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# MONOTONE ITERATIVE TECHNIQUE FOR A COUPLED SYSTEM OF NONLINEAR CONFORMABLE FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we investigate the existence of extremal solutions for a coupled system of nonlinear conformable fractional dynamic equations on time scales, by applying the monotone iterative technique combined with the method of lower and upper solutions. At last, an example is given to illustrate our main result.

## 1. Introduction

Fractional differential equations plays an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, etc., see [16, 20, 23]. In [6], Benkhettou et al. introduced a conformable fractional calculus on an arbitrary time scale, which provides a natural extension of the conformable fractional calculus. The notion of conformable fractional calculus on an arbitrary time scales was developed in [1, 4, 5, 6, 12, 19, 25].

In [24], G. Wang et al. adopted the method of monotone iteration combined with the method of upper and lower solutions to consider the following system of nonlinear

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fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), & t \in (0, T], \\ D^{\alpha}v(t) = g(t, v(t), u(t)), & t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = x_0,\ t^{1-\alpha}v(t)|_{t=0} = y_0, \end{cases}$$

where $0 < T < \infty$, $f, g \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $x_0,\ y_0 \in \mathbb{R}$ and $x_0 \leq\ y_0$, $D^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $\alpha \in (0, 1]$.

S. Liu et al. in [17], studied the existence of extremal iteration solution to the following coupled system of conformable nonlinear fractional differential equations:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t), y(t)), & t \in [a, b], \\ y^{(\alpha)}(t) = g(t, y(t), x(t)), & t \in [a, b], \\ x(a) = x_0^*,\ y(a) = y_0^*, \end{cases} \tag{1.1}$$

where, $x_0^*, y_0^* \in \mathbb{R}$, $x_0^* \leq y_0^*$, $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $x^{(\alpha)}$, $y^{(\alpha)}$ are the conformable fractional derivatives with $\alpha \in (0, 1]$.

Motivated by the above works, this paper is concerned with the existence of extremal solutions for the following coupled system of nonlinear conformable fractional dynamic equations on time scales:

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t), y^{\sigma}(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ y_{\Delta}^{(\alpha)}(t) = g(t, y^{\sigma}(t), x^{\sigma}(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = \lambda_0,\ y(a) = \beta_0. \end{cases} \tag{1.2}$$

Here, $\mathbb{T}$ is an arbitrary bounded time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 < a < b$, $\lambda_0,\ \beta_0 \in \mathbb{R}$, $\lambda_0 \leq\ \beta_0$, $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $x_{\Delta}^{(\alpha)}$, $y_{\Delta}^{(\alpha)}$ are the conformable fractional derivatives (on time scales) with $\alpha \in (0, 1]$. For this purpose, we use the monotone iterative technique combined with the method of upper and lower solutions. For applications of monotone iterative technique combined with the method of upper and lower solutions, one can refer to literatures [2, 3, 9, 11, 13, 14, 17, 18, 21, 22, 24].

This paper is organized as follows. In Section 2, we introduce the definition of conformable fractional calculus on time scales and their important properties. In Section 3, by use of the monotone iterative technique and the method of upper and lower solutions, we prove the existence of extremal solutions of problem (1.2). Finally, an example is given to illustrate our results.

## 2. Preliminaries

In this section, we present some necessary definitions and results from conformable fractional calculus on time scales. These definitions and theorems can be found in the literatures [5, 6, 7, 8, 15, 25].

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

**Definition 2.1.** [7] The function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense point in $\mathbb{T}$ and has a left-sided limits exist at left-dense points in $\mathbb{T}$, write $f \in C_{rd}(\mathbb{T}, \mathbb{R})$.

**Definition 2.2.** [15] Given a function $f : [0, \infty) \to \mathbb{R}$ and a real constant $\alpha \in (0, 1]$. The conformable fractional derivative of $f$ of order $\alpha$ is defined by,

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0$. If $f^{(\alpha)}(t)$ exists and is finite, we say that $f$ is $\alpha$-differentiable at $t$.

**Definition 2.3.** [6] Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^{\kappa}$, and $\alpha \in ]0, 1]$. For $t > 0$, we define $f_{\Delta}^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a $\delta$-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t := ]t - \delta, t + \delta[ \cap \mathbb{T}$) of $t$, $\delta > 0$, such that

$$\left| [f(\sigma(t)) - f(s)]\, t^{1-\alpha} - f_{\Delta}^{(\alpha)}(t)\, [\sigma(t) - s] \right| \leq \epsilon\, |\sigma(t) - s| \ \ \textit{for all } s \in \mathcal{V}_t.$$

We call $f_{\Delta}^{(\alpha)}(t)$ the conformable fractional derivative of $f$ of order $\alpha$ at $t$, and we define the conformable fractional derivative at 0 as $f_{\Delta}^{(\alpha)}(0) = \lim_{t \to 0^+} f_{\Delta}^{(\alpha)}(t)$.

**Remark 1.** (1) *If* $\alpha = 1$, *we have* $f_{\Delta}^{(\alpha)} = f^{\Delta}$, *if* $\alpha = 0$, *we denote* $f_{\Delta}^{(\alpha)} = f$.

(ii) *If* $\mathbb{T}=\mathbb{R}$, *then* $f_{\Delta}^{(\alpha)}=f^{(\alpha)}$ *is the conformable fractional derivative of* $f$ *of order* $\alpha$ *introduced in* [15].

We introduce the following space:

$$C_{rd}^{\alpha}(I,\mathbb{R})=\{f\ is\ conformal\ fractional\ differentiable\ of\ order\ \alpha\ on\ I$$
$$and\ f_{\Delta}^{(\alpha)}\in C_{rd}(I,\mathbb{R})\}.$$

**Theorem 2.1.** [15] *Assume* $f,g:\mathbb{T}\to\mathbb{R}$ *are conformable fractional differentiable of order* $\alpha$. *Then,*

(i) *the sum* $f+g$ *is conformable fractional differentiable with* $(f+g)_{\Delta}^{(\alpha)}=f_{\Delta}^{(\alpha)}+g_{\Delta}^{(\alpha)}$;

(ii) *for any* $\lambda\in\mathbb{R}$, $\lambda f$ *is conformable fractional differentiable with* $(\lambda f)_{\Delta}^{(\alpha)}=\lambda f_{\Delta}^{(\alpha)}$;

(iii) *if* $f$ *and* $g$ *are continuous, then the product* $fg$ *is conformable fractional differentiable with* $(fg)_{\Delta}^{(\alpha)}=f_{\Delta}^{(\alpha)}g+(f\circ\sigma)g_{\Delta}^{(\alpha)}=f_{\Delta}^{(\alpha)}(g\circ\sigma)+fg_{\Delta}^{(\alpha)}$;

(iv) *if* $f$ *and* $g$ *are continuous, then* $f/g$ *is conformable fractional differentiable with*
$$\left(\frac{f}{g}\right)_{\Delta}^{(\alpha)}=\frac{f_{\Delta}^{(\alpha)}g-fg_{\Delta}^{(\alpha)}}{g(g\circ\sigma)},$$
*valid at all points* $t\in\mathbb{T}^{\kappa}$ *for which* $g(t)g(\sigma(t))\neq 0$.

Now we introduce the $\alpha$-conformable fractional integral (or $\alpha$-fractional integral) on time scales.

**Definition 2.4.** [6] Let $f:\mathbb{T}\to\mathbb{R}$ be a regulated function. Then the $\alpha$-fractional integral of $f$, $0<\alpha\leq 1$, is defined by $\int f(t)\Delta^{\alpha}t:=\int f(t)t^{\alpha-1}\Delta t$.

**Theorem 2.2.** [6] *Let* $\alpha\in(0,1]$, $a,b,c\in\mathbb{T}$, $\lambda\in\mathbb{R}$, *and* $f,g$ *be two rd-continuous functions. Then,*

(i) $\displaystyle\int_a^b[\lambda f(t)+g(t)]\Delta^{\alpha}t=\lambda\int_a^b f(t)\Delta^{\alpha}t+\int_a^b g(t)\Delta^{\alpha}t$;

(ii) $\displaystyle\int_a^b f(t)\Delta^{\alpha}t=-\int_b^a f(t)\Delta^{\alpha}t$;

(iii) $\displaystyle\int_a^b f(t)\Delta^{\alpha}t=\int_a^c f(t)\Delta^{\alpha}t+\int_c^b f(t)\Delta^{\alpha}t$;

(iv) *if there exist* $g : \mathbb{T} \to \mathbb{R}$ *with* $|f(t)| \le g(t)$ *for all* $t \in [a,\ b]$, *then* $\left|\int_a^b f(t)\Delta^\alpha t\right| \le \int_a^b g(t)\Delta^\alpha t$;

(v) *if* $f(t) > 0$ *for all* $t \in [a,\ b]$, *then* $\displaystyle\int_a^b f(t)\Delta^\alpha t \ge 0$.

**Definition 2.5.** [25] Let $E \subset \mathbb{T}$ be a $\Delta$-measurable set and let $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$ be a $\Delta$-measurable function. Say that $\varphi$ belongs to $L^1_{\alpha,\Delta}(E,\mathbb{R})$ provided that either

$$\int_E |\varphi(s)|\Delta^\alpha s < +\infty.$$

**Lemma 2.1.** [5] *The initial problem*

$$\begin{cases} x_\Delta^{(\alpha)}(t) - t^{1-\alpha}p\ x(\sigma(t)) = h(t), \quad t \in I = [a,b]_{\mathbb{T}}; \\ x(a) = \lambda_0, \end{cases} \tag{2.2}$$

*with* $-p \in \mathcal{R}_\mu$, $\lambda_0 \in \mathbb{R}$, *and* $h \in L^1_{\alpha,\Delta}(I,\mathbb{R})$, *has a unique solution* $x \in C^\alpha_{rd}(J,\mathbb{R})$, *given by the following expression*

$$x(t) := \int_{[a,t]_{\mathbb{T}}} e_{-p}(s,t)h(s)\Delta^\alpha s + \lambda_0 e_{-p}(a,t), \quad t \in J = [a,\sigma(b)]_{\mathbb{T}}. \tag{2.3}$$

As a direct consequence of expression (2.3), we deduce the following comparison result:

**Lemma 2.2.** *(Comparison principle 1). Let* $x \in C^\alpha_{rd}(J,\mathbb{R})$, *then the following comparison principles hold for every* $-p \in \mathcal{R}^+_\mu$:

(i) *If*

$$\begin{cases} x_\Delta^{(\alpha)}(t) - t^{1-\alpha}p\ x(\sigma(t)) \ge 0, \quad t \in I; \\ x(a) \ge 0, \end{cases}$$

*then* $x \ge 0$ *on* $J$.

(ii) *If*

$$\begin{cases} x_\Delta^{(\alpha)}(t) - t^{1-\alpha}p\ x(\sigma(t)) \le 0, \quad t \in I; \\ x(a) \le 0, \end{cases}$$

*then* $x \le 0$ *on* $J$.

## 3. Main Results

In this section, we prove the existence of extremal solutions for problem (1.2). We introduce the concept of coupled lower and upper solutions of this problem as follows.

**Definition 3.1.** We say that $\gamma$, $\delta \in C^{\alpha}_{rd}(J,\mathbb{R})$ is a pair of coupled lower and upper solutions of the problem (1.2), if $\gamma \leq \delta$ in $J$ and the following inequalities hold:

$$\begin{cases} \gamma^{(\alpha)}_{\Delta}(t) \leq f(t,\gamma^{\sigma}(t),\delta^{\sigma}(t)), \quad \text{for } \ t \in I, \ \gamma(a) \leq \lambda_0, \\ \\ \delta^{(\alpha)}_{\Delta}(t) \geq g(t,\delta^{\sigma}(t),\gamma^{\sigma}(t)), \quad \text{for } \ t \in I, \ \delta(a) \geq \beta_0. \end{cases} \tag{3.1}$$

We assume the following hypothesis:

$(H_1)$ $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

$(H_2)$ There exists $\gamma, \delta \in C^{\alpha}_{rd}(J,\mathbb{R})$, a pair of coupled lower and upper solutions of the problem (1.2).

$(H_3)$ There exist constants $p, q \in \mathbb{R}$ with $-p, -q \in \mathcal{R}^{+}_{\mu}$ and $q \leq 0$ such that

$$\begin{cases} f(t,x,y) - f(t,\overline{x},\overline{y}) \geq t^{1-\alpha}p(x-\overline{x}) + t^{1-\alpha}q(y-\overline{y}), \\ \\ g(t,\overline{y},\overline{x}) - g(t,y,x) \geq t^{1-\alpha}p(\overline{y}-y) + t^{1-\alpha}q(\overline{x}-x), \end{cases}$$

where $\gamma^{\sigma}(t) \leq \overline{x} \leq x \leq \delta^{\sigma}(t)$, $\gamma^{\sigma}(t) \leq y \leq \overline{y} \leq \delta^{\sigma}(t)$ for all $t \in I$, and

$$g(t,y,x) - f(t,x,y) \geq t^{1-\alpha}p(y-x) + t^{1-\alpha}q(x-y),$$

where $\gamma^{\sigma}(t) \leq x \leq y \leq \delta^{\sigma}(t)$ for all $t \in I$.

To study the nonlinear system (1.2), we first consider the associated linear system:

$$\begin{cases} x^{(\alpha)}_{\Delta}(t) = h_1(t) + t^{1-\alpha}p \ x(\sigma(t)) + t^{1-\alpha}q \ y(\sigma(t)), \quad \text{for } t \in I = [a,b]_{\mathbb{T}}, \\ \\ y^{(\alpha)}_{\Delta}(t) = h_2(t) + t^{1-\alpha}p \ y(\sigma(t)) + t^{1-\alpha}q \ x(\sigma(t)), \quad \text{for } t \in I = [a,b]_{\mathbb{T}}, \\ \\ x(a) = \lambda_0, \ y(a) = \beta_0, \end{cases} \tag{3.2}$$

where $\alpha \in (0,1]$, $(\lambda_0, \ \beta_0) \in \mathbb{R}^2$, $\lambda_0 \leq \ \beta_0$, $-p, -q \in \mathcal{R}_{\mu}$, $q \leq 0$ and $h_1, h_2 \in C(I,\mathbb{R})$.

**Lemma 3.1.** *The linear system (3.2) has a unique solution* $(x, y) \in C^{\alpha}_{rd}(J, \mathbb{R}) \times C^{\alpha}_{rd}(J, \mathbb{R})$, *with*

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, \quad t \in J = [a, \sigma(b)]_{\mathbb{T}},$$

*where*

$$z(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p+q)}(s, t)(h_1 + h_2)(s)\Delta^{\alpha} s + (\lambda_0 + \beta_0)e_{-(p+q)}(a, t), \quad t \in J,$$

*and*

$$w(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p-q)}(s, t)(h_1 - h_2)(s)\Delta^{\alpha} s + (\lambda_0 - \beta_0)e_{-(p-q)}(a, t), \quad t \in J.$$

*Proof.* The pair $(x, y) \in C^{\alpha}_{rd}(J, \mathbb{R}) \times C^{\alpha}_{rd}(J, \mathbb{R})$ is a solution to system (3.2) if and only if

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, \text{ for every } t \in J = [a, \sigma(b)]_{\mathbb{T}},$$

where $z(t)$ and $w(t)$ are the solutions to the following problems:

$$\begin{cases} z^{(\alpha)}_{\Delta}(t) = (h_1(t) + h_2(t)) + t^{1-\alpha}(p + q)\ z(\sigma(t)), & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ z(a) = \lambda_0 + \beta_0, \end{cases}$$

and

$$\begin{cases} w^{(\alpha)}_{\Delta}(t) = (h_1(t) - h_2(t)) + t^{1-\alpha}(p - q)\ w(\sigma(t)), & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ w(a) = \lambda_0 - \beta_0. \end{cases}$$

By Lemma 2.1, we have

$$z(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p+q)}(s, t)(h_1 + h_2)(s)\Delta^{\alpha} s + (\lambda_0 + \beta_0)e_{-(p+q)}(a, t), \quad t \in J, \tag{3.3}$$

$$w(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p-q)}(s, t)(h_1 - h_2)(s)\Delta^{\alpha} s + (\lambda_0 - \beta_0)e_{-(p-q)}(a, t), \quad t \in J. \tag{3.4}$$

The proof is finished. $\square$

**Lemma 3.2.** (*Comparison principle 2*). *Let* $(x,y) \in C^{\alpha}_{rd}(J,\mathbb{R}) \times C^{\alpha}_{rd}(J,\mathbb{R})$ *satisfy*

$$\begin{cases} x^{(\alpha)}_{\Delta}(t) - t^{1-\alpha}p\ x(\sigma(t)) + t^{1-\alpha}q\ y(\sigma(t)) \geq 0, & \textit{for } t \in I = [a,b]_{\mathbb{T}}, \\ y^{(\alpha)}_{\Delta}(t) - t^{1-\alpha}p\ y(\sigma(t)) + t^{1-\alpha}q\ x(\sigma(t)) \geq 0, & \textit{for } t \in I = [a,b]_{\mathbb{T}}, \\ x(a) \geq 0,\ y(a) \geq 0, \end{cases} \tag{3.5}$$

*where* $\alpha \in (0,1]$, $-p, -q \in \mathcal{R}^{+}_{\mu}$ *and* $q \leq 0$. *Then* $x(t) \geq 0$, $y(t) \geq 0$ *for all* $t \in J$.

*Proof.* Let $w(t) = x(t) + y(t)$, then (3.5) is equivalent to the following:

$$\begin{cases} w^{(\alpha)}_{\Delta}(t) - t^{1-\alpha}(p-q)\ w(\sigma(t)) \geq 0, & \text{for } t \in I = [a,b]_{\mathbb{T}}, \\ w(a) \geq 0. \end{cases}$$

By Lemma 2.2, we know that $w(t) \geq 0$, for all $t \in J$, i.e., $x(t) + y(t) \geq 0$, for all $t \in J$,

So.

$$\begin{cases} x^{(\alpha)}_{\Delta}(t) - t^{1-\alpha}(p+q)\ x(\sigma(t)) \geq 0, & \text{for } t \in I = [a,b]_{\mathbb{T}}, \\ y^{(\alpha)}_{\Delta}(t) - t^{1-\alpha}(p+q)\ y(\sigma(t)) \geq 0, & \text{for } t \in I = [a,b]_{\mathbb{T}}, \\ x(a) \geq 0,\ y(a) \geq 0. \end{cases}$$

By Lemma 2.2, we have $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \in J$. The proof is completed. □

The obtained result is the following.

**Theorem 3.1.** *Assume that* $(H_1)$, $(H_2)$ *and* $(H_3)$ *hold. Then* (1.2) *has an extremal system of solutions* $(x^*(t), y^*(t)) \in [\gamma(t), \delta(t)] \times [\gamma(t), \delta(t)]$, *and there exist two monotone sequences* $\{y_n\}_{n\in\mathbb{N}}$ *and* $\{z_n\}_{n\in\mathbb{N}}$ *converging uniformly to* $x^*(t), y^*(t)$, *respectively, where* $y_n(t), z_n(t) \in [\gamma(t), \delta(t)]$, *such that*

$$\gamma =: y_0 \leq y_1 \leq ... \leq y_n \leq ... \leq z_n \leq ... \leq z_1 \leq z_0 := \delta, \quad \textit{on } J \textit{ for all } n \in \mathbb{N}.$$

*Proof.* Firstly, for all $y_n, z_n \in C_{rd}^{\alpha}(J,\mathbb{R})$, we consider the linear system:

$$\begin{cases} {y_{n+1}}_{\Delta}^{(\alpha)}(t) = f(t, y_n^{\sigma}(t), z_n^{\sigma}(t)) - t^{1-\alpha}p(y_n^{\sigma}(t) - y_{n+1}^{\sigma}(t)) - t^{1-\alpha}q(z_n^{\sigma}(t) - z_{n+1}^{\sigma}(t)), \\ \qquad \text{for } \Delta\text{-a.e. } t \in I, \\ {z_{n+1}}_{\Delta}^{(\alpha)}(t) = g(t, z_n^{\sigma}(t), y_n^{\sigma}(t)) - t^{1-\alpha}p(z_n^{\sigma}(t) - z_{n+1}^{\sigma}(t)) - t^{1-\alpha}q(y_n^{\sigma}(t) - y_{n+1}^{\sigma}(t)), \\ \qquad \text{for } \Delta\text{-a.e. } t \in I, \\ y_{n+1}(a) = \lambda_0, \ z_{n+1}(a) = \beta_0, \end{cases}$$

i.e.,

$$\begin{cases} {y_{n+1}}_{\Delta}^{(\alpha)}(t) = (f(t, y_n^{\sigma}(t), z_n^{\sigma}(t)) - t^{1-\alpha}(py_n^{\sigma}(t) + qz_n^{\sigma}(t))) + t^{1-\alpha}(py_{n+1}^{\sigma}(t) + qz_{n+1}^{\sigma}(t)), \\ \qquad \text{for } \Delta\text{-a.e. } t \in I, \\ {z_{n+1}}_{\Delta}^{(\alpha)}(t) = (g(t, z_n^{\sigma}(t), y_n^{\sigma}(t)) - t^{1-\alpha}(pz_n^{\sigma}(t) + qy_n^{\sigma}(t))) + t^{1-\alpha}(pz_{n+1}^{\sigma}(t) + qy_{n+1}^{\sigma}(t)), \\ \qquad \text{for } \Delta\text{-a.e. } t \in I, \\ y_{n+1}(a) = \lambda_0, \ z_{n+1}(a) = \beta_0. \end{cases} \tag{3.6}$$

By Lemma 3.1, the linear system (3.6) has a unique solution $(y_{n+1}, z_{n+1}) \in C_{rd}^{\alpha}(J,\mathbb{R}) \times C_{rd}^{\alpha}(J,\mathbb{R})$, with

$$y_{n+1}(t) = \frac{v_{n+1}(t) + w_{n+1}(t)}{2}, \ \ z_{n+1}(t) = \frac{v_{n+1}(t) - w_{n+1}(t)}{2}, \ \text{ for every } t \in J = [a, \sigma(b)]_{\mathbb{T}},$$

where

$$\begin{aligned} v_{n+1}(t) = \int_{[a,t]_{\mathbb{T}}} e_{-(p+q)}(s,t)\Big[ & f(s, y_n^{\sigma}(s), z_n^{\sigma}(s)) + g(s, z_n^{\sigma}(s), y_n^{\sigma}(s)) \\ & - s^{1-\alpha}(p+q)(y_n^{\sigma}(s) + z_n^{\sigma}(s))\Big]\Delta^{\alpha}s + (\lambda_0 + \beta_0)e_{-(p+q)}(a,t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} w_{n+1}(t) = \int_{[a,t]_{\mathbb{T}}} e_{-(p-q)}(s,t)\Big[ & f(s, y_n^{\sigma}(s), z_n^{\sigma}(s)) - g(s, z_n^{\sigma}(s), y_n^{\sigma}(s)) \\ & - s^{1-\alpha}(p-q)(y_n^{\sigma}(s) - z_n^{\sigma}(s))\Big]\Delta^{\alpha}s + (\lambda_0 - \beta_0)e_{-(p-q)}(a,t), \quad t \in J. \end{aligned}$$

Secondly, we shall prove that

$$y_n \le y_{n+1} \le z_{n+1} \le z_n, \quad on\ J\ for\ all\ n \in \mathbb{N}.$$

Let $v := y_1 - y_0 = y_1 - \gamma$, $w := z_0 - z_1 = \delta - z_1$. According to (3.6) and $(H_1)$-$(H_2)$, we have

$$\begin{cases} v_{\Delta}^{(\alpha)}(t) \ge -t^{1-\alpha}p\left(y_0(\sigma(t)) - y_1(\sigma(t))\right) - t^{1-\alpha}q\left(z_0(\sigma(t)) - z_1(\sigma(t))\right), \text{ for } \Delta\text{-a.e. } t \in I, \\ v(a) \ge \lambda_0 - \lambda_0 = 0, \\ w_{\Delta}^{(\alpha)}(t) \ge t^{1-\alpha}p\left(z_0(\sigma(t)) - z_1(\sigma(t))\right) + t^{1-\alpha}q\left(y_0(\sigma(t)) - y_1(\sigma(t))\right), \text{ for } \Delta\text{-a.e. } t \in I, \\ w(a) \ge \beta_0 - \beta_0 = 0, \end{cases}$$

i.e.,

$$\begin{cases} v_{\Delta}^{(\alpha)}(t) \ge t^{1-\alpha}pv(\sigma(t)) - t^{1-\alpha}qw(\sigma(t)), \text{ for } \Delta\text{-a.e. } t \in I, \quad v(a) \ge 0, \\ w_{\Delta}^{(\alpha)}(t) \ge t^{1-\alpha}pw(\sigma(t)) - t^{1-\alpha}qv(\sigma(t)), \text{ for } \Delta\text{-a.e. } t \in I, \quad w(a) \ge 0. \end{cases}$$

Then, by Lemma 3.2, we have $v(t) \ge 0, w(t) \ge 0$, i.e., $y_1 \ge y_0$, $z_0 \ge z_1$.
Let $\xi := z_1 - y_1$. According to (3.6) and $(H_3)$, we have

$$\begin{aligned} \xi_{\Delta}^{(\alpha)}(t) &= {z_1}_{\Delta}^{(\alpha)}(t) - {y_1}_{\Delta}^{(\alpha)}(t) \\ &= g(t, z_0^{\sigma}(t), y_0^{\sigma}(t)) - t^{1-\alpha}(pz_0^{\sigma}(t) + qy_0^{\sigma}(t)) + t^{1-\alpha}(pz_1^{\sigma}(t) + qy_1^{\sigma}(t)) \\ &\quad - f(t, y_0^{\sigma}(t), z_0^{\sigma}(t)) + t^{1-\alpha}(py_0^{\sigma}(t) + qz_0^{\sigma}(t)) - t^{1-\alpha}(py_1^{\sigma}(t) + qz_1^{\sigma}(t)) \\ &\ge t^{1-\alpha}p(z_1^{\sigma}(t) - y_1^{\sigma}(t)) - t^{1-\alpha}q(z_1^{\sigma}(t)) - y_1^{\sigma}(t)) = t^{1-\alpha}(p-q)\xi^{\sigma}(t). \end{aligned}$$

So.

$$\begin{cases} \xi_{\Delta}^{(\alpha)}(t) \ge t^{1-\alpha}(p-q)\xi^{\sigma}(t), \text{ for } \Delta\text{-a.e. } t \in I, \\ \xi(a) = \beta_0 - \lambda_0 \ge 0. \end{cases} \tag{3.7}$$

By Lemma 2.2, we have $\xi(t) \ge 0$, i.e., $z_1(t) \ge y_1(t)$ for all $t \in J$.
By mathematical induction, we can prove that

$$y_n \le y_{n+1} \le z_{n+1} \le z_n, \quad on\ J\ for\ all\ n \in \mathbb{N}.$$

Thirdly, the sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ are monotone and bounded, hence

$$\lim_{n\to\infty} y_n = x^*, \quad \lim_{n\to\infty} z_n = y^*,$$

$(x^*, y^*)$ is an extremal system of solutions to (1.2).

Finally, we prove that (1.2) has at most one extremal system of solutions. Assume that $(x, y) \in [\gamma = y_0, \delta = z_0] \times [y_0, z_0]$ is the system of solutions to (1.2), then

$$y_0 = \gamma \le x, \ y \le z_0 = \delta.$$

For some $k \in \mathbb{N}$, assume that the following relation holds

$$y_k(t) \le x(t), \ y(t) \le z_k(t), \ t \in J.$$

Let $u(t) = x(t) - y_{k+1}(t)$, $\vartheta(t) = z_{k+1}(t) - y(t)$. According to (3.6) and $(H_3)$, we have

$$\begin{aligned}
u_\Delta^{(\alpha)}(t) &= x_\Delta^{(\alpha)}(t) - {y_{k+1}}_\Delta^{(\alpha)}(t) \\
&= f(t, x^\sigma(t), y^\sigma(t)) - f(t, {y_k}^\sigma(t), {z_k}^\sigma(t)) + t^{1-\alpha} p({y_k}^\sigma(t) - {y_{k+1}}^\sigma(t)) \\
&\qquad + t^{1-\alpha} q({z_k}^\sigma(t) - {z_{k+1}}^\sigma(t)) \\
&\ge t^{1-\alpha} p(x^\sigma(t) - {y_k}^\sigma(t)) + t^{1-\alpha} q(y^\sigma(t) - {z_k}^\sigma(t)) + t^{1-\alpha} p({y_k}^\sigma(t) - {y_{k+1}}^\sigma(t)) \\
&\qquad + t^{1-\alpha} q({z_k}^\sigma(t) - {z_{k+1}}^\sigma(t)) \\
&= t^{1-\alpha} p(x^\sigma(t) - {y_{k+1}}^\sigma(t)) + t^{1-\alpha} q(y^\sigma(t) - {z_{k+1}}^\sigma(t)),
\end{aligned}$$

and

$$\begin{aligned}
\vartheta_\Delta^{(\alpha)}(t) &= {z_{k+1}}_\Delta^{(\alpha)}(t) - y_\Delta^{(\alpha)}(t) \\
&= g(t, {z_k}^\sigma(t), {y_k}^\sigma(t)) - g(t, y^\sigma(t), x^\sigma(t)) - t^{1-\alpha} p({z_k}^\sigma(t) - {z_{k+1}}^\sigma(t)) \\
&\qquad - t^{1-\alpha} q({y_k}^\sigma(t) - {y_{k+1}}^\sigma(t)) \\
&\ge t^{1-\alpha} p({z_k}^\sigma(t) - y^\sigma(t)) + t^{1-\alpha} q({y_k}^\sigma(t) - x^\sigma(t)) - t^{1-\alpha} p({z_k}^\sigma(t) - {z_{k+1}}^\sigma(t)) \\
&\qquad - t^{1-\alpha} q({y_k}^\sigma(t) - {y_{k+1}}^\sigma(t)) \\
&= t^{1-\alpha} p({z_{k+1}}^\sigma(t) - y^\sigma(t)) - t^{1-\alpha} q(x^\sigma(t) - {y_{k+1}}^\sigma(t)),
\end{aligned}$$

we can get

$$\begin{cases} u_{\Delta}^{(\alpha)}(t) \geq t^{1-\alpha}pu(\sigma(t)) - t^{1-\alpha}q\vartheta(\sigma(t)), \text{ for } \Delta\text{-a.e. } t \in I, \quad u(a) \geq 0, \\ \vartheta_{\Delta}^{(\alpha)}(t) \geq t^{1-\alpha}p\vartheta(\sigma(t)) - t^{1-\alpha}qu(\sigma(t)), \text{ for } \Delta\text{-a.e. } t \in I, \quad \vartheta(a) \geq 0. \end{cases}$$

Then, by Lemma 3.2, we have $u(t) \geq 0$, $\vartheta(t) \geq 0$, i.e., $y_{k+1}(t) \leq x(t)$, $y(t) \leq z_{k+1}(t)$, $t \in J$,
By the induction arguments, the following relation holds

$$y_n(t) \leq x(t),\ y(t) \leq z_n(t),\ t \in J.$$

Taking the limit as $n \to \infty$, we get that $x^* \leq x$, $y \leq y^*$ *on* $J$. Hence, $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ is the extremal system of solutions to (1.2). So the proof is finished. □

## 4. Example

In this section, we present an example where we apply Theorem 3.1.

**Example 4.1.** *Consider the system of nonlinear conformable fractional dynamic equations:*

$$\begin{cases} x_{\Delta}^{(\frac{1}{3})}(t) = \dfrac{t(2 - x(\sigma(t)))^2 - y^2(\sigma(t))}{\sqrt[3]{t}}, \quad t \in I = [1,2]_{\mathbb{T}}, \\ y_{\Delta}^{(\frac{1}{3})}(t) = t^{\frac{2}{3}}(2 - y(\sigma(t)))^3 - t^{-\frac{1}{3}}x^2(\sigma(t)), \quad t \in I = [1,2]_{\mathbb{T}}, \\ x(1) = 0, \quad y(1) = 0.5, \end{cases} \tag{4.1}$$

*where* $\alpha = \frac{1}{3}$, $f(t, x, y) = \dfrac{t(2-x)^2 - y^2}{\sqrt[3]{t}}$ *and* $g(t, y, x) = t^{\frac{2}{3}}(2-y)^3 - t^{-\frac{1}{3}}x^2$.
*It is clear that* $f, g$ *are continuous functions. Take* $\gamma(t) = 0 \leq \delta(t) = 2$ *for* $t \in [1, \sigma(2)]_{\mathbb{T}}$, *then*

$$\gamma_{\Delta}^{(\frac{1}{3})}(t) = 0 \leq f(t, \gamma^{\sigma}(t), \delta^{\sigma}(t)) = \frac{4(t-1)}{\sqrt[3]{t}} \ for\ t \in [1,2]_{\mathbb{T}},\ \ \gamma(1) = 0 \leq 0,$$

*and*

$$\delta_{\Delta}^{(\frac{1}{3})}(t) = 0 \geq g(t, \delta^{\sigma}(t), \gamma^{\sigma}(t)) = 0 \ for\ t \in [1,2]_{\mathbb{T}},\ \ \delta(1) = 2 \geq 0.5,$$

*then assumptions* $(H_1)$ *and* $(H_2)$ *holds.*

*Let* $x, \overline{x}, y, \overline{y} \in \mathbb{R}$, *then we have:*

$$\begin{aligned} f(t,x,y) - f(t,\overline{x},\overline{y}) &= t^{1-\frac{1}{3}}\left((2-x)^2 - (2-\overline{x})^2\right) - \frac{1}{\sqrt[3]{t}}(y^2 - \overline{y}^2) \\ &\geq t^{1-\frac{1}{3}}\left(-4(x-\overline{x}) + x^2 - \overline{x}^2\right) \\ &\geq -4t^{1-\frac{1}{3}}(x-\overline{x}) \\ &\geq -12t^{1-\frac{1}{3}}(x-\overline{x}) + 0.t^{1-\frac{1}{3}}(y-\overline{y}), \end{aligned}$$

$$\begin{aligned} g(t,\overline{y},\overline{x}) - g(t,y,x) &= t^{\frac{2}{3}}\left((2-\overline{y})^3 - (2-y)^3\right) - t^{-\frac{1}{3}}(\overline{x}^3 - x^3) \\ &\geq t^{\frac{2}{3}}\left(-4(\overline{y}-y) + 2(\overline{y}^2 - y^2) - (\overline{y}^3 - y^3)\right) \\ &\geq -12t^{1-\frac{1}{3}}(\overline{y}-y) + 0.t^{1-\frac{1}{3}}(\overline{x}-x), \end{aligned}$$

*with* $\gamma^\sigma(t) \leq \overline{x} \leq x \leq \delta^\sigma(t)$, $\gamma^\sigma(t) \leq y \leq \overline{y} \leq \delta^\sigma(t)$ *for all* $t \in I$, *and we have*

$$\begin{aligned} g(t,y,x) - f(t,x,y) &= t^{\frac{2}{3}}\left((2-y)^3 - (2-x)^2\right) + t^{-\frac{1}{3}}(y^2 - x^2) \\ &\geq t^{\frac{2}{3}}\left(-4(y-x) + (4 - x^2 + 2y^2) - y^3)\right) \\ &\geq -12t^{1-\frac{1}{3}}(y-x) + 0.t^{1-\frac{1}{3}}(x-y). \end{aligned}$$

*with* $\gamma^\sigma(t) \leq x \leq y \leq \delta^\sigma(t)$, *for all* $t \in I$.

*Hence the assumption* $(H_3)$ *holds with* $p = -12$ *and* $q = 0$. *By Theorem 3.1, the nonlinear system (4.1) has the extremal solution* $(x^*, y^*) \in C_{rd}^{\frac{1}{3}}([1,\sigma(2)]_{\mathbb{T}}) \times C_{rd}^{\frac{1}{3}}([1,\sigma(2)]_{\mathbb{T}})$, *such that* $(x^*, y^*) \in [\gamma,\delta] \times [\gamma,\delta]$ *on* $[1,\sigma(2)]_{\mathbb{T}}$, *which can be obtained by taking limits from the iterative sequences:*

$$\begin{aligned} x_{n+1}(t) = &\int_{[1,t]_{\mathbb{T}}} s^{\frac{-2}{3}} e_{12}(s,t)\left[\frac{t(2-x_n(\sigma(t)))^2 - y_n^2(\sigma(t))}{\sqrt[3]{t}} + 12(x_n^\sigma(s))\right]\Delta s, \\ &t \in J = [1,\sigma(2)]_{\mathbb{T}}, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) = 0.5e_{12}(1,t) + &\int_{[1,t]_{\mathbb{T}}} s^{\frac{-2}{3}} e_{12}(s,t)\left[t^{\frac{2}{3}}(2-y_n(\sigma(t)))^3 - t^{-\frac{1}{3}}x_n^2(\sigma(t)) + 12(y_n^\sigma(s))\right]\Delta s, \\ &t \in J, \end{aligned}$$

## Conclusion

In this paper, we have considered the existence of extremal solutions for a coupled system of nonlinear conformable fractional dynamic equations on time scales. This result will be obtained by using the monotone iterative technique combined with the method of lower and upper solutions.

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