon - Tu^0||_{\mathcal{X}} \\ &\leq ||T^\epsilon u^\epsilon - T^\epsilon u^0|| + ||T^\epsilon u^0 - Tu^0||. \end{aligned} \tag{4.9}$$

On one hand we have:

$$\begin{aligned} &||T^\epsilon u^\epsilon - T^\epsilon u^0||_{\mathcal{X}} \\ &\leq C_s C_f ||u^\epsilon - u^0||_{\mathcal{X}}, \end{aligned}$$

where we used (3.27). Applying (4.3), we infer:

$$||T^\epsilon u^\epsilon - T^\epsilon u^0||_{\mathcal{X}} \to 0 \quad \text{as } \epsilon \to 0. \tag{4.10}$$

On another hand, using (4.4), we obtain:

$$||T^\epsilon u^0 - T^0 u^0|| \to 0, \ \epsilon \to 0. \tag{4.11}$$

Combining (4.10) and (4.11) it yields

$$||T^\epsilon u^\epsilon - Tu^0||_{\mathcal{X}}, \ \epsilon \to 0. \tag{4.12}$$

Combining (4.2), (4.3) and (4.12), we deduce:

$$u^0 = \lim_{\epsilon \to 0} u^\epsilon = \lim_{\epsilon \to 0} T^\epsilon u^\epsilon = Tu^0, \tag{4.13}$$

that is T has a fixed point $u^0 \in \mathcal{X}$. This completes the proof of Theorem 1.1.

Regarding the regularity claimed in Theorem 1.1, one can use estimate (2.22) to infer the regularity of the solid displacement $u_s \in \mathcal{X}$ on Ω_s. Furthermore, one combines the estimates (3.6) and (2.8) to infer the regularity of the fluid velocity v_f on Ω_f. The regularity of the pressure π can be inferred in the following fashion: one considers the regularity of the solution u^0 stated in (3.6), namely that $\partial_t u^0 \in L^2(0, T; H^{-1}(\Omega_f))$ and then applies the energy estimate in [1, Theorem 25, p.226] and thus infer that $\pi \in L^2(0, T; L^2(\Omega_f)/\mathbb{R})$. □

5. Conclusion

According to the procedure adopted by the authors in [5], cf. the bottom of [5, Proof of Theorem 5.1, p.571], they established the globally-in-time existence of a solution by using an iterative method which is based on the linearity of the problem that was considered in their paper. We emphasize that this method is no longer applicable in the non-linear context of the present work. Indeed, it is always possible to prove existence of a solution locally in time, i.e., given a non-linear problem consider on the time interval $[0, T]$, fixed point theory can lead to the existence of a solution on $[0, T_i] \subset [0, T]$. Using the linearity of the differential equations, one can iterate the process to obtain a global in time solution. Given that the problem considered in this paper is non-linear, we had to combine two fixed point theorems, the Banach and Schaeffer theorems.

It should be noted that there is no memory effect in the (iterative) resolution of the presently addressed coupled problem in the sense that the result (in particular the displacement) obtained at a time t does not depend on the displacement at an earlier time. This is notably due to the fact that the deformation of the geometry (in particular that of the contact surface Σ), within time incrementing i.e. during the coupling, is not taken into account, although we assumed in the introduction that the domain $\Omega_f(t)$ depends upon time.

That said, the main theorem established in this paper remains relevant. Indeed, it can be incorporated as an auxiliary result to demonstrate a more complex well-posedness result like for instance [5, Theorem 2.1, p.555]. If one is willing to prove a result analogous to the later in the framework considered in this paper, then one has, in a first step, to generalize Theorem 1.1 to the case when the data in system (1.1)-(1.7) are non-homogeneous along with a non necessary free divergence condition. We believe that this step can be achieved with some slight modification of the procedure described in the present work, one obtains up to this step a result analogous to [5, Theorem 5.1, p.570]. Then one should follow the same method as in the proof of [5, Theorem 2.1, p.555] which deals with the well-posedness of system [5, Problem 2.1, p.551-552] and which requires taking into account the deformation of the geometry during the interaction, especially that of the contact interface.

Finally, a last perspective would be to improve the regularity of the solution of the coupled problem. This has been successfully undertaken in other works assuming that the contact interface Σ^T is flat. This led the authors, by using Fourier analysis techniques, to show a "hidden" regularity. It would be interesting to establish such an additional regularity without restrictive assumption on the shape of the contact interface. This would be an interesting extension of our result.

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