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Γ-BE-ALGEBRAS

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Abstract. We introduce the concept of Γ-BE-algebra as a generalization of a BE-algebra and study its properties. For a fixed binary operation α in a set Γ of binary operations on a non-empty set X, we introduce the concepts of β-reflexive, β-transitive, β-antisymmetric, α-transitive, α-left distributive and γ-left distributive Γ_α-BE-algebra and study the relations between them. We introduce the concept of β-subalgebra and β-filter in a Γ_α-BE-algebra and study the relation between α-subalgebra, α-filter and β-subalgebra in a Γ_α-BE-algebra.

1. Introduction

BE-algebras were introduced by H. S. Kim and Y. H. Kim as a generalization of BCK-algebras(see [4]). Later several authors introduced and studied several substructures of BE-algebras(see [2, 3, 5]). In 1964, N. Nobusawa studied Γ-rings(see [8]). Later, in 1966, Barnes weakened the defining conditions of Γ-ring and studied the resulting structures(see [1]). Γ-rings were also studied by T. S. Ravisankar and U. S. Shukla(see [9]). M. K. Sen and N. K. Saha, studied Γ-semigroups and obtained various generalizations and analogues of corresponding results from semigroup theory(see [10]). Later M. K. Rao introduced the concept of Γ-semiring as a generalization of Γ-ring and studied various types of Γ-semirings(see [6, 7]. This motivated us to introduce the concept of Γ-BE-algebra as a generalization of a BE-algebra and investigate its properties.

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Most algebraic structures are studied in environments where only one binary operation is given, and so is BE-algebra. However, it can be seen that Γ-semigroups, Γ-semirings, and Γ-rings are being studied as algebraic structures with two or more binary operations. We regard these as generalizations of semigroups, semirings, and rings, respectively. As one of the generalizations of BE-algebra, the purpose of this paper is to study BE-algebras with two or more binary operations. We introduce the concept of transitive Γ-BE-algebra and study its properties. For a fixed binary operation α in a set Γ of binary operations on a non-empty set X, we introduce the concepts of β-reflexive, β-transitive, β-antisymmetric, α-transitive, α-left distributive and γ-left distributive Γ_α BE-algebra and study the relations between them. We introduce the concepts of β-subalgebra and β-filter in a Γ_α-BE-algebra and study the relations between α-subalgebra, α-filter and β-subalgebra in a Γ_α-BE-algebra. For every $x, y \in X$, we consider the sets $E_\alpha(x) := \{y \in X \mid x \leq_\alpha y\}$ and $E_\alpha^\beta(x, y) := \{z \in X \mid x \leq_\alpha y\beta z\}$ in a Γ_α-BE-algebra and we observe that $E_\alpha^\beta(x, y)$ is neither an α-filter nor a β-filter of a Γ_α-BE-algebra. We give sufficient conditions for a subset $E_\alpha^\beta(x, y)$ to be an α-filter and a β-filter in a Γ_α-BE-algebra.

2. Preliminaries

A *BE-algebra*, denoted by $(X, 1)_*$, (see [4]) is defined to be a set X together with a binary operation "$*$" and a special element "1" satisfying the conditions:

(BE1) $(\forall a \in X)\ (a * a = 1)$,

(BE2) $(\forall a \in X)\ (a * 1 = 1)$,

(BE3) $(\forall a \in X)\ (1 * a = a)$,

(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

Every BE-algebra X satisfies the following conditions (see [4]):

$$(\forall a, b \in X)\,(a * (b * a) = 1)\,. \tag{2.1}$$

$$(\forall a, b \in X)\,(a * ((a * b) * b) = 1)\,. \tag{2.2}$$

A subset A of a BE-algebra $(X, 1)_*$ is called

- a *subalgebra* of X if it satisfies:

$$(\forall a, b \in A)(a * b \in A), \tag{2.3}$$

- a *filter* of X (see [4]) if it satisfies:

$$1 \in A, \tag{2.4}$$

$$(\forall a, b \in X)(a * b \in A,\ a \in A \ \Rightarrow\ b \in A). \tag{2.5}$$

3. Γ-BE-algebras

Let X be a nonempty set and let Γ be a set of binary operations on X, that is, every element $\alpha \in \Gamma$ is given as follows:

$$\alpha : X \times X \to X,\ (x, y) \mapsto \alpha(x, y). \tag{3.1}$$

In what follows, $\alpha(x, y)$ is denoted by $x\alpha y$, and let $(X, 1)_\Gamma$ be a structure related to a special element "1" and Γ.

Definition 3.1. For a fixed $\alpha \in \Gamma$, a structure $(X, 1)_\Gamma$ is called a *Γ_α-BE-algebra* if it satisfies:

$$(\forall x \in X)(x\alpha x = 1), \tag{3.2}$$

$$(\forall x \in X)(x\alpha 1 = 1), \tag{3.3}$$

$$(\forall x \in X)(1\alpha x = x), \tag{3.4}$$

$$(\forall x, y, z \in X)(\forall \beta \in \Gamma)(x\alpha(y\beta z) = y\beta(x\alpha z)). \tag{3.5}$$

Example 3.1. *Let $X = \{0, 1, 2\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.*

α	0	1	2
0	1	1	1
1	0	1	2
2	1	1	1

β	0	1	2
0	0	1	2
1	1	1	2
2	0	1	0

γ	0	1	2
0	2	1	0
1	0	1	2
2	2	1	2

Then we can easily verify that $(X, 1)_\Gamma$ is a Γ_α-BE-algebra. But $(X, 1)_\Gamma$ is neither a Γ_β-BE-algebra nor a Γ_γ-BE-algebra since $2\beta 2 = 0 \neq 1$ and $2\gamma 2 = 2 \neq 1$.

From Example 3.1, we know that if $\alpha \neq \delta$ in Γ, then a Γ_α-BE-algebra may not be a Γ_δ-BE-algebra.

If $(X, 1)_\Gamma$ is a Γ_α-BE-algebra for all $\alpha \in \Gamma$, we say that $(X, 1)_\Gamma$ is a *Γ-BE-algebra*. That is,

Definition 3.2. A structure $(X,1)_\Gamma$ is called a *Γ-BE-algebra* if it satisfies:

$$(\forall x \in X)(\forall \beta \in \Gamma)(x\beta x = 1), \tag{3.6}$$

$$(\forall x \in X)(\forall \beta \in \Gamma)(x\beta 1 = 1), \tag{3.7}$$

$$(\forall x \in X)(\forall \beta \in \Gamma)(1\beta x = x), \tag{3.8}$$

$$(\forall x, y, z \in X)(\forall \alpha, \beta \in \Gamma)(x\alpha(y\beta z) = y\beta(x\alpha z)). \tag{3.9}$$

For a fixed $\alpha \in \Gamma$, we define a binary relation " $\leq_\alpha$ " on X as follows:

$$(\forall x, y \in X)(x \leq_\alpha y \;\Leftrightarrow\; x\alpha y = 1). \tag{3.10}$$

If $x \leq_\alpha y$ is valid for all $\alpha \in \Gamma$, we present it as $x \leq_\Gamma y$.

Example 3.2. *(1) Every BE-algebra $(X,1)_*$ is a Γ-BE-algebra with $\Gamma = \{*\}$.*

(2) If $|\Gamma| = 2$, then evey pseudo BE-algebra (see [3, 2]) is a Γ-BE-algebra.

(3) Let $X = \{0,1,2,3\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.

α	0	1	2	3
0	1	1	0	1
1	0	1	2	3
2	1	1	1	3
3	0	1	2	1

β	0	1	2	3
0	1	1	0	1
1	0	1	2	3
2	1	1	1	3
3	1	1	0	1

γ	0	1	2	3
0	1	1	0	3
1	0	1	2	3
2	1	1	1	3
3	1	1	0	1

Then $(X,1)_\Gamma$ is a Γ-BE-algebra.

Example 3.3. *Let X be the set of all real numbers greater than or equal to 1 and consider $\Gamma := \{\alpha, \beta\}$ which are given as follows:*

$$\alpha : X \times X \to X,\ (x,y) \mapsto \begin{cases} y & \text{if } x = 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\beta : X \times X \to X,\ (x,y) \mapsto \begin{cases} 1 & \text{if } y = 1 \text{ or } x = y, \\ x & \text{otherwise,} \end{cases}$$

It is easy to verify that $(X,1)_\Gamma$ is a Γ_α-BE-algebra. But it is not a Γ_β-BE-algebra because of $2\beta(3\beta 5) = 2\beta 3 = 2 \neq 3 = 3\beta 2 = 3\beta(2\beta 5)$.

Proposition 3.1. *Every Γ_α-BE-algebra $(X,1)_\Gamma$ satisfies:*

$$(\forall x \in X)(x \leq_\alpha x,\ x \leq_\alpha 1). \tag{3.11}$$

$$(\forall x, y \in X)(x \leq_\alpha y\alpha x,\ x \leq_\alpha (x\alpha y)\alpha y). \tag{3.12}$$

Proof. (3.11) is induced by (3.2) and (3.3). For every $x, y \in X$ and $\beta \in \Gamma$, we have $1 = y\alpha 1 = y\alpha(x\alpha x) = x\alpha(y\alpha x)$ and $x\alpha((x\alpha y)\alpha y) = (x\alpha y)\alpha(x\alpha y) = 1$. Hence (3.12) is valid. □

Question 3.1. If $(X,1)_\alpha$ is a BE-algebra for all $\alpha \in \Gamma$, then is $(X,1)_\Gamma$ a Γ-BE-algebra?

The following example provides a negative answer to Question 3.1.

Example 3.4. *Let $X = \{1,2,3,4\}$ be a set and $\Gamma = \{\alpha,\beta,\gamma\}$ be a set of binary operations on X given in the following tables.*

α	1	2	3	4
1	1	2	3	4
2	1	1	2	4
3	1	1	1	4
4	1	2	3	1

β	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	2	3	1

γ	1	2	3	4
1	1	2	3	4
2	1	1	2	2
3	1	1	1	2
4	1	1	2	1

It is routine to verify that $(X,1)_\alpha$, $(X,1)_\beta$ and $(X,1)_\gamma$ are BE-algebras. But $(X,1)_\Gamma$ is not a Γ-BE-algebra since $2\alpha(3\gamma 4) = 2\alpha 2 = 1 \neq 2 = 3\gamma 4 = 3\gamma(2\alpha 4)$.

Definition 3.3. Let $(X,1)_\Gamma$ be a Γ_α-BE-algebra. The relation "$\leq_\beta$" for $\beta \in \Gamma$ is said to be

- *β-reflexive* if $x\beta x = 1$ for all $x \in X$.
- *β-transitive* if it satisfies:

$$(\forall x, y, z \in X)(y\beta z \leq_\beta (x\beta y)\beta(x\beta z)). \tag{3.13}$$

- *β-antisymmetric* if it satisfies:

$$(\forall x, y \in X)(x \leq_\beta y,\ y \leq_\beta x \ \Rightarrow\ x = y). \tag{3.14}$$

- α-*transitive* if it satisfies:

$$(\forall x, y, z \in X)(y\alpha z \leq_\beta (x\alpha y)\alpha(x\alpha z)). \tag{3.15}$$

Example 3.5. *Let $X = \{0, 1, 2, 3\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.*

α	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	1	1	1	1
3	1	1	1	1

β	0	1	2	3
0	1	1	0	3
1	0	1	0	3
2	0	1	1	3
3	0	1	0	1

γ	0	1	2	3
0	0	1	0	2
1	0	1	0	1
2	0	1	0	2
3	1	1	2	0

Then $(X, 1)_\Gamma$ is a Γ_α-BE-algebra. It can be easily verified that the relation $\leq_\beta$ is β-reflexive, β-transitive, β-antisymmetric and α-transitive. But the relation $\leq_\gamma$ is neither γ-reflexive nor γ-transitive nor γ-antisymmetric since $0\gamma 0 = 0 \neq 1$, $(1\gamma 3)\gamma((0\gamma 1)\gamma(0\gamma 3)) = 1\gamma(1\gamma 2) = 1\gamma 0 = 0 \neq 1$, and $1\gamma 3 = 3\gamma 1 = 1$ but $1 \neq 3$, respectively.

Definition 3.4. Let $(X, 1)_\Gamma$ be a Γ-BE-algebra. The relation " $\leq_\Gamma$ " is said to be Γ-*reflexive* (resp., Γ-*transitive* and Γ-*antisymmetric*) if the relation " $\leq_\beta$ " is β-reflexive (resp., β-transitive and β-antisymmetric) for all $\beta \in \Gamma$. In this case, we say that $(X, 1)_\Gamma$ be a Γ-*reflexive* (resp., Γ-*transitive* and Γ-*antisymmetric*) Γ-BE-algebra.

Example 3.6. *(1) Let $X = \{0, 1, 2, 3, 4\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.*

α	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	3	4
3	0	1	1	1	4
4	0	1	1	3	1

β	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	3	4
3	0	1	1	1	1
4	0	1	1	3	1

γ	0	1	2	3	4
0	1	1	1	3	4
1	0	1	2	3	4
2	0	1	1	3	4
3	0	1	2	1	4
4	1	1	1	1	1

Then $(X, 1)_\Gamma$ is a Γ-BE-algebra. It can be easily verified that the relation $\leq_\Gamma$ is Γ-reflexive, Γ-transitive and Γ-antisymmetric.

(2) Let $X = \{0, 1, 2, 3, 4\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.

α	0	1	2	3	4
0	1	1	1	3	4
1	0	1	2	3	4
2	0	1	1	1	4
3	1	1	2	1	4
4	1	1	1	1	1

β	0	1	2	3	4
0	1	1	1	1	4
1	0	1	2	3	4
2	0	1	1	3	1
3	0	1	1	1	4
4	1	1	2	1	1

γ	0	1	2	3	4
0	1	1	1	1	4
1	0	1	2	3	4
2	0	1	1	3	4
3	0	1	1	1	4
4	1	1	2	3	1

Then $(X, 1)_\Gamma$ is a Γ-BE-algebra. It can be easily verified that the relation $\leq_\Gamma$ is Γ-reflexive and Γ-antisymmetric. But $\leq_\Gamma$ is not Γ-transitive since

$$(0\alpha2)\alpha((3\alpha0)\alpha(3\alpha2)) = 1\alpha(1\alpha2) = 1\alpha2 = 2 \neq 1.$$

(3) Let $X = \{0, 1, 2, 3, 4\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.

α	0	1	2	3	4
0	1	1	1	3	4
1	0	1	2	3	4
2	1	1	1	3	4
3	0	1	2	1	4
4	0	1	2	1	1

β	0	1	2	3	4
0	1	1	0	3	4
1	0	1	2	3	4
2	1	1	1	3	4
3	1	1	1	1	4
4	1	1	1	1	1

γ	0	1	2	3	4
0	1	1	0	3	4
1	0	1	2	3	4
2	1	1	1	3	4
3	1	1	0	1	1
4	1	1	0	3	1

Then $(X, 1)_\Gamma$ is a Γ-BE-algebra. It can be easily verified that the relation $\leq_\Gamma$ is Γ-reflexive and Γ-transitive. But $\leq_\Gamma$ is not Γ-antisymmetric since

$$0\alpha2 = 1 \text{ and } 2\alpha0 = 1 \text{ but } 0 \neq 2.$$

(4) Let $X = \{0, 1, 2, 3, 4\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.

α	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	1	1	1	1
3	1	1	2	1

β	0	1	2	3
0	1	1	2	3
1	0	1	2	3
2	1	1	1	3
3	1	1	2	1

γ	0	1	2	3
0	1	1	2	3
1	0	1	2	3
2	1	1	1	1
3	0	1	1	1

Then $(X,1)_\Gamma$ *is a* Γ*-BE-algebra. It can be easily verified that the relation* $\leq_\Gamma$ *is* Γ*-reflexive. But* $\leq_\Gamma$ *is neither* Γ*-transitive and nor* Γ*-antisymmetric since*

$$(0\alpha 2)\alpha((3\alpha 0)\alpha(3\alpha 2)) = 1\alpha(1\alpha 2) = 1\alpha 2 = 2 \neq 1$$

and $0\alpha 2 = 1$, $2\alpha 0 = 1$ *but* $0 \neq 2$.

It is clear that if $(X,1)_\Gamma$ is a Γ_α-BE-algebra, then the relation " $\leq_\alpha$ " is α-reflexive by (3.2). But " $\leq_\beta$ " for $\beta(\neq \alpha) \in \Gamma$ need not be β-reflexive as seen in the following example.

Example 3.7. *Consider the* Γ_α*-BE-algebra* $(X,1)_\Gamma$ *in Example 3.1. Then the relations " $\leq_\beta$ " and " $\leq_\gamma$ " are not* β*-reflexive and* γ*-reflexive, respectively because of* $2\beta 2 = 0 \neq 1$ *and* $2\gamma 2 = 2 \neq 1$.

Question 3.2. If $(X,1)_\Gamma$ is a Γ_α-BE-algebra, then is the relation " $\leq_\alpha$ " α-transitive or α-antisymmetric?

The following example shows that the answer to Question 3.2 may not be positive.

Example 3.8. *Let* $X = \{0,1,2,3\}$ *be a set and* $\Gamma = \{\alpha,\beta,\gamma\}$ *be a set of binary operations on* X *given in the following tables.*

α	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	1	1	1	1
3	1	1	3	1

β	0	1	2	3
0	0	1	0	1
1	0	1	0	1
2	0	1	2	3
3	1	1	3	1

γ	0	1	2	3
0	0	1	0	1
1	0	1	0	1
2	1	1	2	3
3	3	1	0	1

It is routine to verify that $(X,1)_\Gamma$ *is a* Γ_α*-BE-algebra. But the relation " $\leq_\alpha$ " is not* α*-transitive since* $(0\alpha 2)\alpha((3\alpha 0)\alpha(3\alpha 2)) = 1\alpha(1\alpha 3) = 1\alpha 3 = 3 \neq 1$, *that is,* $0\alpha 2 \nleq_\alpha (3\alpha 0)\alpha(3\alpha 2)$. *Also, it is not* α*-antisymmetric because of* $0 \leq_\alpha 2$ *and* $2 \leq_\alpha 0$, *but* $0 \neq 2$.

Proposition 3.2. *Let $(X,1)_\Gamma$ be a Γ_α-BE-algebra. If the relation " $\leq_\alpha$ " is α-transitive, then*

$$(\forall x \in X)(1 \leq_\alpha x \;\Rightarrow\; x = 1), \tag{3.16}$$

$$(\forall x, y, z \in X)(y \leq_\alpha z \;\Rightarrow\; x\alpha y \leq_\alpha x\alpha z,\; z\alpha x \leq_\alpha y\alpha x). \tag{3.17}$$

Proof. Let $x \in X$ and $1 \leq_\alpha x$. Then $x = 1\alpha x = 1$ by (3.4). Let $x, y, z \in X$ and $y \leq_\alpha z$. Then $y\alpha z = 1$, and so

$$(x\alpha y)\alpha(x\alpha z) = 1\alpha((x\alpha y)\alpha(x\alpha z)) = (y\alpha z)\alpha((x\alpha y)\alpha(x\alpha z)) = 1$$

and

$$\begin{aligned}(z\alpha x)\alpha(y\alpha x) &= 1\alpha((z\alpha x)\alpha(y\alpha x))\\ &= (y\alpha z)\alpha((z\alpha x)\alpha(y\alpha x))\\ &= (z\alpha x)\alpha((y\alpha z)\alpha(y\alpha x)) = 1.\end{aligned}$$

□

Corollary 3.1. *Every Γ-transitive Γ-BE-algebra $(X,1)_\Gamma$ satisfies:*

$$(\forall x \in X)(1 \leq_\Gamma x \;\Rightarrow\; x = 1), \tag{3.18}$$

$$(\forall x, y, z \in X)(\forall \alpha \in \Gamma)(y \leq_\alpha z \;\Rightarrow\; x\alpha y \leq_\alpha x\alpha z,\; z\alpha x \leq_\alpha y\alpha x). \tag{3.19}$$

Definition 3.5. A Γ_α-BE-algebra $(X,1)_\Gamma$ is said to be

- *α-left distributive* if it satisfies:

$$(\forall x, y, z \in X)(\forall \beta \in \Gamma)(x\alpha(y\beta z) = (x\alpha y)\beta(x\alpha z)). \tag{3.20}$$

- *γ-left distributive* for $\gamma \in \Gamma$ if it satisfies:

$$(\forall x, y, z \in X)(\forall \delta \in \Gamma)(x\gamma(y\delta z) = (x\gamma y)\delta(x\gamma z)). \tag{3.21}$$

If a Γ_α-BE-algebra $(X,1)_\Gamma$ is β-left distributive for all $\beta \in \Gamma$, we say it is *Γ-left distributive.*

Example 3.9. *(1) Every self-distributive BE-algebra $(X,1)_*$ is a Γ-left distributive Γ-BE-algebra with $\Gamma = \{*\}$.*

(2) Let $X = \{0, 1, 2\}$ *be a set and* $\Gamma = \{\alpha, \beta, \gamma\}$ *be a set of binary operations on* X *given in the following tables.*

α	0	1	2
0	1	1	1
1	0	1	2
2	1	1	1

β	0	1	2
0	0	1	0
1	2	1	1
2	1	1	0

γ	0	1	2
0	2	1	1
1	0	1	1
2	0	1	2

Then it is routine to verify that $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra and it is* α*-left distributive. Since* $0\beta(0\gamma 0) = 0\beta 2 = 0 \neq 2 = 0\gamma 0 = (0\beta 0)\gamma(0\beta 0)$ *and* $0\gamma(0\beta 0) = 0\gamma 0 = 2 \neq 0 = 2\beta 2 = (0\gamma 0)\beta(0\gamma 0)$*, it is neither* β*-left distributive nor* γ*-left distributive.*

Theorem 3.1. *If a* Γ_α*-BE-algebra* $(X, 1)_\Gamma$ *is* α*-left distributive, then the relation* $\leq_\alpha$ *is transitive.*

Proof. Let $x, y, z \in X$ be such that $x \leq_\alpha y$ and $y \leq_\alpha z$. Then $x\alpha y = 1$ and $y\alpha z = 1$. It follows from (3.20) that

$$x\alpha z = 1\alpha(x\alpha z) = (x\alpha y)\alpha(x\alpha z) = x\alpha(y\alpha z) = x\alpha 1 = 1.$$

Hence $x \leq_\alpha z$, and therefore $\leq_\alpha$ is transitive. □

In the following example, we show that if a Γ_α-BE-algebra $(X, 1)_\Gamma$ is β-left distributive, then the relation $\leq_\beta$ may not be β-transitive.

Example 3.10. *Let* $X = \{0, 1, 2, 3\}$ *be a set and* $\Gamma = \{\alpha, \beta, \gamma\}$ *be a set of binary operations on* X *given in the following tables.*

α	0	1	2	3
0	1	1	1	3
1	0	1	2	3
2	0	1	1	1
3	0	1	2	1

β	0	1	2	3
0	0	1	1	1
1	0	1	2	3
2	0	1	2	1
3	0	1	2	3

γ	0	1	2	3
0	0	1	1	3
1	0	1	2	1
2	0	1	1	3
3	0	1	2	3

It is routine to verify that $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra which is* β*-left distributive. But the relation* $\leq_\beta$ *is not* β*-transitive since* $(0\beta 0)\beta((0\beta 0)\beta(0\beta 0)) = 0\beta(0\beta 0) = 0\beta 0 = 0 \neq 1$.

Proposition 3.3. *For a fixed $\alpha \in \Gamma$, every α-left distributive Γ_α-BE-algebra $(X,1)_\Gamma$ satisfies:*

$$(\forall x, y, z \in X)(\forall \beta \in \Gamma)(x \leq_\beta y \;\Rightarrow\; z\alpha x \leq_\beta z\alpha y). \tag{3.22}$$

Proof. Let $x, y, z \in X$ and $\beta \in \Gamma$ be such that $x \leq_\beta y$. Then $x\beta y = 1$ and so

$$1 = z\alpha 1 = z\alpha(x\beta y) = (z\alpha x)\beta(z\alpha y),$$

that is, $z\alpha x \leq_\beta z\alpha y$ by (3.3) and (3.20). □

Corollary 3.2. *Every Γ-left distributive Γ-BE-algebra $(X,1)_\Gamma$ satisfies:*

$$(\forall x, y, z \in X)(\forall \beta \in \Gamma)(x \leq_\Gamma y \;\Rightarrow\; z\beta x \leq_\Gamma z\beta y). \tag{3.23}$$

Theorem 3.2. *If a Γ-BE-algebra $(X,1)_\Gamma$ is Γ-left distributive, then it is Γ-transitive.*

Proof. Let $(X,1)_\Gamma$ be a Γ-left distributive Γ-BE-algebra. Then $(X,1)_\Gamma$ is $\beta-$left distributive for all $\beta \in \Gamma$. Let $x, y, z \in X$ and $\beta \in \Gamma$. Then

$$\begin{aligned}(y\beta z)\beta((x\beta y)\beta(x\beta z)) &= (y\beta z)\beta(x\beta(y\beta z))\\ &= ((y\beta z)\beta x)\beta((y\beta z)\beta(y\beta z))\\ &= ((y\beta z)\beta x)\beta 1\\ &= 1.\end{aligned}$$

Therefore $(y\beta z) \leq_\beta ((x\beta y)\beta(x\beta z))$ which is true for all $\beta \in \Gamma$. Hence $\leq_\Gamma$ is $\Gamma-$transitive. Thus $(X,1)_\Gamma$ is Γ-transitive. □

Let $(X,1)_\Gamma$ be a Γ_α-BE-algebra. For every $x, y \in X$, we consider the sets

$$E_\alpha(x) := \{y \in X \mid x \leq_\alpha y\} \text{ and } E_\alpha^\beta(x,y) := \{z \in X \mid x \leq_\alpha y\beta z\}.$$

Lemma 3.1. *If $(X,1)_\Gamma$ is a Γ_α-BE-algebra, then $x\beta 1 = 1$ and $x\alpha(y\beta x) = 1$ for all $x, y \in X$ and $\beta \in \Gamma$.*

Proof. For every $x, y \in X$ and $\beta \in \Gamma$, we have

$$x\beta 1 = x\beta((x\beta 1)\alpha 1) = (x\beta 1)\alpha(x\beta 1) = 1$$

and $x\alpha(y\beta x) = y\beta(x\alpha x) = y\beta 1 = 1$. □

It is clear that $1, x \in E_\alpha(x) \cap E_\alpha^\beta(x, y)$ and $E_\alpha^\beta(x, y) = E_\beta^\alpha(y, x)$. In general, however, y does not belong to $E_\alpha^\beta(x, y)$ as seen in the following example.

Example 3.11. *In Example 3.10, we can observe that* $E_\alpha^\beta(1, 0) = \{1, 2, 3\}$ *but* $0 \notin E_\alpha^\beta(1, 0)$.

If $(X, 1)_\Gamma$ is both a Γ_α-BE-algebra and a Γ_β-BE-algebra, then $y \in E_\alpha^\beta(x, y)$ for all $x, y \in X$.

Proposition 3.4. *If* $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra, then* $E_\alpha(x) \subseteq E_\alpha^\beta(x, y)$ *for all* $x, y \in X$.

Proof. Let $z \in E_\alpha(x)$. Then $x \leq_\alpha z$, that is, $x\alpha z = 1$. It follows from (3.5) and Lemma 3.1 that

$$x\alpha(y\beta z) = y\beta(x\alpha z) = y\beta 1 = 1,$$

that is, $x \leq_\alpha y\beta z$. Hence $z \in E_\alpha^\beta(x, y)$. □

Proposition 3.5. *If* $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra, then* $E_\alpha(x) = \bigcap_{y \in X} E_\alpha^\beta(x, y)$.

Proof. It is clear that $E_\alpha(x) \subseteq \bigcap_{y \in X} E_\alpha^\beta(x, y)$ by Proposition 3.4. Let $z \in \bigcap_{y \in X} E_\alpha^\beta(x, y)$. Then $z \in E_\alpha^\beta(x, y)$ for all $y \in X$, in particular, $z \in E_\alpha^\beta(x, 1)$. Hence $x \leq_\alpha 1\beta z$, which implies from (3.4) that $1 = x\alpha(1\beta z) = x\alpha z$, that is, $z \in E_\alpha(x)$. Thus $\bigcap_{y \in X} E_\alpha^\beta(x, y) \subseteq E_\alpha(x)$. □

The combination of Propositions 3.4 and 3.5 leads to the following corollary.

Corollary 3.3. *If* $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra, then* $E_\alpha(x) = E_\alpha^\beta(x, 1) = \bigcap_{y \in X} E_\alpha^\beta(x, y)$ *for all* $x \in X$.

Proposition 3.6. *If* $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra and* $z \in X$*, the following are equivalent to each other.*

$$(\forall x \in X)(z \leq_\alpha x). \tag{3.24}$$

$$X = E_\alpha(z). \tag{3.25}$$

$$(\forall x \in X)(X = E_\alpha^\beta(z, x)). \tag{3.26}$$

Proof. It is straightforward to check that (3.24) and (3.25) are equivalent to each other. Suppose that (3.25) is valid. Then $X = E_\alpha(z) \subseteq E_\alpha^\beta(z,x) \subseteq X$ by Proposition 3.4, and so $X = E_\alpha^\beta(z,x)$. The combination of Corollary 3.3 and (3.26) induces $X = E_\alpha^\beta(z,1) = E_\alpha(z)$. □

4. Filters of Γ-BE-algebras

Definition 4.1. A subset F of a Γ_α-BE-algebra $(X,1)_\Gamma$ is called

- a *β-subalgebra* of $(X,1)_\Gamma$ for $\beta \in \Gamma$ if $x\beta y \in F$ for all $x, y \in F$.
- a *β-filter* of $(X,1)_\Gamma$ for $\beta \in \Gamma$ if it satisfies:

$$1 \in F, \tag{4.1}$$

$$(\forall x, y \in X)(x \in F,\, x\beta y \in F \;\Rightarrow\; y \in F). \tag{4.2}$$

Example 4.1. *(1) Let $X = \{0,1,2\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.*

α	0	1	2
0	1	1	0
1	0	1	2
2	1	1	1

β	0	1	2
0	0	1	2
1	1	1	0
2	0	1	2

γ	0	1	2
0	0	1	2
1	0	1	2
2	1	1	0

Then $(X,1)_\Gamma$ is a Γ_α-BE-algebra. Clearly the set $F = \{0,1\}$ is α-subalgebra of $(X,1)_\Gamma$.

(2) Let $X = \{0,1,2,3\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma\}$ be a set of binary operations on X given in the following tables.

α	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	1	1	1	1
3	0	1	0	1

β	0	1	2	3
0	0	1	0	1
1	1	1	1	1
2	0	1	0	1
3	1	1	1	1

γ	0	1	2	3
0	0	1	0	1
1	0	1	0	1
2	0	1	0	1
3	0	1	0	1

Then $(X,1)_\Gamma$ is a Γ_α-BE-algebra. Clearly the set $F = \{0,2\}$ is β-subalgebra of $(X,1)_\Gamma$, but F is not a β-filter of $(X,1)_\Gamma$ since $1 \notin F$.

(3) From Example 4.1(2), we can observe that the set $F_1 = \{1,3\}$ is an α-filter of $(X,1)_\Gamma$.

(4) Let $X = \{0, 1, 2\}$ *be a set and* $\Gamma = \{\alpha, \beta, \gamma\}$ *be a set of binary operations on* X *given in the following tables.*

α	0	1	2
0	1	1	1
1	0	1	2
2	1	1	1

β	0	1	2
0	0	1	2
1	1	1	2
2	2	1	0

γ	0	1	2
0	2	1	2
1	2	1	2
2	1	1	2

Then $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra. Clearly the set* $F = \{0, 1\}$ *is* β*-filter of* $(X, 1)_\Gamma$.

Theorem 4.1. *If* $(X, 1)_\Gamma$ *is a* Γ_α*-BE-algebra, then every* α*-filter is a* β*-subalgebra for all* $\beta \in \Gamma$.

Proof. Let F be an α-filter of $(X, 1)_\Gamma$ and let $x, y \in F$. Since $x\beta 1 = 1$ for all $x \in X$ and $\beta \in \Gamma$ by Lemma 3.1, we have $y\alpha(x\beta y) = x\beta(y\alpha y) = x\beta 1 = 1 \in F$, and so $x\beta y \in F$. Hence F is a β-subalgebra of $(X, 1)_\Gamma$. □

Corollary 4.1. *In a* Γ_α*-BE-algebra* $(X, 1)_\Gamma$*, every* α*-filter is an* α*-subalgebra.*

The following example shows that the converse of Corollary 4.1 may not be true.

Example 4.2. *In Example 4.1(1), we can observe that the* α*-subalgebra* $F = \{0, 1\}$ *is not an* α*-filter of* $(X, 1)_\Gamma$ *since* $0 \in F$ *and* $0\alpha 2 = 0 \in F$ *but* $2 \notin F$.

In the example below, we show that the set $E_\alpha^\beta(x, y)$ is neither an α-filter nor a β-filter of a Γ_α-BE-algebra $(X, 1)_\Gamma$.

Example 4.3. *(1) From Example 4.1(4), we can observe that* $E_\alpha^\beta(1, 1) = \{0, 1\}$*. But* $E_\alpha^\beta(1, 1)$ *is not an* α*-filter of* $(X, 1)_\Gamma$ *since* $0 \in E_\alpha^\beta(1, 1)$ *and* $0\alpha 2 = 1 \in E_\alpha^\beta(1, 1)$ *but* $2 \notin E_\alpha^\beta(1, 1)$.

(2) Let $X = \{0, 1, 2, 3\}$ *be a set and* $\Gamma = \{\alpha, \beta, \gamma\}$ *be a set of binary operations on* X *given in the following tables.*

α	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	1	1	1	1
3	1	1	1	1

β	0	1	2	3
0	0	1	0	0
1	1	1	1	1
2	0	1	0	0
3	0	1	2	3

γ	0	1	2	3
0	0	1	0	3
1	0	1	0	3
2	0	1	3	0
3	3	1	2	3

Then $(X,1)_\Gamma$ is a Γ_α-BE-algebra and $E_\alpha^\beta(1,3)=\{1\}$. We can observe that $E_\alpha^\beta(1,3)$ is not a β-filter of $(X,1)_\Gamma$ since $1\in E_\alpha^\beta(1,3)$ and $1\beta 2=1\in E_\alpha^\beta(1,3)$ but $2\notin E_\alpha^\beta(1,3)$.

Theorem 4.2. *If a Γ_α-BE-algebra $(X,1)_\Gamma$ is both α-left distributive and β-left distributive, then the set $E_\alpha^\beta(x,y)$ is an α-filter of $(X,1)_\Gamma$.*

Proof. We know that $1\in E_\alpha^\beta(x,y)$. Let $u,v\in X$ be such that $u\alpha v\in E_\alpha^\beta(x,y)$ and $u\in E_\alpha^\beta(x,y)$. Then $x\alpha(y\beta(u\alpha v))=1$ and $x\alpha(y\beta u)=1$. Hence

$$\begin{aligned}1&=x\alpha(y\beta(u\alpha v))=x\alpha((y\beta u)\alpha(y\beta v))\\&=(x\alpha(y\beta u))\alpha(x\alpha(y\beta v))=1\alpha(x\alpha(y\beta v))\\&=x\alpha(y\beta v),\end{aligned}$$

and so $v\in E_\alpha^\beta(x,y)$. Therefore $E_\alpha^\beta(x,y)$ is an α-filter of $(X,1)_\Gamma$. □

The following example shows that if a Γ_α-BE-algebra $(X,1)_\Gamma$ is both α-left distributive and β-left distributive, then the set $E_\alpha^\beta(x,y)$ is not necessarily a β-filter of $(X,1)_\Gamma$.

Example 4.4. *From Example 4.3(2), we can observe that the Γ_α-BE-algebra $(X,1)_\Gamma$ is both α-left distributive and β-left distributive. But the set $E_\alpha^\beta(1,3)=\{1\}$ is not a β-filter of $(X,1)_\Gamma$ since $1\in E_\alpha^\beta(1,3)$ and $1\beta 2=1\in E_\alpha^\beta(1,3)$ but $2\notin E_\alpha^\beta(1,3)$.*

The following theorem is obtained in the same way as the proof in Theorem 4.2.

Theorem 4.3. *If a Γ_β-BE-algebra $(X,1)_\Gamma$ is both α-left distributive and β-left distributive, then the set $E_\alpha^\beta(x,y)$ is a β-filter of $(X,1)_\Gamma$.*

Question 4.1. If F is an α-filter of a Γ_α-BE-algebra $(X,1)_\Gamma$, then does F contain the set $E_\alpha^\beta(x,y)$ for all $x,y\in X$ and $\beta\in\Gamma$?

The answer to Question 4.1 is negative as seen in the following example.

Example 4.5. *In Example 3.10, we can observe that $F=\{1,3\}$ is an α-filter of a Γ_α-BE-algebra $(X,1)_\Gamma$. But F does not contain the set $E_\alpha^\beta(1,0)=\{1,2,3\}$.*

Theorem 4.4. *Let F be a subset of a Γ_α-BE-algebra $(X,1)_\Gamma$. If F is both an α-filter and a β-filter of $(X,1)_\Gamma$, then $E_\alpha^\beta(x,y)\subseteq F$ for all $x,y\in F$ and $\beta\in\Gamma$.*

Proof. Assume that F is both an α-filter and a β-filter of $(X,1)_\Gamma$. Let $\beta \in \Gamma$ and $x, y \in F$, such that $z \in E_\alpha^\beta(x,y)$ for some $z \in X$. Then $x\alpha(y\beta z) = 1 \in F$. Hence $y\beta z \in F$ and so $z \in F$. Therefore $E_\alpha^\beta(x,y) \subseteq F$. □

Corollary 4.2. *Let F be a subset of a Γ_α-BE-algebra $(X,1)_\Gamma$. If F is both an α-filter and a β-filter of $(X,1)_\Gamma$, then* $\bigcup_{x,y\in F} E_\alpha^\beta(x,y) \subseteq F$.

Theorem 4.5. *Let F be a subset of a Γ_α-BE-algebra $(X,1)_\Gamma$ and $\beta \in \Gamma$. If $E_\alpha^\beta(x,y) \subseteq F$, for all $x, y \in F$, then F is a β-filter of $(X,1)_\Gamma$.*

Proof. Let $\beta \in \Gamma$ and assume that $E_\alpha^\beta(x,y) \subseteq F$, for all $x, y \in F$. Since $x\alpha(y\beta 1) = x\alpha 1 = 1$, that is, $x \leq_\alpha y\beta 1$, we get $1 \in E_\alpha^\beta(x,y) \subseteq F$. Let $u, v \in X$ be such that $u \in F$ and $u\beta v \in F$. Then Since $(u\beta v)\alpha(u\beta v) = 1$, that is, $u\beta v \leq_\alpha u\beta v$, we get $v \in E_\alpha^\beta(u\beta v, u) \subseteq F$. Thus F is a β-filter of $(X,1)_\Gamma$. □

Conclusion

We have introduced the concept of Γ-BE-algebra as a generalization of a BE-algebra and studied its properties. For a fixed binary operation α in a set Γ of binary operations on a non-empty set X, we have introduced the concepts of β-reflexive, β-transitive, β-antisymmetric, α-transitive, α-left distributive and γ-left distributive Γ_α-BE-algebra and studied the relations between them. We have introduced the concept of β-subalgebra and β-filter in a Γ_α-BE-algebra and studied the relation between α-subalgebra, α-filter and β-subalgebra in a Γ_α-BE-algebra.

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