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ON A q-ANALOGUE OF THE RIGHT LOCAL GENERAL TRUNCATED M-FRACTIONAL DERIVATIVE

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Abstract. We introduce a q-analogue of the right local general truncated M-fractional derivative for α-differentiable functions. From this newly defined operator, q-analogues of the standard properties and results of the α-right local general truncated M-fractional derivative like the Rolle's theorem, the mean value theorem and its extension, inverse property, the fundamental theorem of calculus and the theorem of integration by parts are obtained. In context with this q-fractional derivative operator, a q-analogue of a physical problem, the falling body problem, is obtained. Also, the q-vertical velocity and the q-distance are obtained from this problem and the solutions has been compared and shown in the graphs for various combination of q-parameter and fractional order α with the classical ordinary solution.

1. Introduction

Let $0 < q < 1$. A q-analogue of the factorial function

$$(a)_n = a(a+1)(a+2)\dots(a+n-1)$$

is defined by [9, Eq.(1.2.15) and (1.2.30), p. 3, 6]

$$(a;q)_n = \begin{cases} 1, & \text{if} \quad n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if} \quad n\in\mathbb{Z}_{>0} \\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})\right]^{-1}, & \text{if} \quad n\in\mathbb{Z}_{<0} \\ \frac{(a;q)_\infty}{(aq^n;q)_\infty}, & \text{if} \quad n\in\mathbb{C}, \end{cases} \tag{1.1}$$

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where $a \in \mathbb{C}$ in general, and $(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$.
For $a \equiv q^a = q$,

$$(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

is a q-analogue of $n!$.
In the theory of q-calculus [14], a q-number (or basic number) is given by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1. \tag{1.2}$$

The q-Gamma function [9, Eq.(1.10.1), p. 20] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x}. \tag{1.3}$$

In a view of the q-calculus, the q-derivative [1, 5, 12, 13] and the general q-integral [9, 11] of a function are defined as follows :

Definition 1.1. For an arbitrary function $f(t)$ and $0 < q < 1$, the q-derivative of $f(t)$ is given by

$$D_q f(t) = \frac{f(t) - f(tq)}{t(1-q)}. \tag{1.4}$$

Note that, if we take $q \to 1$ in (1.4) then $D_q f(t) = f'(t)$.

Definition 1.2. The general q-integral of $f(t)$ is defined as

$$\int_a^b f(t)\,d_q t = \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t, \tag{1.5}$$

where

$$\int_0^a f(t)\,d_q t = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n. \tag{1.6}$$

Note that, if f is continuous on $[0,a]$, then it is easily seen that

$$\lim_{q\to 1}\int_0^a f(t)\,d_q t = \int_0^a f(t)\,dt.$$

With the aid of the q-derivative, in [15, 16], the q_{π_1}-derivative and q^{π_2}-derivative are defined as follows:

Definition 1.3. The q_{π_1}-derivative of a mapping $f:[\pi_1,\pi_2]\to\mathbb{R}$ is defined as

$$ {}_{\pi_1}\mathbb{D}_q f(x)=\frac{f(x)-f(qx+(1-q)\pi_1)}{(1-q)(x-\pi_1)},\ x\neq\pi_1. \tag{1.7}$$

Observe that at $x=\pi_1$, ${}_{\pi_1}\mathbb{D}_q f(\pi_1)=\lim_{x\to\pi_1}{}_{\pi_1}\mathbb{D}_q f(x)$ if it exists and is finite.

Definition 1.4. The q^{π_2}-derivative of a mapping $f:[\pi_1,\pi_2]\to\mathbb{R}$ is defined as

$$ {}^{\pi_2}\mathbb{D}_q f(x)=\frac{f(qx+(1-q)\pi_2)-f(x)}{(1-q)(\pi_2-x)},\ x\neq\pi_2. \tag{1.8}$$

Observe that at $x=\pi_2$, ${}^{\pi_2}\mathbb{D}_q f(\pi_2)=\lim_{x\to\pi_2}{}^{\pi_2}\mathbb{D}_q f(x)$ if it exists and is finite.

2. Main Results

In this section, we first introduce a q-analogue of the α-RLGT M-fractional derivative [6]. In view of the q_{π_1}-derivative [16], we begin with the following definition.

Definition 2.1. Let $f:[\pi_1,b]\to\mathbb{R}$ and $t<b,\ b\in\mathbb{R}$. For $0<\alpha\leq 1$, we define a q-α-right local general truncated M-fractional derivative of order α of f (q-α-RLGT M-fractional derivative) as

$$ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(t):=\frac{1}{\Gamma_q(\beta+1)}\frac{(f((b-t)^\alpha)-f(q(b-t)^\alpha)+(1-q)\pi_1)}{(1-q)((b-t)^\alpha-\pi_1)}. \tag{2.1}$$

Now onwards, for the sake of simplicity, we will denote $(b-t)^\alpha=t_{\alpha,b}$.

Remark 1. *From Definition 2.1, if $f(t)=c$, where c is any constant, then ${}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(t)=0$.*

Remark 2. *For $\pi_1=0$, $\alpha=1$, $b=0$, $\beta=0$ or 1; replacing t by $-t$ and then letting $q\to 1$, (2.1) reduces to $\lim_{q\to 1}{}_0\mathfrak{D}^{1,0}_{q,M,0}f(-t)=f'(t)$.*

Now, we will derive a q-analogue of various properties as given in [6] for q-α-RLGT M-fractional derivative.

Theorem 2.1. *Let $f_1,f_2:[\pi_1,b]\to\mathbb{R}$ be q-α-RLGT M-fractional differentiable at $t, t<b$, $\mu_1,\mu_2\in\mathbb{R}$ and $\beta>0$. Then*

(1) ${}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(\mu_1 f_1+\mu_2 f_2)(t)=\mu_1\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_1(t)+\mu_2\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t)$.

(2) ${}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(f_1\cdot f_2)(t)=f_1(t_{\alpha,b})\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t)+f_2(qt_{\alpha,b}(1-q)+\pi_1)\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_1(t)$.

(3) $_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(\frac{f_1}{f_2}\right)(t)$
$$= \frac{f_2(qt_{\alpha,b}+(1-q)\pi_1)\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_1(t) - f_1(qt_{\alpha,b}+(1-q)\pi_1)\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t)}{f_2(t_{\alpha,b})\ f_2(qt_{\alpha,b}+(1-q)\pi_1)}.$$

(4) $_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(k) = 0$, *where k is a constant.*

(5) *If $f_1(t)$ is q-differentiable* [10, 12] *at $f_2(t)$, then*
$$_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(f_1 o f_2)(t) = D_q(f_1(f_2(t_{\alpha,b}))\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t).$$

Proof. (1) From the Definition 2.1, we have

$$\begin{aligned}
&{}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(\mu_1 f_1 + \mu_2 f_2)(t)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left((\mu_1 f_1+\mu_2 f_2)(t_{\alpha,b}) - (\mu_1 f_1+\mu_2 f_2)(qt_{\alpha,b}+(1-q)\pi_1)\right)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left(\mu_1 f_1(t_{\alpha,b}) + \mu_2 f_2(t_{\alpha,b}) - \mu_1 f_1(qt_{\alpha,b}+(1-q)\pi_1)\right.\\
&\quad \left.-\mu_2 f_2(qt_{\alpha,b}+(1-q)\pi_1)\right)\\
&= \left(\frac{1}{\Gamma_q(\beta+1)}\frac{(\mu_1 f_1(t_{\alpha,b}) - \mu_1 f_1(qt_{\alpha,b}+(1-q)\pi_1))}{(1-q)(t_{\alpha,b}-\pi_1)}\right)\\
&\quad + \left(\frac{1}{\Gamma_q(\beta+1)}\frac{(\mu_2 f_2(t_{\alpha,b}) - \mu_2 f_2(qt_{\alpha,b}+(1-q)\pi_1))}{(1-q)(t_{\alpha,b}-\pi_1)}\right)\\
&= \mu_1\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_1(t) + \mu_2\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t).
\end{aligned}$$

(2) From the Definition 2.1, we have

$$\begin{aligned}
&{}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(f_1\cdot f_2)(t)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left((f_1\cdot f_2)(t_{\alpha,b}) - (f_1\cdot f_2)(qt_{\alpha,b}+(1-q)\pi_1)\right)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left(f_1(t_{\alpha,b})f_2(t_{\alpha,b}) - f_1(qt_{\alpha,b}+(1-q)\pi_1)f_2(qt_{\alpha,b}+(1-q)\pi_1)\right.\\
&\quad \left.+f_1(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1) - f_1(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)\right)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left(f_1(t_{\alpha,b})\left(f_2(t_{\alpha,b}) - f_2(qt_{\alpha,b}+(1-q)\pi_1)\right)\right.\\
&\quad \left.+f_2(qt_{\alpha,b}+(1-q)\pi_1)\left(f_1(t_{\alpha,b}) - f_1(qt_{\alpha,b}+(1-q)\pi_1)\right)\right)\\
&= f_1(t_{\alpha,b})\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t) + f_2(qt_{\alpha,b}+(1-q)\pi_1)\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_1(t).
\end{aligned}$$

(3) Again with the aid of Definition 2.1, we have

$$
\begin{aligned}
&{}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(\frac{f_1}{f_2}\right)(t)\\
&=\frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left(\left(\frac{f_1}{f_2}\right)(t_{\alpha,b})-\left(\frac{f_1}{f_2}\right)(qt_{\alpha,b}+(1-q)\pi_1)\right)\\
&=\frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left(\frac{f_1(t_{\alpha,b})}{f_2(t_{\alpha,b})}-\frac{f_1(qt_{\alpha,b}+(1-q)\pi_1)}{f_2(qt_{\alpha,b}+(1-q)\pi_1)}\right)\\
&=\frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\\
&\quad\times\left(\frac{f_1(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)-f_2(t_{\alpha,b})f_1(qt_{\alpha,b}+(1-q)\pi_1)}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)}\right)\\
&=\frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\\
&\quad\times\left(\frac{f_1(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)-f_1(qt_{\alpha,b}+(1-q)\pi_1)f_2(qt_{\alpha,b}+(1-q)\pi_1)}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1}\right.\\
&\quad\left.+\frac{f_1(qt_{\alpha,b}+(1-q)\pi_1)f_2(qt_{\alpha,b}+(1-q)\pi_1)-f_2(t_{\alpha,b})f_1(t_{\alpha,b}q)}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1}\right)\\
&=\frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\\
&\quad\times\left(\frac{f_2(qt_{\alpha,b}+(1-q)\pi_1)(f_1(t_{\alpha,b})-f_1(qt_{\alpha,b}+(1-q)\pi_1))}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1}\right.\\
&\quad\left.-\frac{f_1(qt_{\alpha,b}+(1-q)\pi_1)(f_2(t_{\alpha,b})-f_2(qt_{\alpha,b}+(1-q)\pi_1))}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1}\right)\\
&=\frac{f_2(qt_{\alpha,b}+(1-q)\pi_1)}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)}\left(\frac{1}{\Gamma_q(\beta+1)}\frac{(f_1(t_{\alpha,b})-f_1(qt_{\alpha,b}+(1-q)\pi_1))}{(1-q)(t_{\alpha,b}-\pi_1)}\right)\\
&\quad-\frac{f_1(qt_{\alpha,b}+(1-q)\pi_1)}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)}\left(\frac{1}{\Gamma_q(\beta+1)}\frac{(f_2(t_{\alpha,b})-f_2(qt_{\alpha,b}+(1-q)\pi_1))}{(1-q)(t_{\alpha,b}-\pi_1)}\right)\\
&=\frac{f_2(qt_{\alpha,b}+(1-q)\pi_1)\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_1(t)-f_1(qt_{\alpha,b}+(1-q)\pi_1)\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}f_2(t)}{f_2(t_{\alpha,b})f_2(qt_{\alpha,b}+(1-q)\pi_1)}.
\end{aligned}
$$

(4) In this case, the proof is directly follows from Remark 1.

(5) This result is proved in two cases: (I) f_2 is constant and (II) f_2 is non constant.

Case-I: Let $f_2(t)=c$, where c is any constant.

Then from Remark 1, we have

$$
{}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}\ (f_1 o f_2)(t)=\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}\ f_1(f_2(t))=\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}\ f_1(c)=0.
$$

Case-II: Let f_2 be not a constant.

Then by Definition 2.1, we have

$$
\begin{aligned}
&{}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b}(f_1 o f_2)(t)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left((f_1 o f_2)(t_{\alpha,b}) - (f_1 o f_2)(q t_{\alpha,b} + (1-q)\pi_1)\right)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\\
&\quad \times \frac{f_1(f_2(t_{\alpha,b})) - f_1(f_2(q t_{\alpha,b} + (1-q)\pi_1))}{f_2(t_{\alpha,b}) - f_2(q t_{\alpha,b} + (1-q)\pi_1)}\left(f_2(t_{\alpha,b}) - f_2(q t_{\alpha,b} + (1-q)\pi_1)\right)\\
&= \frac{f_1(f_2(t_{\alpha,b})) - f_1(f_2(q t_{\alpha,b} + (1-q)\pi_1))}{f_2(t_{\alpha,b}) - f_2(q t_{\alpha,b} + (1-q)\pi_1)}\\
&\quad \times \left(\frac{1}{\Gamma_q(\beta+1)}\frac{1}{(1-q)(t_{\alpha,b}-\pi_1)}\left(f_2(t_{\alpha,b}) - f_2(q t_{\alpha,b} + (1-q)\pi_1)\right)\right)\\
&= D_q(f_1(f_2(t_{\alpha,b})))\ {}_{\pi_1}\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_2(t).
\end{aligned}
$$

□

In the next theorem, a relation between q-α-RLGT M-fractional derivative and q-difference operator is obtained.

Theorem 2.2. *If $f : [0,b] \to \mathbb{R}$ has the q-α-RLGT M-fractional derivative at t, $t < b$ with $\beta > 0$ and $\pi_1 = 0$, then*

$$
{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(t) = \frac{1}{\Gamma_q(\beta+1)} D_q f(t_{\alpha,b}). \tag{2.2}
$$

Proof. For $t < b$ and $\pi_1 = 0$, from Definition 2.1, we have

$$
\begin{aligned}
&{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(t)\\
&= \frac{1}{\Gamma_q(\beta+1)}\frac{(f(t_{\alpha,b}) - f(q t_{\alpha,b}))}{(1-q)t_{\alpha,b}}\\
&= \frac{1}{\Gamma_q(\beta+1)} D_q f(t_{\alpha,b}).
\end{aligned}
$$

□

Now, as a consequence of Theorem 2.2, we have the following q-α-RLGT M-fractional derivatives of various q-analogues of some well-known functions.

Theorem 2.3. *Let $\mu \in \mathbb{R}$, $\beta > 0$, $\alpha \in (0,1]$ and $t < b$. Then*

(1) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(1) = 0$;

(2) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(t^\mu) = \dfrac{[\mu]_q}{\Gamma_q(\beta+1)} t^{\mu-1}_{\alpha,b}$;

(3) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(e_q(\mu t)) = \dfrac{\mu}{\Gamma_q(\beta+1)}\ e_q(\mu t_{\alpha,b})$;

(4) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(E_q(\mu t)) = \frac{\mu}{\Gamma_q(\beta+1)}\ E_q(\mu q t_{\alpha,b})$;

(5) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(\sin_q(\mu t)) = \dfrac{\mu}{\Gamma_q(\beta+1)}\ cos_q(\mu t_{\alpha,b})$;

(6) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(\cos_q(\mu t)) = -\dfrac{\mu}{\Gamma_q(\beta+1)}\ sin_q(\mu t_{\alpha,b})$;

(7) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(Sin_q(\mu t)) = \dfrac{\mu}{\Gamma_q(\beta+1)}\ Cos_q(\mu q t_{\alpha,b})$;

(8) ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(Cos_q(\mu t)) = -\dfrac{\mu}{\Gamma_q(\beta+1)}\ Sin_q(\mu q t_{\alpha,b})$.

Proof.

(1) The proof is directly follows from Remark 1.

(2) From Theorem 2.2 and q-derivative of $t^\mu_{\alpha,b}$ [12, p. 7], we have

$$
\begin{aligned}
{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(t^\mu) &= \frac{1}{\Gamma_q(\beta+1)} D_q(t^\mu_{\alpha,b}) \\
&= \frac{[\mu]_q}{\Gamma_q(\beta+1)}\ t^{\mu-1}_{\alpha,b}.
\end{aligned}
$$

(3) Using Theorem 2.2 and q-derivative of $e_q(\mu t_{\alpha,b})$ [12, Eq.(9.11), p. 31], we have

$$
\begin{aligned}
{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(e_q(\mu t)) &= \frac{1}{\Gamma_q(\beta+1)} D_q(e_q(\mu t_{\alpha,b})) \\
&= \frac{\mu}{\Gamma_q(\beta+1)}\ e_q(\mu t_{\alpha,b}).
\end{aligned}
$$

(4) Using Theorem 2.2 and q-derivative of $E_q(\mu t_{\alpha,b})$ [12, Eq.(9.11), p. 31], we get

$$
\begin{aligned}
{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(E_q(\mu t)) &= \frac{1}{\Gamma_q(\beta+1)} D_q(E_q(\mu t_{\alpha,b})) \\
&= \frac{\mu}{\Gamma_q(\beta+1)}\ E_q(\mu q t_{\alpha,b}).
\end{aligned}
$$

(5) Using Theorem 2.2 and q-derivative of $\sin_q(\mu t_{\alpha,b})$ [12, Eq.(10.4), p. 34], we have

$$
\begin{aligned}
{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(sin_q(\mu t)) &= \frac{1}{\Gamma_q(\beta+1)} D_q(sin_q(\mu t_{\alpha,b})) \\
&= \frac{\mu}{\Gamma_q(\beta+1)}\ cos_q(\mu t_{\alpha,b}).
\end{aligned}
$$

(6) Using Theorem 2.2 and q-derivative of $\cos_q(\mu t_{\alpha,b})$ [12, Eq.(10.5), p. 34], we obtain

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(cos_q(\mu t)) = \frac{1}{\Gamma_q(\beta+1)} D_q(cos_q(\mu t_{\alpha,b})) = -\frac{\mu}{\Gamma_q(\beta+1)}\ sin_q(\mu t_{\alpha,b}). $$

(7) Using Theorem 2.2 and q-derivative of $Sin_q(\mu t_{\alpha,b})$ [12, Eq.(10.4), p. 34], we arrive at

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(Sin_q(\mu t)) = \frac{1}{\Gamma_q(\beta+1)} D_q(Sin_q(\mu t_{\alpha,b})) = \frac{\mu}{\Gamma_q(\beta+1)}\ Cos_q(\mu q t_{\alpha,b}). $$

(8) Using Theorem 2.2 and q-derivative of $Cos_q(\mu t_{\alpha,b})$ [12, Eq.(10.5), p. 34], we have

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(Cos_q(\mu t)) = \frac{1}{\Gamma_q(\beta+1)} D_q(Cos_q(\mu t_{\alpha,b})) = -\frac{\mu}{\Gamma_q(\beta+1)}\ Sin_q(\mu q t_{\alpha,b}). $$

□

2.1. **Generalization of fundamental results of calculus through q-α-RLGT M-fractional derivative.** Further, we have observed that a q-α-RLGT M-fractional derivative also has various important theorems similar to the α-RLGT M-fractional derivative [6]. We have derived the q-analogues of Rolle's theorem, the mean value theorem and its extension using this newly defined q-analogue of a fractional derivative operator in the next three theorems.

Theorem 2.4. *Let $f : [\gamma, \rho] \to \mathbb{R}$, where $\rho < b$. If*

(1) *f is continuous on $[\gamma, \rho]$,*
(2) *f is q-α-RLGT M-fractional differentiable on (γ, ρ),*
(3) *$f(\gamma) = f(\rho)$,*

then there exists $c \in (\gamma, \rho)$ such that ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(c) = 0,\ \beta > 0$.

Proof. We will prove this theorem in three cases:

Case-I: When $f(x) = k$ on $[\gamma, \rho]$, where k is any constant.

Then from Remark 1, ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(x) = 0$ for all $x \in [\gamma, \rho]$.

That is, in other words, we can say that there exists $c \in (\gamma, \rho)$ such that

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(c) = 0. $$

Case-II: Let f be non-constant. In this case, suppose that there is some $d \in (\gamma, \rho)$ such that $f(d) > f(\gamma)$.

Since f is continuous on $[\gamma, \rho]$, by a q-analogue of the extreme value theorem [13, Thm. 2.1, p. 172], $f(x)$ has local maximum in $[\gamma, \rho]$.

Also, as $f(\gamma) = f(\rho)$ and $f(d) > f(\gamma)$, we have the maximum value of f at some c in (γ, ρ).

Here, c occurs in the interior of the interval means that $f(x)$ has relative maximum at $x = c$ and by the second hypothesis, ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(x)$ exists.

Therefore, ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(c) = 0$.

Case-III: Let f be non-constant, but in this case, suppose that there is some $d \in (\gamma, \rho)$ such that $f(d) < f(\gamma)$.

Now, in the similar manner of Case-II, by a q-analogue of the extreme value theorem [13, Thm. 2.1, p. 172], $f(x)$ has local minimum in $[\gamma, \rho]$.

Also, as $f(\gamma) = f(\rho)$ and $f(d) < f(\gamma)$, we have the minimum value of f at some c in (γ, ρ).

Hence, ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(c) = 0$. □

Theorem 2.5. *Let $f : [\gamma, \rho] \to \mathbb{R}$, where $\rho < b,\ 0 \notin [\gamma, \rho]$. If*

(1) *f is continuous on $[\gamma, \rho]$,*

(2) *f is q-α-RLGT M-fractional differentiable on (γ, ρ),*

then there exists $c \in (\gamma, \rho)$ such that

$$ \frac{f(\rho) - f(\gamma)}{\rho - \gamma} = \Gamma_q(\beta + 1)\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}f(c). \tag{2.3} $$

Proof. For $x \in [\gamma, \rho]$, let

$$ g(x) := f(x) - f(\gamma) - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma} \right)(x - \gamma). \tag{2.4} $$

Since f is continuous on $[\rho, \gamma]$, g is continuous on $[\rho, \gamma]$ too.

Also, it can be easily verified that $g(\gamma) = 0 = g(\rho)$.

Therefore, from Theorem 2.1, we can say that g is q-α-RLGT M-fractional differentiable on (γ, ρ).

Now, from Theorem 2.4, there exists $c \in (\gamma, \rho)$ such that

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} g(c) = 0. \tag{2.5}$$

Taking ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}$ on both the sides of (2.4), we get

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} g(x) = {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(x) - {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(\gamma) - \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}(x-\gamma).$$

Applying Theorem 2.2 and then Theorem 2.3, we obtain

$$\begin{aligned} &{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} g(x) \\ &= {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(x) - \ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(\gamma) - \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{1}{\Gamma_q(\beta+1)} D_q(x_{\alpha,b}-\gamma) \\ &= {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(x) - 0 - \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{1}{\Gamma_q(\beta+1)}. \end{aligned}$$

Whence at $x = c$,

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} g(c) = {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(c) - \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \ \frac{1}{\Gamma_q(\beta+1)}.$$

Then using (2.5), we get

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(c) - \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{1}{\Gamma_q(\beta+1)} = 0.$$

Therefore,

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(c) = \left(\frac{f(\rho)-f(\gamma)}{\rho-\gamma}\right) \frac{1}{\Gamma_q(\beta+1)}.$$

Hence,

$$ \frac{f(\rho)-f(\gamma)}{\rho-\gamma} = \Gamma_q(\beta+1) \ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(c).$$

□

Theorem 2.6. *Let* $\rho < b$, $0 \notin [\gamma, \rho]$ *and* $f_1, f_2 : [\gamma, \rho] \to \mathbb{R}$. *If*

(1) f_1, f_2 *are continuous on* $[\gamma, \rho]$ *and* $f_2(\gamma) \neq f_2(\rho)$,

(2) f_1, f_2 *is* q-α-*RLGT* M-*fractional differentiable on* (γ, ρ),

then there exists $c \in (\gamma, \rho)$ *such that*

$$ \frac{{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_1(c)}{{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_2(c)} = \frac{f_1(\rho)-f_1(\gamma)}{f_2(\rho)-f_2(\gamma)} \quad \textit{with } \beta > 0.$$

Proof. For $x \in [\gamma, \rho]$, define

$$G(x) := f_1(x) - f_2(\gamma) - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right)(f_2(x) - f_2(\gamma)). \tag{2.6}$$

Since f_1, f_2 are continuous on $[\rho, \gamma]$, G is continuous on $[\rho, \gamma]$ too.
Also, it can be easily seen that $G(\gamma) = 0 = G(\rho)$.
Therefore, from Theorem 2.1, we can say that G is a q-α-RLGT M-fractional differentiable function on (γ, ρ).
Now, from Theorem 2.4, there exists $c \in (\gamma, \rho)$ such that

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} G(c) = 0. \tag{2.7}$$

Taking ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}$ on both the sides of (2.6), we get

$$\begin{aligned} &{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} G(x) \\ &= {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_1(x) - {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_2(\gamma) - \left({}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\,(f_2(x) - f_2(\gamma))\right)\left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right). \end{aligned}$$

Writing the above expression at $x = c$ and then again applying Theorem 2.2 and Remark 1, we have

$${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} G(c) = {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_1(c) - 0 - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right) {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_2(c) - 0,$$

which implies from (2.7),

$${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_1(c) - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right)\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_2(c) = 0.$$

Therefore,

$$\frac{{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_1(c)}{{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f_2(c)} = \frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}.$$

□

Next, we will show some results pertaining to the corresponding q-integral. For that, we have defined a corresponding q-analogue of the right M-integral as follows.

Definition 2.2. Let f be a function defined in $[t, b)$ and $\alpha \in (0, 1]$. Then a q-right M-integral of order α of f is denoted and defined as

$$\mathfrak{I}^{\beta}_{q,M,b} f(t) = -\Gamma_q(\beta + 1) \int_t^b f(x)\, d_q x = -\Gamma_q(\beta + 1)\ I_{q,b} f(t), \tag{2.8}$$

with $\beta > 0$, where

$$I_{q,b}f(t) = \int_t^b f(x)\, d_qx = \int_0^b f(x)\, d_qx - \int_0^t f(x)\, d_qx,$$

and here

$$\int_0^b f(x)\, d_qx = b(1-q)\sum_{n=0}^{\infty} f(bq^n)\; q^n.$$

In context with the above definition, we have obtained a q-analogue of the inverse property, fundamental theorem of calculus and the theorem of integration by parts in the upcoming theorems.

Theorem 2.7. *Let $b \in \mathbb{R}$, $\alpha \in (0,1]$ and f be a continuous function such that there exists $\mathfrak{I}^{\beta}_{q,M,b}f$. Then*

$$ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(\mathfrak{I}^{\beta}_{q,M,b}f(t)\right) = f(t_{\alpha,b}), \tag{2.9}$$

with $0 \neq t < b$ and $\beta > 0$.

Proof. From Theorem 2.2, we have

$$\begin{aligned}
&{}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(\mathfrak{I}^{\beta}_{q,M,b}f(t)\right)\\
&= \frac{1}{\Gamma_q(\beta+1)} D_q\left(\mathfrak{I}^{\beta}_{q,M,b}f(t_{\alpha,b})\right)\\
&= \frac{1}{\Gamma_q(\beta+1)} D_q\left(-\Gamma_q(\beta+1)\int_t^b f(x)\, d_qx\right)\\
&= -D_q\left(\int_0^b f(x)\, d_qx - \int_0^t f(x)\, d_qx\right)\\
&= -\left[D_q\left(\int_0^b f(x)\, d_qx\right) - D_q\left(\int_0^t f(x)\, d_qx\right)\right]\\
&= -\left[0 - f(t_{\alpha,b})\right]\\
&= f(t_{\alpha,b}),
\end{aligned}$$

as $(D_qI_{q,b}f)(t) = f(t)$ from [14, Eq. (11), p. 313]. ∎

Theorem 2.8. *Let $f : [0, b] \to \mathbb{R}$ be a continuously differentiable function such that ${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f$ exists and $\alpha \in (0, 1]$. Then for all $t < b$,*

$$\mathfrak{I}^{\beta}_{q,M,b}\left({}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(t)\right) = f(t_{\alpha,b}) - f(b), \tag{2.10}$$

with $\beta > 0$.

Proof. From Definition 2.2 and then applying Theorem 2.2, we have

$$\begin{aligned}
&\mathfrak{I}^{\beta}_{q,M,b}\left({}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(t)\right) \\
&= -\Gamma_q(\beta+1)\int_t^b {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(x)\, d_q x \\
&= -\Gamma_q(\beta+1)\int_t^b \frac{1}{\Gamma_q(\beta+1)} D_q f(x_{\alpha,b})\, d_q x \\
&= \int_b^t D_q f(x_{\alpha,b})\, d_q x \\
&= I_{q,b}\ D_q f(t_{\alpha,b}) \\
&= f(t_{\alpha,b}) - f(b),
\end{aligned}$$

by [14, Eq. (11), p. 313]. □

It can be easily observed that, if $f(b) = 0$, then by (2.10) for all $t < b$,

$$\mathfrak{I}^{\beta}_{q,M,b}\left({}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} f(t)\right) = f(t_{\alpha,b}).$$

Now, for the sake of brevity, we denote

$$\mathfrak{I}^{\beta}_{q,M,b} f(t) = -\int_t^b f(x)\ d_{q,\beta}\ x, \quad \text{where } d_{q,\beta}\ x = \Gamma_q(\beta+1)\, d_q x.$$

In this notation, we derive a q-analogue of the generalization of the integration by parts in the following theorem for a q-right M-integral.

Theorem 2.9. *Let $f_1, f_2 : [c,d] \to \mathbb{R}$ be continuously differentiable and $\alpha \in (0,1]$. Then*

$$\int_c^d f_1(x_{\alpha,b})\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(f_2(x)\right)\ d_{q,\beta}\ x$$

$$= [f_1(x_{\alpha,b})f_2(x_{\alpha,b})]_c^d - \int_c^d f_2(qx_{\alpha,b})\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(f_1(x)\right)\ d_{q,\beta}\ x, \quad \beta > 0.$$

Proof. In the stated notations,

$$\int_c^d f_1(x_{\alpha,b})\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(f_2(x)\right)\ d_{q,\beta}\ x$$

$$= \int_c^d f_1(x_{\alpha,b})\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(f_2(x)\right)\Gamma_q(\beta+1)\, d_q x$$

$$= \int_c^d f_1(x_{\alpha,b})\ \left(\frac{1}{\Gamma_q(\beta+1)} D_q\left(f_2(x_{\alpha,b})\right)\right)\Gamma_q(\beta+1)\, d_q x,$$

by Theorem 2.2.

Now, applying the formula for q-integration by parts [14, p. 313], we get

$$\int_c^d f_1(x)\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\left(f_2(x)\right)\ d_{q,\beta}\ x$$

$$= \int_c^d f_1(x_{\alpha,b})\ D_q\left(f_2(x_{\alpha,b})\right)\ d_q x$$

$$= [f_1(x_{\alpha,b})f_2(x_{\alpha,b})]_c^d - \int_c^d f_2(qx_{\alpha,b}) D_q(f_1(x_{\alpha,b}))\, d_q x$$

$$= [f_1(x_{\alpha,b})f_2(x_{\alpha,b})]_c^d - \int_c^d f_2(qx_{\alpha,b})\ D_q(f_1(x_{\alpha,b}))\left(\frac{1}{\Gamma_q(\beta+1)}\right)\ d_{q,\beta}\ x$$

$$= [f_1(x_{\alpha,b})f_2(x_{\alpha,b})]_c^d - \int_c^d f_2(qx_{\alpha,b})\ {}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b}\ \left(f_1(x)\right) d_{q,\beta}\ x,$$

by again using Theorem 2.2. ∎

3. Application

In this section, we have derived a q-analogue of the falling body problem [2] using the q-α-RLGT M-fractional derivative.

Consider the falling of an object of mass m on the earth gravitational field through the air from a height h with initial velocity v_0. The classical equation of motion for the particle is given by [7, 8]

$$m\frac{dv}{dt} = -mg - mkv, \tag{3.1}$$

where k is a positive constant and g is the gravitational force. The initial conditions are given as

$$v(0) = v_0, \quad z(0) = h,$$

where $z(t)$ is the vertical distance of the particle at aritrary time t and $\dfrac{dz(t)}{dt} = v(t)$. It can be easily observe that the solution of (3.1) for the velocity and distance are obtained as

$$v = \frac{mg}{k} - \frac{A}{k}e^{-\frac{kt}{m}}, \text{ where } A \neq 0 \tag{3.2}$$

and

$$z = \frac{A}{m}e^{-\frac{kt}{m}} \tag{3.3}$$

respectively.

A q-analogue of (3.1) using the q-α-RLGT M-fractional derivative is

$${}_0\mathfrak{D}^{\alpha,\beta}_{q,M,b} v_q = -g - kv_q, \quad q \in (0,1), \tag{3.4}$$

with $v_q(0) = 0,\ z_q(0) = h$.

Using Theorem 2.2, the above equation becomes

$$\frac{1}{\Gamma_q(\beta+1)} D_q v_q(t_{\alpha,b}) = -g - kv_q. \tag{3.5}$$

In order to solve (3.5), we assume the solution in the series form

$$v_q(t_{\alpha,b}) = \sum_{n=0}^{\infty} a_{n,q}\ t^n_{\alpha,b}. \tag{3.6}$$

Therefore,

$$\begin{aligned}
\frac{1}{\Gamma_q(\beta+1)}\ D_q\left(\sum_{n=0}^{\infty} a_{n,q}\ t_{\alpha,b}^{n}\right) &= \frac{1}{\Gamma_q(\beta+1)}\sum_{n=0}^{\infty}[n]_q\ a_{n,q}\ t_{\alpha,b}^{n-1} \\
&= \frac{1}{\Gamma_q(\beta+1)}\sum_{n=1}^{\infty}[n]_q\ a_{n,q}\ t_{\alpha,b}^{n-1}, \quad \text{where } [0]_q = 0, \\
&= \frac{1}{\Gamma_q(\beta+1)}\sum_{n=0}^{\infty}[n+1]_q\ a_{n+1,q}\ t_{\alpha,b}^{n}.
\end{aligned}$$

Now, from (3.5), we have

$$\frac{1}{\Gamma_q(\beta+1)}\sum_{n=0}^{\infty}[n+1]_q\ a_{n+1,q}\ t_{\alpha,b}^{n} = -g - k\sum_{n=0}^{\infty} a_{n,q}\ t_{\alpha,b}^{n},$$

which gives

$$\begin{aligned}
a_1, q &= \frac{-g - ka_{0,q}}{[1]_q}\ \Gamma_q(\beta+1), \\
a_{n+1}, q &= \frac{-ka_{n,q}}{[n+1]_q}\ \Gamma_q(\beta+1),\ n \geq 1.
\end{aligned} \tag{3.7}$$

From this comparison, the n-term coefficient can be expressed as

$$a_{n,q} = \frac{(-1)^n k^{n-1} g + (-k)^n a_{0,q}}{[n]_q!}\ \Gamma_q^n(\beta+1),\ n \geq 1. \tag{3.8}$$

Therefore, the instantaneous q-α-RLGT velocity is obtained as

$$\begin{aligned}
v_q(t_{\alpha,b}) &= a_{0,q} + \sum_{n=1}^{\infty} a_{n,q} t_{\alpha,b}^{n} \\
&= a_{0,q} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n k^{n-1} g + (-k)^n a_{0,q}}{[n]_q!}\ \Gamma_q^n(\beta+1)\right) t_{\alpha,b}^{n} \\
&= a_{0,q} + \sum_{n=1}^{\infty}\left(\frac{\frac{g}{k}(-kt_{\alpha,b})^n + (-kt_{\alpha,b})^n a_{0,q}}{[n]_q!}\ \Gamma_q^n(\beta+1)\right) \\
&= a_{0,q} + \left(\frac{g}{k} + a_{0,q}\right)\sum_{n=1}^{\infty}\frac{(-kt_{\alpha,b})^n}{[n]_q!}\ \Gamma_q^n(\beta+1).
\end{aligned}$$

For $v_q(0)=0$, we have

$$\begin{aligned}
v_q(t_{\alpha,b}) &= \frac{g}{k}\left(\sum_{n=1}^{\infty}\frac{(-kt_{\alpha,b})^n}{[n]_q!}\ \Gamma_q^n(\beta+1)\right)\\
&= \frac{g}{k}\left(\sum_{n=1}^{\infty}\frac{(-k)^n(t_{\alpha,b})^n}{[n]_q!}\ \Gamma_q^n(\beta+1)\right)\\
&= \frac{g}{k}\left(\sum_{n=1}^{\infty}\frac{(-k)^n(b-t)^{\alpha n}}{[n]_q!}\ \Gamma_q^n(\beta+1)\right)\\
&= \frac{g}{k}\left(\sum_{n=0}^{\infty}\frac{(-k)^{n+1}(b-t)^{\alpha(n+1)}}{[n+1]_q!}\ \Gamma_q^{n+1}(\beta+1)\right)\\
&= -\frac{g}{k}\ \Gamma_q(\beta+1)(b-t)^{\alpha}+\frac{g}{k}\sum_{n=1}^{\infty}\frac{(-k)^n\Gamma_q^n(\beta+1)}{[n]_q!}\ (b-t)^{\alpha n}. \qquad (3.9)
\end{aligned}$$

The vertical distance $z_q(t_{\alpha,b})$ in quantum calculus is governed by

$$D_q z_q(t_{\alpha,b}) = -\frac{g}{k}\ \Gamma_q(\beta+1)(b-t)^{\alpha}+\frac{g}{k}\sum_{n=1}^{\infty}\frac{(-k)^n\Gamma_q^n(\beta+1)}{[n]_q!}\ (b-t)^{\alpha n}. \qquad (3.10)$$

Taking q-right M-integral on both the sides of (3.10), we get

$$\begin{aligned}
&\mathfrak{I}^{\beta}_{q,M,b}\left(D_q z_q(t_{\alpha,b})\right)\\
&= \mathfrak{I}^{\beta}_{q,M,b}\left(-\frac{g}{k}\ \Gamma_q(\beta+1)(b-t)^{\alpha}+\frac{g}{k}\sum_{n=1}^{\infty}\frac{(-k)^n\Gamma_q^n(\beta+1)}{[n]_q!}\ (b-t)^{\alpha n}\right).
\end{aligned}$$

Using Theorem 2.8 and Definition 2.2, we obtain

$$\begin{aligned}
&z_q(t_{\alpha,b})-z_q(b)\\
&= -\frac{g}{k}\ \Gamma_q(\beta+1)\left(\Gamma_q(\beta+1)\int_b^t (b-x)^{-\alpha}\,d_qx\right)\\
&\quad+\frac{g}{k}\ \left(\Gamma_q(\beta+1)\int_b^t\sum_{n=1}^{\infty}\frac{(-k)^n\ \Gamma_q^n(\beta+1)}{[n]_q!}(b-x)^{\alpha n}\,d_qx\right)\\
&= -\frac{g}{k}\ \Gamma_q^2(\beta+1)\left[-\frac{(b-x)^{\alpha+1}}{[\alpha+1]_q}\right]_b^t\\
&\quad+\frac{g}{k}\ \sum_{n=1}^{\infty}\frac{(-k)^n\ \Gamma_q^{n+1}(\beta+1)}{[n]_q!}\left[-\frac{(b-x)^{\alpha n+1}}{[\alpha n+1]_q}\right]_b^t\\
&= \frac{g}{k}\ \Gamma_q^2(\beta+1)\ \frac{(b-t)^{\alpha+1}}{[\alpha+1]_q}-\frac{g}{k}\ \sum_{n=1}^{\infty}\frac{(-k)^n\ \Gamma_q^{n+1}(\beta+1)}{[n]_q!}\frac{(b-t)^{\alpha n+1}}{[\alpha n+1]_q}.
\end{aligned}$$

With $z_q(b) = h$, we have

$$z_q(t_{\alpha,b}) = h + \frac{g}{k}\, \Gamma_q^2(\beta+1)\, \frac{(b-t)^{\alpha+1}}{[\alpha+1]_q} - \frac{g}{k} \sum_{n=1}^{\infty} \frac{(-k)^n\, \Gamma_q^{n+1}(\beta+1)}{[n]_q!} \frac{(b-t)^{\alpha n+1}}{[\alpha n+1]_q}.$$

The comparison of the solutions through Newton derivative and traditional q-derivative with the q-α-RLGT M-fractional derivative are shown in Figures 1 and 2 for various values of α and q. It can be seen from the graphs that we can control the q-distance by choosing suitable parameters of the q-α-RLGT M-fractional derivative operator.

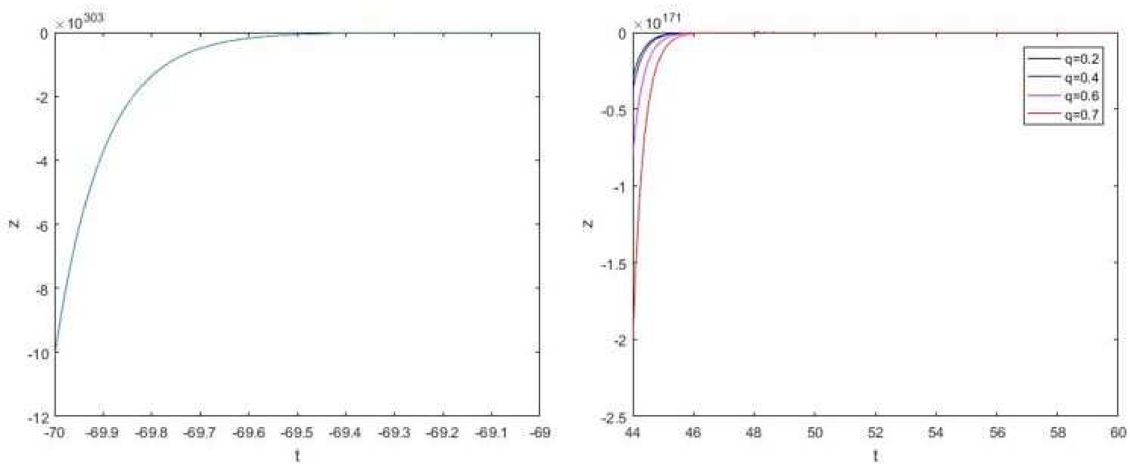

FIGURE 1. Newton derivative and traditional q-derivative ($\alpha = 1$) solutions for the q-distance

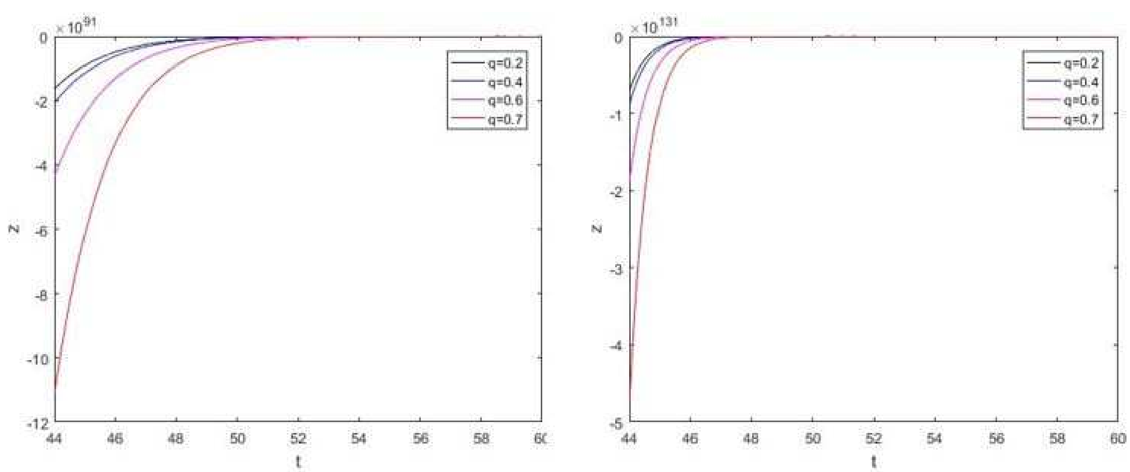

FIGURE 2. Solutions of (3.4) for the q-distance for $\alpha = 0.2$ and $\alpha = 0.6$

Figures 3 and 4 show the comparison of the solutions through Newton derivative and traditional q-derivative with the q-α-RLGT M-fractional derivative for various values of α and q. It can be seen from the graphs that we can control the q-velocity by choosing suitable parameters of the q-α-RLGT M-fractional derivative operator.

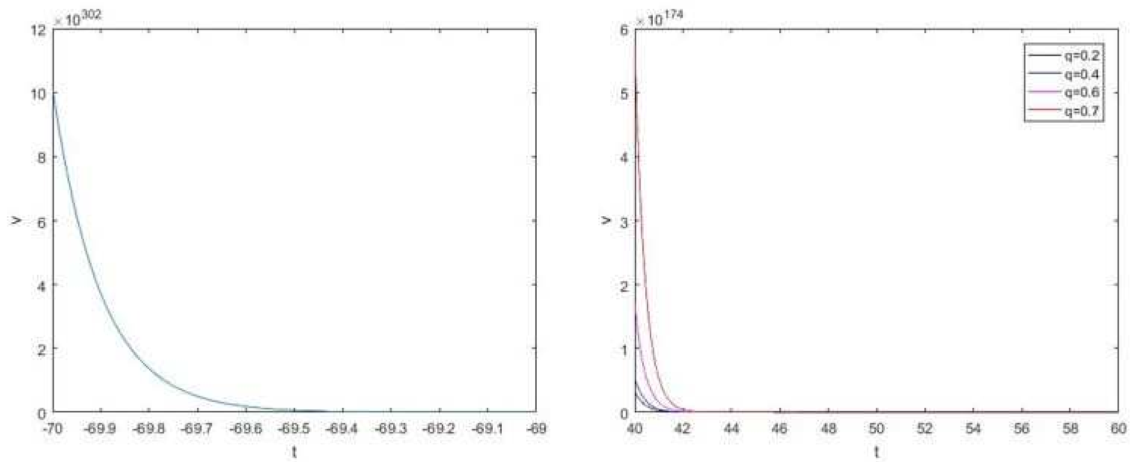

FIGURE 3. Newton derivative and traditional q-derivative ($\alpha = 1$) solutions of (3.4) for the q-vertical velocity

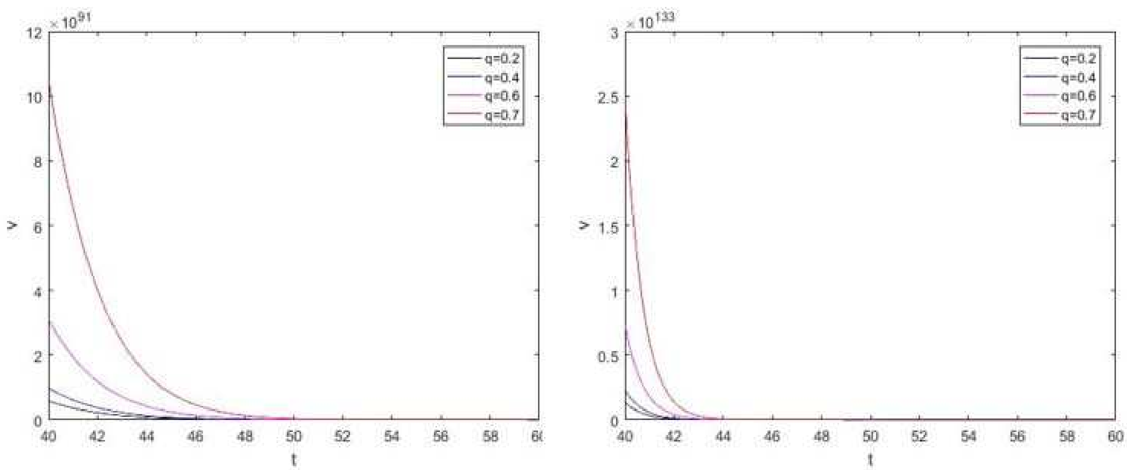

FIGURE 4. Solutions of (3.4) for the q-vertical velocity for $\alpha = 0.2$ and $\alpha = 0.6$

4. CONCLUSION

We have established a q-analogue of the α-right local general truncated M-fractional derivative [6] and a q-right M-integral. Additionally, we could find the associations between the q-α-RLGT M-fractional derivative and q-right M-integral. The

q-analogues of the well known results like the Rolle's theorem, the mean value theorem, the fundamental theorem of calculus and the theorem containing integration by parts are also obtained for our newly defined q-fractional derivative operator.

Also, using the proved results in the previous sections, we have obtained the exact solutions for the q-vertical velocity and the q-distance of a q-analogue of the well known physical problem, *the falling body problem*, by our newly defined q-α-RLGT M-fractional derivative operator. With the use of MATLAB software, we have compared the solutions of q-falling body problem for the q-vertical velocity and the q-distance for various fractional order α and an integer order by considering different values of q. Also, these solutions are compared with with the Newton derivative and the traditional q-derivative.

As a future perspective, one can define and study the results of q-α-left local general truncated M-fractional derivative which is defined as follows:

Let $f : [a, \pi_2] \to \mathbb{R}$ and $t < \pi_2,\ a \in \mathbb{R}$. For $0 < \alpha \leq 1$, we define a *q-α-left local general truncated M-fractional derivative* of order α of f as

$$ {}^{\pi_2}\mathfrak{D}^{\alpha,\beta}_{q,M,a} f(t) := \frac{1}{\Gamma_q(\beta+1)} \frac{(f((t-a)^{\alpha}) - f(q(t-a)^{\alpha}) + (1-q)\pi_2)}{(1-q)((t-a)^{\alpha} - \pi_2)}. \tag{4.1} $$

Also, one can work on the possible open problems by analyzing and studying the results and inequalities proved in [16, 17] for our newly defined q-α-RLGT M-fractional derivative.

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References

[1] Haydar Akça, Jamal Benbourenane and Hichem Eleuch, The q-derivative and differential equation, *Journal of Physics: Conference Series*, **1411** (2019), 012002.

[2] Abdulaziz Alanazi, Abdelhalim Ebaid, Wadha Alhawiti and Ghulam Muhiuddin, The falling body problem in quantum calculus, *Frontiers in Physics*, **8(43)** (2020), 1–5.

[3] Khaled A. Aldwoah, Agnieszka B. Malinowska, and Delfim F. M. Torres, The power quantum calculus and variational problems, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms*, **19 (2012)** (2012), 93–116.

[4] Mahmoud H. Annaby and Zeinab S. Mansour, q-Fractional Calculus and Equations, Springer, 2012.

[5] Ali ARAL, Vijay Gupta and Ravi P. Agarwal, Applications of q-Calculus in Operator Theory, Springer, 2013.

[6] Rajendrakumar B. Chauhan and Meera H. Chudasama, A study of the right local general truncated M-fractional derivative, *Communications in Korean Mathematical Society*, (2021). https://doi.org/10.4134/CKMS.c210098

[7] A. Ebaid, B. Masaedeh and E. El-Zahar, A new fractional model for the falling body problem. *Chinese Physics Letters*, **34(2)** (2017), 020201.

[8] JJR Garcia, M. G. Calderon, J. M. Ortiz and D. Baleanu, Motion of a particle in a resisting medium using fractional calculus approach, *Proceedings of the romanian academy, Series A*, **14(1)** (2013), 42–47.

[9] G. Gasper and M. Rahman, Basic hypergeometric Series, Cambridge University press, 1990.

[10] H. Gauchman, Integral inequalities in q-calculus, *Computers and Mathematics with Applications*, **47(2)**(2004), 281–300. https://doi.org/10.1016/S0898-1221(04)90025-9

[11] F. H. Jackson, On q-functions and a certain difference operator, *Transactions of the Royal Society of Edinburgh*, **46(2)** (1909), 253–281. http://dx.doi.org/10.1017/S0080456800002751

[12] Victor Kac and Pokman Cheung, Quantum Calculus, Springer, 2002.

[13] Predrag M. Rajković, Miomir S. Stanković and Sladana D. Marinković, Mean value theorems in q-calculus, *MATEMATHQKHBECHHKatematicki Vensik*, **54** (2002), 171–178.

[14] Predrag M. Rajković, Sladana D. Marinković and Miomir S. Stanković, Fractional integrals and derivatives in q-calculus. *Applicable Analysis and Discrete Mathematics*, **1(1)** (2007), 311–323.

[15] J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, *Advances in Difference Equations*, **2013** (2013), 1–19.

[16] Miguel Vivas-Cortez, Artion Kashuri, Rozana Liko and Jorge E. Hernández Hernández, Some new q—integral inequalities using generalized quantum montgomery identity via Preinvex Functions, *Symmetry*, **12(4)** (2020), 553. https://doi.org/10.3390/e23070828

[17] Miguel Vivas-Cortez, Muhammad Aamir Ali, Hüseyin Budak, Humaira Kalsoom and Praveen Agarwal, Some new Hermite–Hadamard and related inequalities for Convex functions via (p, q)-integral, *Entropy*, **23(7)** (2021), 828. https://doi.org/10.3390/e23070828

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