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BAHADUR'S STOCHASTIC COMPARISON OF ASYMPTOTIC RELATIVE EFFICIENCY IN COMBINING INFINITELY MANY INDEPENDENT TESTS IN CASE OF CONDITIONAL EXTREME VALUE DISTRIBUTION

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Abstract. Bahadur's stochastic comparison of asymptotic relative efficiency of combining Infinitely many independent tests in case of conditional extreme value distribution is proposed. Six distribution-free combination producers namely; Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values were studied. Several comparisons among the six procedures using the exact Bahadur's slopes were obtained. Results showed that the logistic producer is the best procedure.

1. INTRODUCTION

Bahadur's stochastic comparison is one of the most common approach in asymptotic relative efficiency for two test procedures in which the *Type I* and *Type II* error probabilities changes with increasing sample size, and also with respect to the manner in which the alternatives under consideration are required to behave.

In comparison of test procedures, let $H_0 : F \in \mathscr{F}_0$ is to be tested, where $\mathscr{F}_0$ is a family of distributions, for any test procedure T_n. The function $\gamma_n(T, F) = P_F(T_n \text{ rejects } H_0)$, for distribution functions F, represents the power function of T_n. Under H_0, $\gamma_n(T, F)$ represents the probability of a *Type I* error. The size of the test is $\alpha_n(T, \mathscr{F}_0) = \sup\limits_{F \in \mathscr{F}_0} \gamma_n(T, F)$. For $F \notin \mathscr{F}_0$, the probability of a *Type II* error is $\beta_n(T, F) = 1 - \gamma_n(T, F)$. We are interested in studying consistent tests, that is

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for fixed $F \notin \mathscr{F}_0$, $\beta_n(T, F) \to 0$ as $n \to \infty$, and unbiased tests that is $F \notin \mathscr{F}_0$, $\gamma_n(T, F) \geq \alpha_n(T, \mathscr{F}_0)$. To compare two test procedures through their power functions, we will use the asymptotic relative efficiency (ARE) for two test procedures T_A and T_B, with sample sizes n_1 and n_2 respectively, then the ratio n_1/n_2 goes to some limit. This limit is the ARE of T_B relative to T_A. In Bahadur approach, the following behaviors are satisfied: the *Type I* error is $\alpha_n \to 0$, the *Type II* error is $\beta_n \to 0$, and the alternatives is $F^n = F$ fixed.

Asymptotic relative efficiency have been considered by many authors. [2] studied six free-distribution methods (sum of p-values, inverse normal, logistic, Fisher, minimum of p-values and maximum of p-values) of combining infinitely number of independent tests when the p-values are IID rv's distributed with uniform distribution under the null hypothesis versus triangular distribution with essential support $(0, 1)$ under the alternative hypothesis. They proved that the sum of p-values method is the best method. [1] they combined infinite number of independent tests for testing simple hypotheses against one-sided alternative for normal and logistic distributions, they used four methods of combining (Fisher, logistic, sum of p-values and inverse normal). [3] studied six methods of combining independent tests. He showed under conditional shifted Exponential distribution that the inverse normal method is the best among six combination methods. [4] considered combining independent tests in case of conditional normal distribution with probability density function $X|\theta \sim N(\gamma\theta, 1)$, $\theta \in [a, \infty]$, $a \geq 0$ when $\theta_1, \theta_2, ...$ have a distribution function (DF) F_θ. They concluded that the inverse normal procedure is the best procedure.

2. Extreme Value (Gumbel) Distribution

The extreme value (Gumbel) distribution (EV(θ,1)) is used as the distribution of the maximum, or the minimum, of a number of samples of many distributions. Also, it used in the estimation of the magnitude chance of earthquakes and food levels.

The $EV(\theta, 1)$ distribution with location parameter θ, has distribution function (DF) and probability density function (pdf) that are given, respectively, by

$$F(x; \theta) = e^{-e^{-(x-\theta)}}, x \in \Re, \theta \in \Re \tag{2.1}$$

$$f(x;\theta) = e^{-(x-\theta)-e^{-(x-\theta)}} = -F(x;\theta)\ln F(x;\theta), x \in \Re, \theta \in \Re. \quad (2.2)$$

The conditional probability density function of X given Λ is

$$f(x|\Lambda) = e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}} = -F(x;\Lambda\vartheta)\ln F(x;\Lambda\vartheta), x \in \Re. \quad (2.3)$$

3. The Basic Prooblem

Consider testing the hypothesis

$$H_0^{(i)} : \eta_i = \eta_0^i, \ \ vs \ , H_1^{(i)} : \eta_i \in \Omega_i - \{\eta_0^i\} \quad (3.1)$$

such that $H_0^{(i)}$ becomes rejected for large values of some real valued continuous random variable $T^{(i)}$, $i = 1, 2, \ldots, n$. The n hypotheses are combined into one as

(3.2)

$$H_0^{(i)} : (\eta_1, ..., \eta_n) = (\eta_0^1, ..., \eta_0^n), \ \ vs \ , H_1^{(i)} : (\eta_1, ..., \eta_n) \in \left\{ \prod_{i=1}^{n} \Omega_i - \{(\eta_0^1, ..., \eta_0^n)\} \right\}$$

Where $\prod_{i=1}^{n} \Omega_i = \Omega_1 \times \Omega_2 \times ... \times \Omega_n$ is the cartesian product of sets.
For $i = 1, 2, \ldots, n$ the p-value of the i-th test is given by

$$P_i(t) = P_{H_0^{(i)}}\left(T^{(i)} > t\right) = 1 - F_{H_0^{(i)}}(t) \quad (3.3)$$

where $F_{H_0^{(i)}}(t)$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0,1)$ under $H_0^{(i)}$.

In this paper, we will consider the special case where: $\eta_i = \vartheta\Lambda_i, \ i = 1, \ldots, n$. Then our proposed model will be $W|\Lambda \sim EV(\Lambda\vartheta, 1)$, $\Lambda \in \Re\backslash(-\infty, \kappa), \kappa \geq 0$ where $\Lambda_1, \Lambda_2, ...$ are independent identically distributed with DF H_Λ with support defined on $\Lambda \in \Re\backslash(-\infty, \kappa), \kappa \geq 0$, assuming that $T^{(1)}, \ldots, T^{(n)}$ are independent,then (3.1) reduces to

$$H_0 : \vartheta = 0 \ \ vs \ \ H_1 : \vartheta > 0, \quad (3.4)$$

It follows that the p-values $P_1, \ldots, P_n$ are also iid rv's that have a $U(0,1)$ distribution under H_0, and under H_1 have a distribution whose support is a subset of the interval $(0,1)$ and is not a $U(0,1)$ distribution. Therefore, if f is the probability density function (pdf) of P, then (3.4) is equivalent to

$$H_0 : P \sim U(0,1), \ \ vs \ , H_1 : P \nsim U(0,1) \quad (3.5)$$

where P has a pdf f with support subset of the interval $(0,1)$.
By sufficiency we may assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \ldots, n$. Then we consider the sequence $\left\{T^{(n)}\right\}$ of independent test statistics, thus is we will take a random sample $X_1, \ldots, X_n$ of size n and let $n \to \infty$ and compare the six non-parametric methods via exact Bahadur slope (EBS).
The producers that we will used in this paper are Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values. These producers are based on p-values of the individual statistics T_i, and reject H_0 if

$$\Psi_{Fisher} = -2\sum_{i=1}^{n} \ln(P_i) > \chi^2_{2n,\alpha}, \Psi_{logistic} = -\sum_{i=1}^{n} \ln\left(\frac{P_i}{1-P_i}\right) > b_\alpha,$$

$$\Psi_{Normal} = -\sum_{i=1}^{n} \Phi^{-1}(P_i) > \sqrt{n}\Phi^{-1}(1-\alpha),$$

$$\Psi_{Sum} = -\sum_{i=1}^{n} P_i > C_\alpha, \Psi_{Max} = -max\ P_i < \alpha^{\frac{1}{n}}, \Psi_T = -min\ P_i < 1-(1-\alpha)^{\frac{1}{n}}.$$

where Φ is the DF of standard normal distribution.

4. Difinitions

This section lays out some basic tools to Bahadur's stochastic comparison theory that used in this article

Definition 4.1. [6] *(Bahadur efficiency and exact Bahadur slope (EBS))* Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with a probability density function $f(x,\theta)$, and we want to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \Theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \le t_i\right)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that

$$C_i(\theta) = \lim_{\theta\to\infty} -2n^{-1}\ln\left(L_n^i\right)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B\left(T_1, T_2\right) = C_1(\theta)/C_2(\theta)$.

Theorem 4.1. [6] (Large deviation theorem) *Let $X_1, X_2, \ldots, X_n$ be IID, with distribution F and put $S_n = \sum_{i=1}^{n} X_i$. Assume existence of the moment generating function (mgf) $M(z) = E_F\left(e^{zX}\right)$, z real, and put $m(t) = \inf_z e^{-zt} M(z)$. The behavior of large deviation probabilities $P\left(S_n \geq t_n\right)$, where $t_n \to \infty$ at rates slower than $O(n)$. The case $t_n = tn$, if $-\infty < t \leq EY$, then $P\left(S_n \leq nt\right) \leq [m(t)]^n$, the*

$$-2n^{-1} \ln P_F\left(S_n \geq nt\right) \to -2 \ln m(t) \;\; a.s. \;\; \left(F_\theta\right).$$

Theorem 4.2. [5] (Bahadur theorem) *Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

(1) *Under $H_1 : \theta \in \Theta - \{\theta_0\}$:*

$$n^{-\frac{1}{2}} T_n \to b(\theta) \;\; a.s. \;\; \left(F_\theta\right),$$

where $b(\theta) \in \Re$.

(2) *There exists an open interval I containing $\{b(\theta) : \theta \in \Theta - \{\theta_0\}\}$, and a function g continuous on I, such that*

$$\lim_n -2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = \lim_n -2n^{-1} \log \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = g(t), \;\; t \in I.$$

If $\{T_n\}$ satisfied (1)-(2), then for $\theta \in \Theta - \{\theta_0\}$

$$-2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(T_n)\right] \to C(\theta) \;\; a.s. \;\; \left(F_\theta\right).$$

Theorem 4.3. [3] *Let $X_1, \ldots, X_n$ be i.i.d. with probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. For $j = 1, 2$, let $T_{n,j} = \sum_{i=1}^{n} f_i(x_i)/\sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let φ_j be the test based on $T_{n,j}$. Assume $\mathbb{E}_\theta(f_i(x)) > 0, \forall \theta \in \Theta$, $\mathbb{E}_0(f_i(x)) = 0$, $Var(f_i(x)) > 0$ for $j = 1, 2$. Then*

1. If the derivative $b_j'(0)$ is finite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\frac{b_1'(0)}{b_2'(0)}\right]^2,$$

where $b_i(\theta) = \mathbb{E}_\theta(f_j(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.

2. If the derivative $b_j'(0)$ is infinite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\lim_{\theta \to 0} \frac{b_1'(\theta)}{b_2'(\theta)}\right]^2.$$

Theorem 4.4. [6] *If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0, respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any α, then if φ_j is the test based on $T_n^{(j)}$, $j = 1, 2$, then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta).$$

Corollary 4.1. [6] *If T_n is the uniformly most powerful test for all α, then it is the best via EBS.*

Theorem 4.5. [3]

$$2t \leq m_S(t) \leq et, \ \forall : 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z>0} e^{-zt} \frac{e^z - 1}{z}.$$

Theorem 4.6. [3]

(1) $m_L(t) \geq 2te^{-t}, \ \forall t \geq 0,$

(2) $m_L(t) \leq te^{1-t}, \ \forall t \geq 0.852,$

(3) $m_L(t) \leq t \left(\frac{t^2}{1+t^2} \right)^3 e^{1-t}, \ \forall t \geq 4,$

where $m_L(t) = \inf_{z \in (0,1)} e^{-zt} \pi z \ csc(\pi z)$ and csc is an abbreviation for cosecant function.

Theorem 4.7. *For $x > 0$,*

$$\phi(x) \left[\frac{1}{x} - \frac{1}{x^3} \right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ is the pdf of standard normal distribution.

Theorem 4.8. [3] *For $x > 0$,*

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

Lemma 1. [3]

(1) $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$

(2) $m_L(t) \leq \dfrac{e^{-t^2/(t+1)} \left(\frac{\pi t}{t+1} \right)}{sin \left(\frac{\pi t}{t+1} \right)}$

(3) $\begin{cases} m_s(t) = \inf_{z>0} \frac{e^{-zt}(1-e^{-z})}{z} \leq \inf_{z>0} \frac{e^{-zt}}{z} \leq -et, & t < 0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$

5. Derivation of the EBS with general DF H_Λ

In this section we will study testing problem (3.4). We will compare the six methods Fisher, logistic, sum of p-values,the inverse normal, Tippett's method and maximum of p-values using EBS.

Let $X_1, \dots, X_n$ be IID with probability density function (2.3) and we want to test (3.4). Then by (2.1), the p-value is given by

$$P_n(X_n) = 1 - F^{H_0}(X_n) = 1 - e^{-e^{-x}} \tag{5.1}$$

The next three lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), sum of p-values (C_S), Tippett's method (C_T) and maximum of p-values (C_{max})methods.

Lemma 2. The exact Bahadurs slope (EBSs) result for the tests, which is given at the end of Section 3, are as follows:

B1. Fisher method. $C_F(\vartheta) = b_F(\vartheta) - 2\ln(b_F(\vartheta)) + 2\ln(2) - 2$, where

$$b_F(\vartheta) = -2\left(\psi(1) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta} + 1)\right),$$

and $\psi(\cdot) = \frac{\Gamma'(\cdot)}{\Gamma(\cdot)}$ is the digamma function.

B2. Logistic method. $C_L(\vartheta) = -2\ln(m(b_L(\vartheta)))$, where

$$m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\ csc(\pi z)$$

and

$$b_L(\vartheta) = \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta} + 1) - \mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta} - \psi(1).$$

B3. Sum of p-values method. $C_S(\vartheta) = -2\ln(m(b_S(\vartheta)))$, where

$$m_S(t) = \inf_{z>0} e^{-zt}\frac{1 - e^{-z}}{z}$$

and

$$b_S(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta} + 1\right)^{-1}.$$

B4. Inverse Normal method. $C_N(\vartheta) = -2\ln(m(b_N(\vartheta))) = b_N^2(\vartheta)$, where

$$b_N(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\left[e^{\Lambda\vartheta}\,\mathbb{E}_{Beta(e^{\Lambda\vartheta}-1,1)}\,\phi\left(\Phi^{-1}(1-V)\right)\right]$$

Proof of B1. For Fisher procedure,

$$T_F = -2\sum_{i=1}^{n}\frac{\ln\left[1-e^{-e^{-x}}\right]}{\sqrt{n}}.$$

By Theorem 4.2 (1) and by the strong law of large number (SLLN), we have

$$\frac{T_F}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_F(\vartheta) = -2\,\mathbb{E}^{H_1}\ln\left[1-e^{-e^{-x}}\right]$$

then

$$b_F(\vartheta) = -2\,\mathbb{E}_{H_\Lambda}\,\mathbb{E}_{EV(\Lambda\vartheta,1)}\left(\ln\left[1-e^{-e^{-x}}\right]|\Lambda\right).$$

Now, let $U = e^{-(X-\Lambda\vartheta)}$, and $Z = 1-e^{-e^{-\Lambda\vartheta}U}$, then

$$\mathbb{E}_{H_\Lambda}\int_{\Re}\ln\left[1-e^{-e^{-x}}\right]e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}}\,dx$$

$$= \mathbb{E}_{H_\Lambda}\,e^{\Lambda\vartheta}\int_0^1 \ln(z)(1-z)^{e^{\Lambda\vartheta}-1}\,dz = \mathbb{E}_{H_\Lambda}\,\mathbb{E}_{Beta(1,e^{\Lambda\vartheta})}\ln Z$$

$$= \psi(1) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1).$$

Thus, $b_F(\vartheta) = -2\left(\psi(1)-\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)\right)$.

Now under H_0, then using Theorem 4.1, we have $m_S(t) = \inf_{z>0} e^{-zt}M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\left(1-e^{-e^{-x}}\right) \sim U(-1,0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by Theorem 4.2 (2), we complete the proof, that is

$$C_F(\vartheta) = -2\ln(m_F(b_F(\vartheta))) = -2\ln\left(\frac{b_F(\vartheta)}{2}e^{1-\frac{b_F(\vartheta)}{2}}\right) = b_F(\vartheta)-2\ln(b_F(\vartheta))+2\ln(2)-2.$$

□

Proof of B2. For logistic procedure,

$$T_L = -\sum_{i=1}^{n}\frac{\ln\left[\frac{1-e^{-e^{-x}}}{e^{-e^{-x}}}\right]}{\sqrt{n}}.$$

By Theorem 4.2 (1) and by the strong law of large number (SLLN), we have

$$\frac{T_L}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_L(\vartheta) = -\,\mathbb{E}^{H_1} \ln\left[\frac{1-e^{-e^{-x}}}{e^{-e^{-x}}}\right]$$

then

$$b_L(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\,\mathbb{E}_{EV(\Lambda\vartheta,1)}\left(\ln\left[\frac{1-e^{-e^{-x}}}{e^{-e^{-x}}}\right]|\Lambda\right)$$

$$= -\,\mathbb{E}_{H_\Lambda}\int_{\Re} \ln\left[1-e^{-e^{-x}}\right] e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}}\,dx - \mathbb{E}_{H_\Lambda}\int_{\Re} e^{-x}e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}}\,dx.$$

Now,
$\int_{\Re} e^{-x}e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}}\,dx = e^{-\Lambda\vartheta}$,
and from Proof (B1), $\int_{\Re} \ln\left[1-e^{-e^{-x}}\right] e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}}\,dx = \psi(1)-\psi(e^{\Lambda\vartheta}+1)$. Thus

$$b_L(\vartheta) = \mathbb{E}_{H_\Lambda}\left(\psi(e^{\Lambda\vartheta}+1)\right) - \mathbb{E}_{H_\Lambda}\left(e^{-\Lambda\vartheta}\right) - \psi(1)$$

□

Proof of B3. For sum of p-values procedure,

$$T_S = -\sum_{i=1}^{n}\frac{1-e^{-e^{-x}}}{\sqrt{n}}.$$

It follows from Theorem 4.2 (1) and by the strong law of large number (SLLN) that

$$\frac{T_S}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_S(\theta) = -\,\mathbb{E}^{H_1}\left(1-e^{-e^{-x}}\right)$$

then

$$b_S(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\,\mathbb{E}_{EV(\Lambda\vartheta,1)}\left\{\left(1-e^{-e^{-x}}\right)|\Lambda\right\} = -\,\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}+1\right)^{-1}.$$

Now, by Theorem 4.1, we have $m_S(t) = \inf_{z>0} e^{-zt}M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\left(1-e^{-e^{-x}}\right) \sim U(-1,0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 4.2 we complete the proof, we conclude that $C_S(\vartheta) = -2\ln(m_S(b_S(\vartheta)))$. □

Proof of B4. For the inverse normal procedure,

$$T_N = -\sum_{i=1}^{n}\frac{\Phi^{-1}\left(1-e^{-e^{-x}}\right)}{\sqrt{n}}.$$

By Theorem 4.2 (1) and the strong law of large number (SLLN), we have

$$n^{-\frac{1}{2}}T_N \xrightarrow{\text{w.p.1}} b_N(\vartheta) = -\,\mathbb{E}^{H_1}\,\Phi^{-1}\left(1-e^{-e^{-x}}\right),$$

$$b_N(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\,\mathbb{E}_{EV(\Lambda\vartheta,1)}\left\{\Phi^{-1}\left(1-e^{-e^{-x}}\right)|\Lambda\right\},$$

let $U = \Phi^{-1}\left(1-e^{-e^{-x}}\right)$ so we have

$$b_N(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\left\{\int_{\Re} e^{\Lambda\vartheta} u\phi(u)\left(1-\Phi(u)\right)^{e^{\Lambda\vartheta}-1}\,du\right\}$$

$$= \mathbb{E}_{H_\Lambda}\left\{\int_{\Re} e^{\Lambda\vartheta}\frac{d\phi(u)}{du}\left(1-\Phi(u)\right)^{e^{\Lambda\vartheta}-1}\,du\right\},$$

where $-u\phi(u) = \frac{d}{du}\phi(u)$. Now, by using integration by parts and substituting $V = 1-\Phi\left(U\right)$, we get

$$b_N(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\left\{e^{\Lambda\vartheta}\left(e^{\Lambda\vartheta}-1\right)\int_0^1 v^{e^{\Lambda\vartheta}-2}\phi\left(\Phi^{-1}(1-v)\right)\,dv\right\}$$

$$= -\,\mathbb{E}_{H_\Lambda}\left\{e^{\Lambda\vartheta}\,\mathbb{E}_{Beta(e^{\Lambda\vartheta}-1,1)}\,\phi\left(\Phi^{-1}(1-v)\right)\right\}$$

where $\phi^2\left(\zeta\right) = \frac{1}{\sqrt{2\pi}}\phi\left(\sqrt{2}\zeta\right)$ and $\frac{\phi\left(\sqrt{2}\zeta\right)}{\phi(\zeta)} = e^{-\frac{1}{2}\zeta^2} = \sqrt{2\pi}\phi(\zeta)$.
Now, by Theorem 1, we have $m_N(t) = \inf_{z>0} e^{-zt}M_N(z)$, where $M_N(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\left(1-e^{-e^{-x}}\right) \sim N(0,1)$, so $M_N(z) = e^{z^2/2}$, by part (2) of Theorem 4.2, $C_N(\vartheta) = -2\ln(m_N(b_N(\vartheta))) = b_N^2(\vartheta)$. □

Theorem 1. Let $U_1, U_2, \ldots$ be i.i.d. with probability density function f and suppose that we want to test $H_0 : U_i \sim U(0,1)$ vs. $H_1 : U_i \sim f$ on $(0,1)$ but not $U(0,1)$. Then $C_{max}(f) = -2\ln\left(ess.sup_f(u)\right)$
where $ess.sup_f(u) = sup\left\{u : f(u) > 0\right\}$ w.p.1 under f. [3]

Lemma 3.

$$C_{max}(\vartheta) = 0.$$

[3]

Proof. Assume that $\frac{d}{d\Lambda}H_\Lambda = g_\Lambda$ the probability density function of the DF H_Λ, then the joint probability density function of X and Λ is

$$h(x,\Lambda) = f(x|\Lambda)g_\Lambda$$

$$h(x,\Lambda) = e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}}g_\Lambda, x \in \Re.$$

The marginal probability density function of X is

$$f(x) = \int_{(\kappa,\infty)} e^{-(x-\Lambda\vartheta)-e^{-(x-\Lambda\vartheta)}} g_\Lambda \, d\Lambda, x \in \Re, \kappa \geq 0$$
$$= e^{-x} \int_{(\kappa,\infty)} e^{\Lambda\vartheta} \left(e^{-e^{-x}}\right)^{e^{\Lambda\vartheta}} dH_\Lambda.$$

Now, under ϑ the p-value $P = 1 - e^{-e^{-x}}$, so

$$h(p) = \int_{(\kappa,\infty)} e^{\Lambda\vartheta} (1-P)^{e^{\Lambda\vartheta}-1} \, dH_\Lambda, \; p \in (0,1). \tag{5.2}$$

Then by Theorem 1 we have $ess.sup_f(p) = 1$. Therefore, $C_{max}(\vartheta) = 0$. □

Theorem 2. If $\pi(\ln\pi)^2 f(\pi) \to 0$ as $\pi \to 0$, then $C_T(f) = 0$.

Lemma 4.

$$C_T(\vartheta) = 0.$$

Proof. From (5.2), we have

$$h(p) = -\int_{(\kappa,\infty)} \frac{d}{de^{\Lambda\vartheta}} (1-p)^{e^{\Lambda\vartheta}} \, dH_\Lambda = -\frac{d}{de^{\Lambda\vartheta}} \mathbb{E}_{H_\Lambda} (1-p)^{e^{\Lambda\vartheta}}. \tag{5.3}$$

So by Theorem 2, we get

$$\lim_{p\to 0} p(\ln p)^2 h(p) = -\lim_{p\to 0} p(\ln p)^2 \left\{\frac{d}{de^{\Lambda\vartheta}} \mathbb{E}_{H_\Lambda} (1-p)^{e^{\Lambda\vartheta}}\right\}.$$

Clearly, applying by L'Hopital rule twice we have, $\lim_{p\to 0} p(\ln p)^2 = 0$, also,

$$-\lim_{p\to 0} \left\{\frac{d}{de^{\Lambda\vartheta}} \mathbb{E}_{H_\Lambda} (1-p)^{e^{\Lambda\vartheta}}\right\} = 0.$$

Which implies $C_T(\vartheta) = 0$. □

6. Comparison of the EBSs when $\vartheta \to 0$

In this section, we will compare the EBSs that obtained in Section (5). We will find the limit of the ratio of the EBSs of any two methods when $\vartheta \to 0$.

Corollary 1. The limits of ratios of different tests are as follows:

C1. $\frac{C_T(\vartheta)}{C_{\mathfrak{D}}(\vartheta)} = \frac{C_{max}(\vartheta)}{C_{\mathfrak{D}}(\vartheta)} = 0$, where $C_{\mathfrak{D}}(\vartheta) \in \{C_F(\vartheta), C_L(\vartheta), C_S(\vartheta), C_N(\vartheta)\}$.

C2. $e_B(T_S, T_F) \to 1.80314$

C3. $e_B\left(T_L, T_F\right) \to 1.97729$

C4. $e_B\left(T_N, T_F\right) \to 1.96121$

C5. $e_B\left(T_L, T_N\right) \to 1.0082$

C6. $e_B\left(T_N, T_S\right) \to 1.08764$

C7. $e_B\left(T_L, T_S\right) \to 1.09656$

Proof of C2.

$$b_F(\vartheta) = -2\left(\psi(1) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)\right).$$

Therefore,

$$b_F'(\vartheta) = 2\,\mathbb{E}_{H_\Lambda}\left(\Lambda e^{\Lambda\vartheta}\psi_1(1+e^{\Lambda\vartheta})\right),$$

where $\psi_1(z) = \dfrac{d}{dz}\psi(z)$ is the trigamma function.

$$\lim_{\vartheta\to 0} b_F'(\vartheta) = 2\left(\frac{\pi^2}{6} - 1\right)\mathbb{E}_{H_\Lambda}\left(\Lambda\right) < \infty.$$

Also

$$b_S(\vartheta) = -\,\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}+1\right)^{-1},$$

then

$$\lim_{\vartheta\to 0} b_S'(\vartheta) = \lim_{\vartheta\to 0} \tfrac{1}{4}\,\mathbb{E}_{H_\Lambda}\left(\Lambda\cosh^{-2}\left(\frac{\Lambda\vartheta}{2}\right)\right) = \tfrac{1}{4}\,\mathbb{E}_{H_\Lambda}\left(\Lambda\right) < \infty.$$

Now under $H_0 : h_F(x) = -2\ln\left[1 - e^{-e^{-x}}\right] \sim \chi_2^2$ and $h_S(x) = -\left(1 - e^{-e^{-x}}\right) \sim U(-1,0)$, so $Var_{\vartheta=0}(h_F(x)) = 4$ and $Var_{\vartheta=0}(h_S(x)) = \frac{1}{12}$, also, $\dfrac{b_S'(0)}{b_F'(0)} = \left(\dfrac{8\pi^2}{6} - 8\right)^{-1}$. By applying Theorem (4.3) we get $\lim\limits_{\vartheta\to 0}\dfrac{C_S(\vartheta)}{C_F(\vartheta)} = \dfrac{27}{\left(\pi^2-6\right)^2} = 1.80314$. Similarly we can prove other parts.

□

6.1. **The Limiting ratio of the EBS for different tests when $\vartheta \to \infty$.** Now, we will compare the limit of the ratio of EBSs for any two methods when $\vartheta \to \infty$.

Corollary 2. The limits of ratios for different tests are as follows:

D1. $e_B(T_L, T_F) \to 1$

D2. $e_B(T_S, T_F) \to 1$

D3. $e_B(T_N, T_S) \to 0$

D4. $\lim_{\vartheta\to\infty} \{C_F(\vartheta) - C_L(\vartheta)\} \leq 0$

D5. $\lim_{\vartheta\to\infty} \{C_S(\vartheta) - C_L(\vartheta)\} < 0$

D6. $e_B(T_N, T_F) \to 0, e_B(T_N, T_L) \to 0, e_B(T_L, T_S) \to 1.$

Proof of D1. By Lemma (1) part (1) $C_L(\vartheta) \leq 2b_L(\vartheta)$. So

$$\frac{C_L(\vartheta)}{C_F(\vartheta)} \leq \frac{2b_L(\vartheta)}{b_F(\vartheta) - 2\ln(b_F(\vartheta)) + 2\ln(2) - 2}.$$

It is sufficient to obtain $\lim_{\vartheta\to\infty} \frac{2b_L(\vartheta)}{b_F(\vartheta)}$.
Therefore,

$$\lim_{\vartheta\to\infty} \frac{2b_L(\vartheta)}{b_F(\vartheta)} = -\lim_{\vartheta\to\infty} \frac{\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1) - \mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta} - \psi(1)}{\psi(1) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)} = 1.$$

So,

$$\lim_{\vartheta\to\infty} \frac{C_L(\vartheta)}{C_F(\vartheta)} \leq 1.$$

Also, by Theorem (4.6) part (2), we have $C_L(\vartheta) \geq 2b_L(\vartheta) - 2\ln(b_L(\vartheta)) - 2$. So

$$\lim_{\vartheta\to\infty} \frac{C_L(\vartheta)}{C_F(\vartheta)} \geq \lim_{\vartheta\to\infty} \frac{2b_L(\vartheta) - 2\ln(b_L(\vartheta)) - 2}{b_F(\vartheta) - 2\ln(b_F(\vartheta)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)}{b_F(\vartheta)}$.

Therefore,

$$\lim_{\vartheta\to\infty}\frac{2b_L(\vartheta)}{b_F(\vartheta)}=-\lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)-\mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta}-\psi(1)}{\psi(1)-\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)}=1.$$

Then,

$$\lim_{\vartheta\to\infty}\frac{C_L(\vartheta)}{C_F(\vartheta)}\geq 1$$

Thus, by pinching theorem, we have $\lim_{\vartheta\to\infty}\frac{C_L(\vartheta)}{C_F(\vartheta)}=1$. □

Proof of D2. By Lemma (1) part (3) $C_S(\vartheta)\leq -2\ln(2)-2\ln(-b_S(\vartheta))$. So

$$\lim_{\vartheta\to\infty}\frac{C_S(\vartheta)}{C_F(\vartheta)}\leq\lim_{\vartheta\to\infty}\frac{-2\ln(2)-2\ln(-b_S(\vartheta))}{b_F(\vartheta)-2\ln(b_F(\vartheta))+2\ln(2)-2}.$$

It is sufficient to obtain the limit of $\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)}$.

Then

$$\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)}=\lim_{\vartheta\to\infty}\frac{-\ln\mathbb{E}_{H_\Lambda}\left(1+e^{\Lambda\vartheta}\right)^{-1}}{\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)-\psi(1)}.$$

Now, by Jensen's inequality where the logarithm is concave function, then

$$-\ln\mathbb{E}_{H_\Lambda}\left(1+e^{\Lambda\vartheta}\right)^{-1}\leq\mathbb{E}_{H_\Lambda}\ln\left(1+e^{\Lambda\vartheta}\right),$$

so

$$\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)}\leq\lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}\ln\left(1+e^{\Lambda\vartheta}\right)}{\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)-\psi(1)}.$$

Now, by using Gauss's integral for asymptotic expansion of ψ

$$\psi(z)=\ln z-\frac{1}{2z}-\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tz}\,dt,$$

we get

$$\psi(1+e^{\Lambda\vartheta})=\ln\left(1+e^{\Lambda\vartheta}\right)-\frac{1}{2\left(1+e^{\Lambda\vartheta}\right)}-\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-t\left(1+e^{\Lambda\vartheta}\right)}\,dt$$
$$\asymp\ln\left(1+e^{\Lambda\vartheta}\right)\text{ as }\vartheta\to\infty.$$

Therefore,

$$\lim_{\vartheta\to\infty}\frac{-2\ln(-b_S(\vartheta))}{b_F(\vartheta)}\leq\lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}\ln\left(1+e^{\Lambda\vartheta}\right)}{\mathbb{E}_{H_\Lambda}\ln(e^{\Lambda\vartheta}+1)-\psi(1)}=1.$$

So

$$\lim_{\vartheta\to\infty}\frac{C_S(\vartheta)}{C_F(\vartheta)}\leq 1.$$

Also, by Lemma (1) part (3), we have $C_S(\vartheta)\geq -2-2\ln(-b_S(\vartheta))$. So, in the same manner, we get

$$\lim_{\vartheta\to\infty}\frac{C_S(\vartheta)}{C_F(\vartheta)}\geq 1.$$

Clearly, by pinching theorem, we have $\lim\limits_{\vartheta\to\infty}\dfrac{C_S(\vartheta)}{C_F(\vartheta)}=1.$ □

Proof of D3. From B4 we have

$$C_N(\vartheta)=\mathbb{E}^2_{H_\Lambda}\left[e^{\Lambda\vartheta}\,\mathbb{E}_{Beta(e^{\Lambda\vartheta}-1,1)}\,\phi\left(\Phi^{-1}(1-V)\right)\right]$$

By Lemma (1) part (3) $C_S(\vartheta)\geq -2-2\ln(-b_S(\vartheta))$, we have

$$\lim_{\vartheta\to\infty}\frac{C_N(\vartheta)}{C_S(\vartheta)}\leq\lim_{\vartheta\to\infty}\frac{\mathbb{E}^2_{H_\Lambda}\left[e^{\Lambda\vartheta}\,\mathbb{E}_{Beta(e^{\Lambda\vartheta}-1,1)}\,\phi\left(\Phi^{-1}(1-V)\right)\right]}{-2-2\ln(-b_S(\vartheta))}$$

$$=\lim_{\vartheta\to\infty}\frac{\mathbb{E}^2_{H_\Lambda}\left[e^{\Lambda\vartheta}\,\mathbb{E}_{Beta(e^{\Lambda\vartheta}-1,1)}\,\phi\left(\Phi^{-1}(1-V)\right)\right]}{-2-2\ln\mathbb{E}_{H_\Lambda}\left(1+e^{\Lambda\vartheta}\right)^{-1}}.$$

Now by using reflection symmetry, then $V\sim Beta\left(e^{\Lambda\vartheta}-1,1\right)$ then $1-V\sim Beta\left(1,e^{\Lambda\vartheta}-1\right)$, then

$$\lim_{\vartheta\to\infty}\frac{C_N(\vartheta)}{C_S(\vartheta)}\leq\lim_{\vartheta\to\infty}\frac{\mathbb{E}^2_{H_\Lambda}\left[e^{\Lambda\vartheta}\,\mathbb{E}_{Beta(1,e^{\Lambda\vartheta}-1)}\,\phi\left(\Phi^{-1}(V)\right)\right]}{-2-2\ln\mathbb{E}_{H_\Lambda}\left(1+e^{\Lambda\vartheta}\right)^{-1}}.$$

Now we will find the limiting distribution for $Z_\vartheta=e^{\Lambda\vartheta}V_\vartheta$ when $e^{\Lambda\vartheta}\to\infty$. Let,

$$G_{Z_\vartheta}\left(z_\vartheta\right)=P_\vartheta\left[Z_\vartheta\leq z_\vartheta\right]$$

$$=P_\vartheta\left[V_\vartheta\leq z_\vartheta e^{-\Lambda\vartheta}\right]=F_{Y_\vartheta}\left(z_\vartheta e^{-\Lambda\vartheta}\right)=\left(e^{\Lambda\vartheta}-1\right)\int_0^{z_\vartheta e^{-\Lambda\vartheta}}\left(1-v_\vartheta\right)^{e^{\Lambda\vartheta}-2}dv_\vartheta$$

$$=1-\left[1-\frac{z_\vartheta}{e^{\Lambda\vartheta}}\right]^{e^{\Lambda\vartheta}-1},\ 0<z_\vartheta<e^{\Lambda\vartheta}.$$

Now,

$$\lim_{e^{\Lambda\vartheta}\to\infty}G_{Z_\vartheta}\left(z_\vartheta\right)=1-\frac{\lim_{e^{\Lambda\vartheta}\to\infty}\left[1-\frac{z_\vartheta}{e^{\Lambda\vartheta}}\right]^{e^{\Lambda\vartheta}}}{\lim_{e^{\Lambda\vartheta}\to\infty}\left[1-\frac{z_\vartheta}{e^{\Lambda\vartheta}}\right]}=1-e^{-z_\vartheta},z>0.$$

Thus, $\lim\limits_{e^{\Lambda\vartheta}\to\infty}e^{\Lambda\vartheta}\,\mathrm{Beta}(1,e^{\Lambda\vartheta}-1)=\mathrm{Exponential}(1)$ and by Jensen's inequality where the logarithm is concave function, we get

$$\lim_{\vartheta\to\infty}\frac{C_N(\vartheta)}{C_S(\vartheta)}\leq\lim_{\vartheta\to\infty}\frac{\mathbb{E}^2_{\mathrm{Exp}(1)}\,\phi\left(\Phi^{-1}(e^{-\Lambda\vartheta}V_\vartheta)\right)}{2+2\,\mathbb{E}_{H_\Lambda}\ln\left(1+e^{\Lambda\vartheta}\right)}=0.$$

Hence,

$$\lim_{\vartheta\to\infty}\frac{C_N(\vartheta)}{C_S(\vartheta)}=0.$$

□

Proof of D4. By Theorem 4.6 (2), we have

$$\begin{aligned}C_F(\vartheta)-C_L(\vartheta)&\le b_F(\vartheta)-2\ln b_F(\vartheta)+2\ln(2)+2\ln b_L(\vartheta)-2b_L(\vartheta)\\&=b_F(\vartheta)-2b_L(\vartheta)+2\ln\left(\frac{b_L(\vartheta)}{b_F(\vartheta)}\right)+2\ln(2).\end{aligned}$$

Now,

$$b_F(\vartheta)-2b_L(\vartheta)=2\,\mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta}.$$

Also,

$$\lim_{\vartheta\to\infty}\frac{b_L(\vartheta)}{b_F(\vartheta)}=-\lim_{\vartheta\to\infty}\frac{\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)-\mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta}-\psi(1)}{2\left(\psi(1)-\mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1)\right)}=\frac{1}{2}.$$

Then,

$$\begin{aligned}\lim_{\vartheta\to\infty}\left(C_F(\vartheta)-C_L(\vartheta)\right)&\le\lim_{\vartheta\to\infty}\left(b_F(\vartheta)-2\ln b_F(\vartheta)\right)+2\lim_{\vartheta\to\infty}\ln\left(\frac{b_L(\vartheta)}{b_F(\vartheta)}\right)+2\ln(2)\\&=0-2\ln(2)+2ln(2)=0.\end{aligned}$$

So, $C_F(\vartheta)\le C_L(\vartheta)$ for large ϑ. □

Proof of D5. By Theorem (4.6) part (2), we have

$$C_L(\vartheta)\ge 2b_L(\vartheta)-2\ln\left(b_L(\vartheta)\right)-2$$

also by Lemma (1) part (3), we have

$$C_S(\vartheta)\le-2\ln(2)-2\ln(-b_S(\theta)),$$

we get

$$C_S(\vartheta)-C_L(\vartheta)\le d(\vartheta)$$

where

$$d(\vartheta)\equiv-2\ln(2)-2\ln(-b_S(\vartheta))-2b_L(\vartheta)+2\ln\left(b_L(\vartheta)\right)+2.$$

Since, the term $b_L(\vartheta)$ dominates the term $\ln b_L(\vartheta)$. Thus,

$$d(\vartheta)=-\ln(-b_S(\vartheta))-b_L(\vartheta).$$

Now, by (B2) and (B3), we have

$$d(\vartheta) \equiv -\ln\left(\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}+1\right)^{-1}\right) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1) + \mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta} + \psi(1).$$

Again by using Jensen's inequality, we have

$$-\ln\left(\mathbb{E}_{H_\Lambda}\left(e^{\Lambda\vartheta}+1\right)^{-1}\right) \le \mathbb{E}_{H_\Lambda}\ln\left(e^{\Lambda\vartheta}+1\right).$$

From proof (D2) we proved

$$\mathbb{E}_{H_\Lambda}\ln\left(e^{\Lambda\vartheta}+1\right) \asymp \mathbb{E}_{H_\Lambda}\,\psi\left(e^{\Lambda\vartheta}+1\right),$$

then

$$\begin{aligned} d(\vartheta) &\le \mathbb{E}_{H_\Lambda}\ln\left(e^{\Lambda\vartheta}+1\right) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1) + \mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta} + \psi(1) \\ &\asymp \mathbb{E}_{H_\Lambda}\,\psi\left(e^{\Lambda\vartheta}+1\right) - \mathbb{E}_{H_\Lambda}\,\psi(e^{\Lambda\vartheta}+1) + \mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta} + \psi(1). \end{aligned}$$

So,

$$d(\vartheta) \le \mathbb{E}_{H_\Lambda}\,e^{-\Lambda\vartheta} + \psi(1).$$

Now, when $\vartheta \to \infty$, we get

$$d(\vartheta) \le \psi(1) = -0.577216.$$

Which implies

$$\lim_{\vartheta\to\infty}\left(C_S(\vartheta) - C_L(\vartheta)\right) \le -0.577216 < 0$$

□

Proof of D6. Straight forward by using D1 to D3. □

7. Conclusion

In this section we will compare the EBS for the six combination producers. From the relations in section (6) we conclude that locally as $\vartheta \to 0$, the logistic procedure is better than all other procedures since it has the highest EBS, followed in decreasing order by the inverse normal, sum of p-values procedure and the Fisher's procedure. The worst two are the Tippett's and the maximum of p-values procedures, i.e,

$$C_L(\vartheta) > C_N(\vartheta) > C_S(\vartheta) > C_F(\vartheta) > C_T(\vartheta) = C_{max}(\vartheta).$$

Whereas, from result of Section (6.1) as $\vartheta \to \infty$ the worst methods are Tippett's and the maximum of p-values. The logistic is better than all other procedures, followed in decreasing order by sum of p-values procedure, Fisher's and the inverse normal procedures, i.e,

$$C_L(\vartheta) > C_S(\vartheta) > C_F(\vartheta) > C_N(\vartheta) > C_T(\vartheta) = C_{max}(\vartheta).$$

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