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PROPERTIES OF RATIONALIZED TOEPLITZ HANKEL OPERATORS

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Abstract. In this paper, we introduce and study the notion of Rationalized Toeplitz Hankel Matrix of order (k_1, k_2) as the two way infinite matrix (α_{ij}) such that

$$\alpha_{ij} = \alpha_{i+k_2, j+k_1}$$

where k_1 and k_2 are relatively prime non zero integers. It is proved that a bounded linear operator R on L^2 is a Rationalized Toeplitz Hankel operator [5] of order (k_1, k_2) if and only if its matrix w.r.t. the orthonormal basis $\left\{z^i : i \in \mathbb{Z}\right\}$ is a Rationalized Toeplitz Hankel matrix of the same order. Some algebraic properties of the Rationalized Toeplitz Hankel operator R_φ like normality, hyponormality and compactness are also discussed.

1. Introduction

The study of Toeplitz operators began by O. Toeplitz in 1911. A lot of work on Toeplitz and Hankel operators has been done by different mathematicians in the world. Toeplitz operators and Hankel operators became a subject of investigations for the researchers. Motivated by these, M. C. Ho [7] in the year 1995 introduced slant Toeplitz operators and later the notion of slant Hankel operators was introduced. Further, these notions have been generalized [2] to kth order slant Toeplitz and slant Hankel operators and studied simultaneously.

Motivated by all these, in [5] we introduced Rationalized Toeplitz Hankel operators which give the rationalization of all kinds of Toeplitz and Hankel operators.

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For $\varphi \in L^\infty$ a Rationalized Toeplitz Hankel operator on the space L^2 of order (k_1, k_2) is defined as

$$R_\varphi : L^2 \to L^2$$
$$R_\varphi(f) = W_{k_1} M_\varphi W_{k_2}^*(f) \ \forall f \in L^2$$

where k_1 and k_2 are non zero integers and $W_k(z^i) = \left\{ \begin{array}{ll} z^{i|k}, & i \text{ is divisible by } k \\ 0, & \text{otherwise} \end{array} \right\}$.

It is proved in [5] that if k_1 and k_2 are relatively prime then a bounded linear operator R on L^2 is a Rationalized Toeplitz Hankel operator if and only if $M_{z^{k_2}} R = R M_{z^{k_1}}$. Therefore if (α_{ij}) in the matrix of R_φ w.r.t the orthonormal basis $\{z^i : i \in \mathbb{Z}\}$ then if $\varphi = \sum a_i z^i$ is the Fourier expansion of φ.

$$\begin{aligned} \alpha_{ij} &= \left\langle R_\varphi z^j, z^i \right\rangle \\ &= \left\langle W_{k_1} M_\varphi W_{k_2}^* z^j, z^i \right\rangle \\ &= \left\langle M_\varphi z^{k_2 j}, z^{k_1 i} \right\rangle \\ &= \left\langle \varphi, z^{k_1 i - k_2 j} \right\rangle \\ &= a_{k_1 i - k_2 j} \ . \end{aligned}$$

That is,

$$\left(\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ \ldots a_{-k_1+k_2} & a_{-k_1} & a_{-k_1-k_2} & a_{-k_1-2k_2}\ldots \\ \hline & & & \\ \ldots a_{k_2} & a_0 & a_{-k_2} & a_{-2k_2}\ldots \\ \ldots a_{k_1+k_2} & a_{k_1} & a_{k_1-k_2} & a_{k_1-2k_2}\ldots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right).$$

2. Rationalized Toeplitz Hankel Matrix

We begin with the following definition :

Definition 2.1. If k_1 and k_2 are relatively prime integers then the $(k_1, k_2)^{th}$ order Rationalized Toeplitz Hankel Matrix is defined as a two way infinite matrix (α_{ij}) such that

$$\alpha_{ij} = \alpha_{i+k_2,j+k_1} \ .$$

Note: We note that this Rationalized Toeplitz Hankel matrix is the Generalized Matrix of all kinds of Toeplitz, Hankel matrices. This can be observed with the following

(1) If $k_1 = k_2 = 1$, then $R_\varphi = M_\varphi$, and so matrix
$(\alpha_{ij}) = (\alpha_{i+1,j+1})$.
That is,

$$\left(\begin{array}{c|cccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ ....a_0 & a_{-1} & a_{-2} & a_{-3} & ..... \\ \hline \\ ....a_1 & a_0 & a_{-1} & a_{-2} & ..... \\ ....a_2 & a_1 & a_0 & a_{-1} & ..... \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right)$$

which is the matrix which is constant along diagonals. i.e. two way infinite Toeplitz matrix .

(2) if $k_1 = -1, k_2 = 1$, then $\alpha_{ij} = \alpha_{i+1,j-1}$ which is a two way infinite Hankel matrix on L^2

$$\left(\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ ....a_2 & a_1 & a_0 & a_{-1}.... \\ \hline \\ ....a_1 & a_0 & a_{-1} & a_{-2}..... \\ ....a_0 & a_{-1} & a_{-2} & a_{-3}..... \\ \vdots & \vdots & \vdots & \vdots \end{array}\right) .$$

(3) If $k_1 = 2, k_2 = 1$, then $R_\varphi = W_2 M_\varphi$ which is the two way infinite slant Toeplitz matrix $\alpha_{ij} = \alpha_{i+1,j+2}$

$$\left(\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ \ldots.a_{-1} & a_{-2} & a_{-3} & a_{-4}\ldots. \\ \hline & & & \\ \ldots.a_1 & a_0 & a_{-1} & a_{-2}\ldots. \\ \ldots.a_3 & a_2 & a_1 & a_0\ldots. \\ \vdots & \vdots & \vdots & \vdots \end{array}\right).$$

(4) If $k_1 = -2$ and $k_2 = 1$, then it would be a two way infinite slant Hankel matrix

$\alpha_{ij} = \alpha_{i+1,j-2}$

$$\left(\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ \ldots.a_3 & a_2 & a_1 & a_0\ldots. \\ \hline & & & \\ \ldots.a_1 & a_0 & a_{-1} & a_{-2}\ldots. \\ \ldots.a_{-1} & a_{-2} & a_{-3} & a_{-4}\ldots. \\ \vdots & \vdots & \vdots & \vdots \end{array}\right).$$

Remark 1. *If k_1 and k_2 are not relatively prime that is if k_1 and k_2 have the greatest common divisor as d, i.e. let $k_1 = dn$ and $k_2 = dm$, then also it is proved in* [5], *that a bounded operator R on L^2, (for a pair of non zero integers k_1 and k_2), is Rationalized Toeplitz Hankel operator if and only if*

$$R \mid \widetilde{N_i} = W_m M_{\tilde{\varphi}_i} W_n^* / \tilde{N}_i$$

for some $\tilde{\varphi}_i$ in L^∞ and for $i = 1, 2, ...d-1, k_1 = dn, k_2 = dm$

$L^2 = \tilde{N_0} \oplus \tilde{N_1} \oplus .... \oplus \tilde{N_{d-1}}$, *where* $\tilde{N_0} = N_0 \oplus N_1 \oplus ......N_{m-1}$

$\tilde{N_1} = N_m \oplus N_{m+1} \oplus ......N_{2m-1}$

$\vdots$

$\tilde{N_{d-1}} = N_{(d-1)m} \oplus N_{(d-1)m+1} \oplus ......N_{dm-1}$

$N_i =$ *The closed linear span of* $\{z^{k_1 t+1} : t \in \mathbb{Z}\}$ *for* $i = 0, 1, 2.......k_1 - 1$.

Thus for any non zero integers k_1 and k_2 we can define the Rationalized Toeplitz

Hankel matrix of order (k_1, k_2) *as*

$$\alpha_{ij} = \alpha_{i+\frac{k_2}{d}, j+\frac{k_1}{d}}$$

where $d = gcd(k_1, k_2)$.

We proceed with the following result.

Theorem 2.1. *A necessary and sufficient condition for a bounded operator* R *on* L^2 *is a Rationalized Toeplitz Hankel operator of order* (k_1, k_2) *is that its matrix w.r.t. the orthonormal basis* $\{z^i : i \in \mathbb{Z}\}$ *is a Rationalized Toeplitz Hankel matrix of order* (k_1, k_2)*, where* k_1 *and* k_2 *are nonzero relatively prime integers.*

Proof. Let R be a Rationaliznd Toeplitz Hankel operator of order (k_1, k_2) and (α_{ij}) be the matrix of R w.r.t. the orthonormal basis $\{z : i \in \mathbb{Z}\}$

$$\begin{aligned}
\alpha_{ij} &= \left\langle R_\varphi z^j, z^i \right\rangle \\
&= \left\langle W_{k_1} M_\varphi W_{k_2}^* z^j, z^i \right\rangle \\
&= \left\langle M_\varphi W_{k_2}^* z^j, W_{k_1}^* z^i \right\rangle \\
&= \left\langle \varphi, z^{k_1 i - k_2 j} \right\rangle \\
&= a_{k_1 i - k_2 j} \\
&= a_{k_1(i+k_2) - k_2(j+k_1)} \\
&= \alpha_{i+k_2, j+k_1} \ .
\end{aligned}$$

$\Rightarrow$ (α_{ij}) is the Rationalized Toeplitz Hankel matrix of order (k_1, k_2).
Conversely, let (α_{ij}) be the Rationalized Toeplitz Hankel operator of order (k_1, k_2).
This implies for all i, j,

$$\begin{aligned}
\left\langle R z^j, z^i \right\rangle &= \alpha_{ij} \\
&= \alpha_{i+k_2, j+k_1} \\
&= \left\langle R z^{j+k_1}, z^{i+k_2} \right\rangle \ .
\end{aligned}$$

Now

$$\begin{aligned}\left\langle M_{z^{k_2}} R z^j, z^i \right\rangle &= \left\langle R z^j, z^{i-k_2} \right\rangle \\ &= \left\langle R z^{j+k_1}, z^i \right\rangle \\ &= \left\langle R M_{z^{k_1}} z^j, z^i \right\rangle \ .\end{aligned}$$

This is true for all i, j

$$M_{z^{k_2}} R z^j = R M_{z^{k_1}} z^j \quad \forall j \in \mathbb{Z}$$

$$\Rightarrow \quad M_{z^{k_2}} R = R M_{z^{k_1}} \ .$$

Thus by [5] R is a Rationalized Toeplitz Hankel operator of order (k_1, k_2). This completes the proof. In view of the above, for $\varphi \in L^\infty$, $R_\varphi : L^2 \to L^2$ is defined as

$$R_\varphi(z^i) = \sum_{t=-\infty}^{\infty} a_{k_1 t - k_2 i} z^t$$

where $\varphi = \sum a_i z^i$ is the Fourier expansion of φ. The adjoint R_φ^*, of the operator R_φ, is defined as

$$\begin{aligned}\left\langle R_\varphi^* z^j, z^i \right\rangle &= \left\langle z^j, R_\varphi z^i \right\rangle \\ &= \overline{\langle R_\varphi z^i, z^j \rangle} \\ R_\varphi^*(z^j) &= \sum_{t=-\infty}^{\infty} \bar{a}_{k_1 j - k_2 t} z^t \ .\end{aligned}$$

We can see that the adjoint R_φ^* in not a Rationalized Toeplitz Hankel operator. □

3. Properties of R_φ

We begin with the following

Theorem 3.1. *Let M_φ be the Multiplication operator. Then the product $M_\varphi R_\psi$ is the Rationalized Toeplitz Hankel operator and $M_{\varphi(z)} R_{\psi(z)} = R_{\varphi(z^{k_1})\psi(z)}$.*

Proof. M_φ in the Multiplication operator and R_ψ in the Rationalized Toeplitz Hankel operator, so consider

$$\begin{aligned}M_{z^{k_2}}\left(M_\varphi R_\psi\right) &= M_\varphi M_{z^{k_2}} R_\psi \\ &= \left(M_\varphi R_\psi\right) M_{z^{k_1}} \ .\end{aligned}$$

This gives by [5] that $M_\varphi R_\psi$ in Rationatized Toeplitz Hankel operator. Moreover

$$\begin{aligned} M_{\varphi(z)}R_{\psi(z)} &= M_{\varphi(z)}W_{k_1}M_{\psi(z)}W^*_{k_2} \\ &= W_{k_1}M_{\varphi\left(z^{k_1}\right)}M_{\psi(z)}W^*_{k_2} \\ &= W_{k_1}M_{\varphi\left(z^{k_1}\right)\psi(z)}W^*_{k_2} \\ &= R_{\varphi(z^{k_1})\psi(z)} \ . \end{aligned}$$

Thus $M_\varphi R_\psi$ is a Rationalized Toeplitz Hankel operater with

$$M_{\varphi(z)}R_{\psi(z)} = R_{\varphi\left(z^{k_1}\right)\psi(z)} \ .$$

(ii) $R_\psi M_\varphi = M_\varphi R_\psi$ iff φ is constant

$W_{k_1}M_{\psi(z)}W^*_{k_2}M_\varphi = M_\varphi W_{k_1}M_\psi W^*_{k_2} \Leftrightarrow W_{k_1}M_{\psi(z)\varphi(z^{k_2})}W^*_{k_2} = W_{k_1}M_{\varphi\left(z^{k_1}\right)\psi(z)}W^*_{k_2}$

$\Leftrightarrow \psi(z)\varphi\left(z^{k_2}\right) = \varphi_{(z^{k_1})}\psi(z)$.

As $\psi(z)$ is invertible, $\varphi \to R_\psi$ is one one, thus $\varphi =$ constant. □

Theorem 3.2. *If for some φ in L^∞, φ^{-1} also belongs to L^∞, then*

$$\sigma_p\left(R_{\varphi\left(z^{k_1}\right)}\right) = \sigma_p\left(R_{\varphi\left(z^{k_2}\right)}\right)$$

where σ_p is the point spectrum.

Proof. Let $\lambda \in \sigma_p\left(R_{\varphi\left(z^{k_2}\right)}\right)$

$$\begin{aligned} R_{\varphi\left(z^{k_2}\right)}f &= \lambda f \text{ Let } F = \varphi f. \text{ Then} \\ R_{\varphi\left(z^{k_1}\right)}F &= W_{k_1}\left(M_{\varphi\left(z^{k_1}\right)}W^*_{k_2}\right)F \\ &= M_\varphi W_{k_1}{W_{k_2}}^*\varphi f \\ &= M_\varphi W_{k_1}M_{\varphi\left(z^{k_2}\right)}W^*_{k_2}f \\ &= \varphi R_{\varphi\left(z^{k_2}\right)}f \\ &= \varphi\lambda f \\ &= \lambda\varphi f \\ &= \lambda f \ . \end{aligned}$$

Since $\varphi \neq 0$ i.e. we have $F = \varphi f$ ia also non zero $\Rightarrow \lambda \in \sigma_p(R_{\varphi(z^{k_1})})$.

Conversely, suppose that $\mu \in \sigma_p\left(R_{\varphi(z^{k_1})}\right)$ so there exists a non zero function g in L^2

such that $R_{\varphi\left(z^{k_1}\right)} g = \mu g$. Let $G = \varphi^{-1} g$

$$\begin{aligned} R_{\varphi\left(z^{k_2}\right)} G &= W_{k_1} M_{\varphi\left(z^{k_2}\right)} W_{k_2}^* G \\ &= W_{k_1} \varphi\left(z^{k_2}\right) W_{k_2}^* \varphi^{-1} g \\ &= W_{k_1} \varphi\left(z^{k_2}\right) \varphi^{-1}\left(z^{k_2}\right) W_{k_2}{}^* g \\ &= W_{k_1} W_{k_2}^* g \\ &= W_{k_1} W_{k_2}^* \varphi \varphi^{-1} g \\ &= \varphi^{-1} \varphi W_{k_1} W_{k_2}^* g \\ &= \varphi^{-1} W_{k_1} \varphi(z^{k_1}) W_{k_2}^* g \\ &= \varphi^{-1} W_{k_1} M_{\varphi\left(z^{k_1}\right)} W_{k_2}^* g \\ &= \varphi^{-1} R_{\varphi\left(z^{k_1}\right)} g \\ &= \varphi^{-1} \mu g \\ &= \mu \varphi^{-1} g \\ &= \mu G\ . \end{aligned}$$

Since $\varphi^{-1} g$ is also non zero $\Rightarrow \mu \in \sigma_p\left(R_{\varphi\left(z^{k_2}\right)}\right)$.
Hence $\sigma_p\left(R_{\varphi\left(z^{k_1}\right)}\right) = \sigma_p\left(R_{\varphi\left(z^{k_2}\right)}\right)$. □

Further we will discuss some properties as follows

Theorem 3.3. *For any non zero integer k, the Rationalized Toeplitz Hankel operator* $R_\varphi = W_k M_\varphi W_k^*$ *is normal. Moreover* $R_\varphi = W_k M_\varphi W_k^*$ *is self adjoint if and only if* $W_k \varphi = W_k \bar{\varphi}$.

Proof. Since we proved that in [5]

$$W_k M_\varphi W_k^* = M_{W_k \varphi}$$

It follows that R_ϕ is normal. It is self adjoint if only if

$$W_k \varphi = \overline{W_k \varphi} = W_k \bar{\varphi}\ .$$

□

Theorem 3.4. *$W_kM_\varphi W^*_{-k}$ is self adjoint if and only if $W_k\varphi = W_{-k}\bar{\varphi}$ Also it is hyponormal if and only if it is normal, ie., if and only if*

$$|W_k\varphi|^2 = |W_{-k}\varphi|^2$$

Proof.

$$\begin{aligned} As\ ||W_k|| &= 1, \text{ so} \\ W_kM_\varphi W^*_{-k} &= (W_kM_\varphi W^*_k)W^*_{-1} \\ &= M_{W_k\varphi}W_{-1} \\ \text{and } (M_{W_k\varphi}W_{-1})^* &= W_{-1}M_{W_k\bar{\varphi}} \\ &= M_{W_{-k}\bar{\varphi}}W_{-1}\ . \end{aligned}$$

It follows that $M_{W_{k\varphi}}W_{-1}$ is self adjoint if and only if $W_k\varphi = W_{-k}\bar{\varphi}$. We also observe that $(M_{W_k\varphi}W_{-1})(M_{W_k\varphi}W_{-1})^* = M_{|W_k\varphi|^2}$ and

$$\left(M_{W_{-k}}\varphi W_{-1}\right)\left(M_{W_{-k}\varphi}W_{-1}\right)^* = M_{|W_{-k}\varphi|^2}\ .$$

So the Rationalized Toeplitz Hankel. $W_kM_\varphi W^*_{-k}$ is hyponormal if and only if $M_{|W_{-k}\varphi|^2-|W_k\varphi|^2} \geq 0$.
Or equivalenty

$$|W_k\varphi|^2 \geq |W_{-k}\varphi|^2\ .$$

Therefore, $W_kM_\varphi W^*_{-k}$ is hyponormal if and only if

$$|W_k\varphi|^2 = |W_{-k}\varphi|^2\ .$$

i.e., if only if it is normal. Thus Rationalized Toeplitz Hankel of all of this type is hyponormal if and only if it is normal. But, we can not get the same result if k_1 and k_2 are two different integers. For this we have the following □

Theorem 3.5. *Let k_1 and k_2 are non zero integers such that $|k_1| \neq |k_2|$, then*

1. *If k_1 and k_2 are relatively prime $R_\varphi = W_{k_1}M_\varphi W^*_{k_2}$ is normal if and only if $\varphi = 0$.*
2. *If k_1 and k_2 are not relatively prime, then $R_\varphi = W_{k_1}R_\varphi W^*_{k_2}$ is normal if and only if $W_d\varphi = 0$, where d is the greatest common divisor of k_1 and k_2.*

Proof. i) It is given that k_1 and k_2 are non zero integers with $|k_1| \neq |k_2|$. Without loss of generality, we can assume that $|k_1| < |k_2|$. Suppose that $R_\varphi = W_{k_1} M_\varphi W_{k_2}^*$ is normal. If we consider the basis elements z^0, z^1, z^2, $\cdots z^{|k_1|-1}$, then $||W_{k_1} M_\varphi W_{k_2}^* z^n|| = ||W_{k_2} M_{\bar{\varphi}} W_{k_1}^* z^n||$ for $n = 0, 1, 2, \cdots |k_1|-1$. That is $\sum_{i=-\infty}^{\infty} |a_{k_1 i - k_2 n}|^2 = \sum_{i=-\infty}^{\infty} |a_{-k_2 i + k_1 n}|^2$ where $\sum_{l=-\infty}^{\infty} a_i z^i$ is the Fouriers expansion of φ. Since k_1 and k_2 are relatively prime, we have

$$
\begin{aligned}
||\varphi||^2 &= \sum_{n=0}^{|k_1|-1} \sum_{i=-\infty}^{\infty} |a_{k_1 i - k_2 n}|^2 \\
&= \sum_{n=0}^{|k_1|-1} \left(\sum_{i=-\infty}^{\infty} |a_{-k_2 i + k_1 n}|^2 \right)
\end{aligned} .
$$

This implies that $a_{-k_2 i + k_1 n} = 0$ for $n = |k_1|, |k_1|+1, \ldots |k_2|$. Again consider the basis elements $z^{|k_1|}$, $z^{|k_1|+1}$,.......$z^{|2k_1|+1|}$. Then by similar arguments, we get for some $n \in \mathbb{Z}$

$$a_{-k_2 i + k_1 n} = 0 \ .$$

Continue like this, we get $a_i = 0$ for all i. Thus if k_1 and k_2 are relatively prime integer, then the Rationalized Toeplitz Hankel operator $R_\varphi = W_{k_1} M_\varphi W_{k_2}$ is Normal if and only if $\varphi = 0$

ii) If k_1 and k_2 are not relatively prime and let d be the greatest common divisor of k_1 and k_2 suppose $k_1 = dn$ and $k_2 = dm$, then n and m are relatively prime. Therefore,

$$
\begin{aligned}
W_{k_1} M_\varphi W_{k_2}^* &= W_n (W_d M_\varphi W_d^*) W_m^* \\
&= W_n M_{W_d \varphi} W_m^* \ .
\end{aligned}
$$

So by previous the case, it is normal if and only if $W_d \varphi = 0$. If $|k_1| > |k_2|$, then in a similar way, we can show that $R_\varphi = W_{k_1} M_\varphi W_{k_2}^*$ hyponormal if and only if it is zero operator. □

Remark 2. *If $|k_1| \leq |k_2|$, then $W_{k_1} M_\varphi W_{k_2}^*$ can be hyponormal without being the zero operater. We have the following example. If*

$$k_1 = k_2 = 1$$

$$M_\varphi = W_1 M_\varphi W_1^* \text{ is always hyponormal}$$

Also $W_{k_2}^* = W_1 M_1 W_{k_2}^*$ is hyponormal, because it is an isometry.

Theorem 3.6. *Given a pair of nonzero integer k_1 and k_2, the Rationalized Toeplitz Hankel operator $R_\varphi = W_{k_1} M_\varphi W^*_{k_2}$ is compact if and anly if it is zero operater.*

Proof. Let n be a positive integer and i and j be any integers. Then

$$\begin{aligned}\left\langle W_{k_1} M_\varphi W^*_{k_2} z^{j+nk_1}, z^{i+nk_2}\right\rangle &= \left\langle W_{k_1} M_\varphi W^*_{k_2} z^j z^i\right\rangle \\ &= a_{k_1 i - k_2 j}\ .\end{aligned}$$

where $\sum_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of φ.
This implies that

$$|a_{k_1 i - k_2 j}| \leq ||W_{k_1} M_\varphi W^*_{k_2} z^{j+nk_1}||\ .$$

Suppose $W_{k_1} M_\varphi W^*_{k_2}$ is compact. Since $z^{j+k_1 n} \to 0$ weakly as $k \to \infty$. This implies that

$$||W_{k_1} M_\varphi W^*_{k_2} z^{j+nk_1}|| \to 0\ .$$

Therefore for all $i, j \in Z$

$$|a_{k_1 i - k_2 j}| = 0\ .$$

$\Rightarrow \varphi = 0$. That is $W_{k_1} M_\varphi W^*_{k_2} = R_\varphi = 0$. Hence the Rationalized Toeplitz Hankel operator $R_\varphi = W_{k_1} M_\varphi W^*_{k_2}$ is compact if and only if $\varphi = 0$. □

Conclusion

In this paper, we characterize Rationalized Toeplitz Hankel operators in terms of their matrix. We have got some conditions under which R_φ is hyponormal and normal. It is also proved that the only compact operator R_φ is the zero operator. In addition, it is proved that for an invertible symbol φ in L^∞,

$$\sigma_p\left(R_{\varphi\left(z^{k_1}\right)}\right) = \sigma_p\left(R_{\varphi\left(z^{k_2}\right)}\right)$$

where σ_p is the point spectrum, which will be useful in finding the spectral values of the operator R_φ.

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