Jordan Journal of Mathematics and Statistics (JJMS), 16(1), 2023, pp 57 - 66
DOI: https://doi.org/10.47013/16.1.4

ON $*$-STRONG COMMUTATIVITY PRESERVING WITH ENDOMORPHISMS

S. DAKIR[(1)] AND A. MAMOUNI[(1)]

Abstract. In this paper, we investigate commutativity of a prime ring with involution. More specifically, we introduce certain algebraic identities of $*$-strong commutativity with two endomorphisms, and study their connection with the commutativity of these rings. Finally, we provide examples to show that the various restrictions imposed in the hypothesis of our theorems are necessary.

1. Introduction

Throughout this paper, $R \neq \{0\}$ will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. R is 2-torsion free if whenever $2x = 0$, with $x \in R$ implies $x = 0$. R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $*$ induces the identity map on the center $Z(R)$ of R, otherwise it is said to be of the second kind. In the latter case, it is worthwhile to mentioned that $S(R) \cap Z(R) \neq \{0\}$.

Over the last 30 years, several authors have investigated the relationship between the commutativity of a ring R and certain special types of mappings defined on R (see [1], [2], [3], [5], [6], [8], [12], [15], [16], [17], [18] where further references can be

2010 *Mathematics Subject Classification.* 16N60, 16W10, 16W25.

Key words and phrases. Prime ring, involution, commutativity, endomorphism.

Received: Jan. 24, 2022 Accepted: March 12, 2022 .

found).

Recall that a mapping $f : R \longrightarrow R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [10], [19] for references). A mapping $f : R \longrightarrow R$ is said to be strong commutativity preserving (SCP) on a subset S of R if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [7], Bell and Daif investigated the commutativity of rings admitting a derivation that is SCP on a nonzero right ideal. Indeed, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subset Z(R)$. In particular, R is commutative if $I = R$. Later, Deng and Ashraf [11] proved that if there exists a derivation d of a semiprime ring R and a mapping $f : I \longrightarrow R$ defined on a nonzero ideal I such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contained a nonzero central ideal. In particular, they showed that R is commutative if $I = R$. Further, Ali and Huang [4] showed that if R is a 2-torsion free semiprime ring an d is a derivation of R satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of R, then R contains a nonzero central ideal. Many related generalizations of these results can be found in the literature (see for instance [9]). In [5] Ashraf and Rehman proved that, if R is a prime ring, I is a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) - (x \circ y) = 0$ for all $x, y \in I$, then R is commutative.

Recently, the authors in [14] are continued this line of investigation by considering new classes of mappings rather than derivations.

The present paper is motivated by the previous results, we here continue this line of investigation by considering two endomorphisms.

2. Main result

The following facts are very crucial for developing the proof of our main result. We leave its proof to the reader.

Fact 1. Let $(R, *)$ be a 2-torsion free noncommutative prime ring with involution of the second kind.

(i) If $a[x, x^*]b = 0$ for all $x \in R$, then $a = 0$ or $b = 0$.

(ii) If $[x, x^*] \circ a = 0$ for all $x \in R$, then $a = 0$

Main Theorem

Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits two endomorphisms, then the following assertions are equivalent:

(1) $[g_1(x), g_2(x^*)] - [x, x^*] = 0$;

(2) $[g_1(x), g_2(x^*)] + [x, x^*] = 0$;

(3) R is a commutative integral domain.

Remark 1. *In (1), we must suppose that g_1 and g_2 are not both the identity endomorphism of R.*

Proof. Obviously, (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2).

We must prove that (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3).

(1) $\Longrightarrow$ (3) Assume that R is a noncommutative ring. We are given that

$$[g_1(x), g_2(x^*)] - [x, x^*] = 0. \tag{2.1}$$

Substituting x by $x + y$ in (2.1), we find that

$$[g_1(x), g_2(y^*)] + [g_1(y), g_2(x^*)] - [x, y^*] - [y, x^*] = 0. \tag{2.2}$$

Replacing y by y^* in (2.2), we obtain

$$[g_1(x), g_2(y)] + [g_1(y^*), g_2(x^*)] - [x, y] - [y^*, x^*] = 0.$$

Take $(x^*)^2$ instead y in the last expression, we leads to

$$[x, x^*](g_2(x^*) + g_1(x) - x^* - x) + (g_2(x^*) + g_1(x) - x^* - x)[x, x^*] = 0$$

That is

$$[x, x^*] \circ (g_2(x^*) + g_1(x) - x^* - x) = 0$$

And replacing x by $x + h$, where $h \in H(R) \cap Z(R) \setminus \{0\}$, we get

$$[x, x^*] \circ (g_2(h) + g_1(h) - 2h) = 0$$

In view of Fact 1 (ii), we conclude that

$$g_2(h) + g_1(h) - 2h = 0$$

So $g_2(h)$ commutes with all element of $g_1(R)$ and $g_1(h)$ commutes with all element of $g_2(R)$

Now, replacing y by yh in (2.2), where $h \in Z(R) \cap H(R)$, we obtain

$$[g_1(x), g_2(y^*)]g_2(h) + [g_1(y), g_2(x^*)]g_1(h) - [x, y^*]h - [y, x^*]h = 0. \tag{2.3}$$

Right multiplying (2.2) by $g_2(h)$, we get

$$[g_1(x), g_2(y^*)]g_2(h) + [g_1(y), g_2(x^*)]g_2(h) - [x, y^*]g_2(h) - [y, x^*]g_2(h) = 0. \tag{2.4}$$

Using equations (2.3) and (2.4) one can easily see that

$$([g_1(x), g_2(y^*)])(g_1(h) - g_2(h)) + ([x, y^*] + [y, x^*])(g_2(h) - h) = 0. \tag{2.5}$$

Replacing now x by xh in the last expression, we get

$$([g_1(x), g_2(y^*)])(g_2(h) - g_1(h))g_1(h) + ([x, y^*] + [y, x^*])(g_2(h) - h)h = 0. \tag{2.6}$$

Right multiplying (2.5) by $g_1(h)$, we get

$$([g_1(x), g_2(y^*)])(g_2(h) - g_1(h))g_1(h) + ([x, y^*] + [y, x^*])(g_2(h) - h)g_1(h) = 0. \tag{2.7}$$

From equations (2.6) and (2.7), we can conclude that

$$([x, y^*] + [y, x^*])(g_2(h) - h)(g_1(h) - h) = 0. \tag{2.8}$$

Since $[g_1(u), g_2(u^*)]$ commutes with $g_2(h) - h$ and with $g_1(h) - h$, then we have $([x, y^*] + [y, x^*])[g_1(u), g_2(u^*)](g_2(h) - h)(g_1(h) - h) = 0$ for any $u, x, y \in R$.

In view of our hypothesis, we can see that $([x, y^*] + [y, x^*])[u, u^*](g_2(h) - h)(g_1(h) - h) = 0$. And applying Fact 1 (i), we have either $(g_2(h) - h)(g_1(h) - h) = 0$ or $[x, y^*] + [y, x^*] = 0$.

Assume that $[x, y^*] + [y, x^*] = 0$ for any $x, y \in R$. Replacing y by ys, where $s \in Z(R) \cap S(R) \backslash \{0\}$, we obtain

$$-[x, y^*]s + [y, x^*]s = 0. \tag{2.9}$$

The primness of R yields

$$-[x, y^*] + [y, x^*] = 0. \tag{2.10}$$

Accordingly, $[x, y] = 0$ for any $x, y \in R$, which contradicts the fact that R is not commutative. So $(g_2(h) - h)(g_1(h) - h) = 0$ for any $h \in Z(R) \cap H(R)$. Reasoning as above, we get $g_1(h) = h$ or $g_2(h) = h$ for any $h \in Z(R) \cap H(R)$.

If $g_1(h) = h$ for any $h \in Z(R) \cap H(R)$, then the equation (2.3) reduces to

$$[g_1(x), g_2(y^*)]g_2(h) + [g_1(y), g_2(x^*)]h - [x, y^*]h - [y, x^*]h = 0,$$

and so

$$[g_1(x), g_2(x^*)](g_2(h) - h) = 0 \text{ for any } x \in R.$$

Hence

$$[x, x^*](g_2(h) - h) = 0 \text{ for any } x \in R.$$

Fact 1 (i), together with the fact that R is noncommutative prime ring, we conclude that $g_2(h) = h$. By the same proof we can prove that $g_1(h) = h$ if and only if $g_2(h) = h$.

In conclusion, we have

$$g_1(h) = g_2(h) = h \text{ for any } h \in Z(R) \cap H(R).$$

Hence $g_1(s^2) = s^2$ and $g_2(s^2) = s^2$ for any $s \in Z(R) \cap S(R)$. Therefore $(g_1(s) - s)(g_1(s) + s) = 0$ and $(g_2(s) - s)(g_2(s) + s) = 0$. By using the same arguments as using above, and using Brauer's trick, we conclude that $g_1(s) = g_2(s) = s$ for any $s \in S(R) \cap Z(R)$ or $g_1(s) = g_2(s) = -s$ for any $s \in S(R) \cap Z(R)$ or $g_1(s) = -g_2(s) = s$ for any $s \in S(R) \cap Z(R)$ or $g_1(s) = -g_2(s) = -s$ for any $s \in S(R) \cap Z(R)$.

On the other hand, replacing x by xs in equation (2.1) where $s \in S(R) \cap Z(R) \backslash \{0\}$, we find

$$[g_1(x), g_2(x^*)]g_1(s)g_2(s) - [x, x^*]s^2 = 0, \tag{2.11}$$

and thus

$$[x, x^*](g_1(s)g_2(s) - s^2) = 0. \tag{2.12}$$

Since R is a noncommutative prime ring, so $g_1(s)g_2(s) = s^2$. Which reduces the four above cases to two cases as follows:

$$g_1(s) = g_2(s) = s \text{ for any } s \in S(R) \cap Z(R) \text{ or } g_1(s) = g_2(s) = -s \text{ for any } s \in S(R) \cap Z(R).$$

If $g_1(s) = g_2(s) = s$ for any $s \in S(R) \cap Z(R)$, then substituting ys for y in equation (2.2), where $s \in S(R) \cap Z(R) \backslash \{0\}$, we arrive at

$$\Big(- [g_1(x), g_2(y^*)] + [g_1(y), g_2(x^*)] + [x, y^*] - [y, x^*] \Big) s = 0. \tag{2.13}$$

Since R is prime, the last equation assures that

$$-[g_1(x), g_2(y^*)] + [g_1(y), g_2(x^*)] + [x, y^*] - [y, x^*] = 0. \tag{2.14}$$

Using (2.2) together with (2.14), we may write

$$[g_1(x), g_2(y)] - [x, y] = 0. \tag{2.15}$$

And R is a commutative ring by ([11], Theorem 4), a contradiction.

If $g_1(s) = g_2(s) = -s$ for any $s \in S(R) \cap Z(R)$ and substituting ys for y in equation (2.2), where $s \in S(R) \cap Z(R) \backslash \{0\}$, we get

$$([g_1(x), g_2(y^*)] - [g_1(y), g_2(x^*)] + [x, y^*] - [y, x^*])s = 0. \tag{2.16}$$

Since R is prime, the last equation assures that

$$[g_1(x), g_2(y^*)] - [g_1(y), g_2(x^*)] + [x, y^*] - [y, x^*] = 0. \tag{2.17}$$

Using (2.2) together with (2.17), we obtain

$$[g_1(x), g_2(y)] = [x, y]^*. \tag{2.18}$$

Replacing x by xy in (2.18), we get

$$[g_1(x), g_2(y)]g_1(y) = y^*[x, y]^*. \tag{2.19}$$

Hence

$$[x, y]g_1(y^*) = y[x, y]. \tag{2.20}$$

Replacing x by xu in (2.20), we obtain

$$[x, y]ug_1(y^*) + x[u, y]g_1(y^*) = yx[u, y] + y[x, y]u. \tag{2.21}$$

Using the equation (2.20), we find that

$$[x,y]ug_1(y^*) + xy[u,y] = yx[u,y] + y[x,y]u. \tag{2.22}$$

Then

$$[x,y]ug_1(y^*) + [x,y][u,y] = y[x,y]u. \tag{2.23}$$

Replacing u by ut in (2.23), we obtain

$$[x,y]utg_1(y^*) + [x,y][u,y]t + [x,y]u[t,y] = y[x,y]ut. \tag{2.24}$$

Therefore

$$[x,y]u[t, y + g_1(y^*)] = 0. \tag{2.25}$$

So $[x,y] = 0$ or $[t, y + g_1(y^*)] = 0$. By using the Brauer's trick, we conclude that either $[x,y] = 0$ for all $x, y \in R$ or $[t, y + g_1(y^*)] = 0$ for all $x, y \in R$. Since R is not commutative, then we have

$$y + g_1(y^*) \in Z(R) \text{ for any } y \in R. \tag{2.26}$$

So replacing y by yy^* in (2.26), we have $yy^* + g_1(yy^*) \in Z(R)$ for any $y \in R$. By ([13], Theorem 1) we conclude that R is commutative, a contradiction. Consequently, R is commutative.

(2) $\Longrightarrow$ (3) Similar to the proof of (1) $\Longrightarrow$ (3) with slight modifications. □

As a consequence of ***Main Theorem***, it follows:

Corollary 2.1 ([14], Theorem 1)**.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. Then the following assertions are equivalent:*

(1) R admits a nontrivial $*$-SCP endomorphism;

(2) R admits a nontrivial $*$-Skew SCP endomorphism;

(3) R is a commutative integral domain.

The following example proves that the condition `* is of the second kind` in ***Main Theorem*** is necessary.

Example 2.1. Let us consider $R = M_2(\mathbb{Q})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

It is straightforward to check that $(R, *)$ is a prime ring and $*$ is an involution of the first kind.
For the first Theorem, it is easily to prove that

$$[X, X^*] = 0 \text{ for all } X \in R.$$

If we take $g_1 = 0$, and g_2 an arbitrary endomorphism, then the conditions of ***Main Theorem*** are satisfied, but R is not commutative.

The following example proves that the `primeness` hypothesis in ***Main Theorem*** is necessary .

Example 2.2. Let R_1 be a commutative domain, g_1 an endomorphism on R_1 and R_2 a noncommutative ring provided with an involution of the second kind σ. If we take $*$ the involution defined on the non-prime ring $\mathcal{R} = R_1 \times R_2$ by $* = (I_d, \sigma)$, then $*$ is an involution of the second kind. Moreover, the endomorphism defined on $\mathcal{R}$ by $g(x, y) = (g_1(x), y)$ satisfies

$$[g(X), g(X^*)] = [X, X^*] \text{ for any } X \in \mathcal{R},$$

but $\mathcal{R}$ is a noncommutative ring.

Acknowledgement

We would like to thank the editor and the referees for their remarks, their time and the careful reading of the manuscript.

References

[1] S. Ali and N. A. Dar, On *-centralizing mappings in rings with involution, *Georgian Math. J.* **21(1)**(2014), 25–28 DOI 10.1515/gmj-2014-0006.

[2] S. Ali and N. A. Dar, On *-commuting mapping and derivations in rings with involution, *Turk. J. Math.* **40**(2016), 884-894 DIO:10.3906/mat.1508-61.

[3] S. Ali, N. A. Dar and A. N. Khan , On strong commutativity preserving like maps in rings with involution, *Miskolc Math. Notes.* **16(1)** (2015), 17-24.

[4] S. Ali, and S. Huang, On derivation in semiprime rings, *Algebr. Represent. Theor.* **15** (2012), no. 6, 1023-1033.

[5] M. Ashraf and N. Rehman, On commutativity of rings with derivation, *Results Math.* **42** (2002), no. 1-2, 3-8.

[6] H. E. Bell and M. N. Daif, On derivations and commutativity in prime rings, *Acta Math. Hungar.* **66** (1995), 337-343.

[7] H. E. Bell and M. N. Daif, On commutativity and strong commutativity preserving maps, *Canad. Math. Bull.* **37** (1994), 443-447.

[8] H. E. Bell and W. S. Martindale, Centralizing mappings of semiprime rings, *Canad. Math. Bull.* **30** (1987), no. 1, 92-101.

[9] M. Brešar and C. R. Miers, Strong commutativity preserving mappings of semiprime rings, *Canad. Math. Bull.* **37** (1994), 457-460.

[10] M. Brešar, Commuting traces of biaddiitive mapping, commutativity preserving mapping and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), no. 2, 525-546.

[11] Q. Deng and M. Ashraf, On strong commutativity preserving mappings, *Results. Math.* **30** (1996), no.3-4, 259-263. https://doi.org/10.1007/BF03322194.

[12] N. J. Divinsky, On commuting automorphisms of rings, *Trans. Roy. Soc. Canada. Sect. III.* **49** (1955), no. 3, 19-22.

[13] H. El Mir, A. Mamouni and L. Oukhtite, Special mappings with central values on prime rings, *Algebra Colloq.* **27**(2020), no. 3, 405-414.

[14] H. El Mir, A. Mamouni and L. Oukhtite, New classes of endomorphisms and some classification theorems, *Comm. Algebra*, doi.org/10.1080/00927872.2019.1632330.

[15] A. N. Khan and S. Ali, Involution in prime rings with endomorphisms, *AIMS Mathematics*, **5(4)**(2020) , 32-74.

[16] L. Oukhtite and A. Mamouni, Generalized derivations centralizing on Jordan ideals of rings with involution, *Turkish J. Math.* **38** (2014), no. 2, 225-232.

[17] L. Oukhtite, A. Mamouni and M. Ashraf, Commutativity theorems for rings with differential identities on Jordan ideals, *Comment. Math. Univ. Carolin.* **54** (2013), no. 4, 447-457.

[18] E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1093-1100.

[19] P. Semrl, Commutativity preserving maps, *Linear Algebra Appl.* **429** (2008), 1051-1070.

(1) Department of Mathematics, Department of Mathematics, Faculty of Sciences, Moulay Ismaïl University, Meknes; Morocco

Email address: dakirsouad0@gmail.com

(2) Department of Mathematics, Department of Mathematics, Faculty of Sciences, Moulay Ismaïl University, Meknes; Morocco

Email address: a.mamouni.fste@gmail.com