Jordan Journal of Mathematics and Statistics (JJMS), 16(1), 2023, pp 79 - 98
DOI: https://doi.org/10.47013/16.1.6

3/8-SIMPSON TYPE INEQUALITIES FOR FUNCTIONS WHOSE MODULUS OF FIRST DERIVATIVES AND ITS q-TH POWERS ARE s-CONVEX IN THE SECOND SENSE

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Abstract. The purpose of this study is to improve certain existing results concerning the Simpson type inequalities involving four point called Simpson second formula. First, we prove a new integral identity. Then, we use this identity to come up with a new Simpson second formula inequalities for functions whose first derivatives are s-convex. We also deal with situations in which the first derivatives are bounded and Lipschitzian. In addition, some applications are given to show how well our main results work.

1. Introduction

Definition 1.1. [22] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Due to its rapid growth, a number of researchers have introduced new classes of convex functions, including the class of s-convex functions established by Breckner.

Definition 1.2. [4] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \;\leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

2010 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. 3/8-Simpson inequality, Newton-Cotes quadrature, s-convex functions, Lipschitzian functions, bounded functions.

Received: Feb. 3, 2022 Accepted: Oct. 30, 2022 .

There is no doubt that convexity plays a significant role in the evolution of the theory of inequalities, which explains why there have been so many inequalities-related studies published. Gronwall type inequalities [1, 14], Lyapunov type inequality [2], Minkowski inequality [13], Opial integral inequalities [19], Hadamard type inequalities [17] and Ostrowski type inequalities [15, 16, 18].

Note that this mathematical tool is heavily used to estimate the error of quadrature formulas. Among the well known and widely used of this quadratures are those of Simpson. Simpson's quadratures are among the most popular and commonly utilized of this type.

The first formula also called 1/8-Simpson inequality is as follows:

$$\left|\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty. \tag{1.1}$$

The second formula called 3/8-Simpson inequality is as follows:

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-a)^4}{6480}\left\|f^{(4)}\right\|_\infty, \tag{1.2}$$

where f is four-times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|$.

Regarding some papers dealing with inequality (1.1)-(1.2) we refer readers to [3, 5–7, 9–12, 21, 24, 25] and references therein.

Recently, Noor et al. [20], established the following 3/8-Simpson type inequalities for differentiable convex functions:

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\\&\leq(b-a)\left\{\frac{17}{756}\left(\frac{973|f'(a)|^q+251|f'(b)|^q}{1224}\right)^{\frac{1}{q}}\right.\\&\left.\quad+\frac{1}{36}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\frac{17}{756}\left(\frac{251|f'(a)|^q+973|f'(b)|^q}{1224}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq (b-a)\left\{\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{6}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right\}$$

and

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq (b-a)\left\{\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|f'(a)|^q+|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+5|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right\}.$$

where $q,p>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In [8], Erden et al. showed that if the first derivatives are bounded, then the inequality (1.2) satisfies the following estimate:

Theorem 1.1. *Let Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping whose derivative is continuous on (a,b). Then, for all $x\in[a,b]$, we have the inequality*

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{25(b-a)}{288}\|f\|_\infty.$$

Also they discussed the cases where the first derivative is of bounded variation, absolutely continuous and Lipschitzian.

Inspired by the above results, in this paper we first prove a new identity as an auxiliary result. Based on this identity we establish new 3/8-Simpson type inequalities for functions whose first derivatives are s-convex in the second sense. We also discuss the cases where the first derivatives are bounded as well as Lipschitzian functions.

Some applications are given to show the validity of the fundamental results. It should be noted that the results obtained are an improvement above [20, 23].

2. Main results

In order to prove our results, we need the following lemma

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L^1[a, b]$, then the following equality holds*

$$\begin{aligned}&\tfrac{1}{8}\left(f(a) + 3f\left(\tfrac{2a+b}{3}\right) + 3f\left(\tfrac{a+2b}{3}\right) + f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\,du\\
=&\tfrac{b-a}{9}\left(\int_0^1 \left(t - \tfrac{3}{8}\right) f'\left((1-t)a + t\tfrac{2a+b}{3}\right) dt\right.\\
&+\int_0^1 \left(t - \tfrac{1}{2}\right) f'\left((1-t)\tfrac{2a+b}{3} + t\tfrac{a+2b}{3}\right) dt\\
&\left.+\int_0^1 \left(t - \tfrac{5}{8}\right) f'\left((1-t)\tfrac{a+2b}{3} + tb\right) dt\right).\end{aligned}$$

Proof. Let

$$I_1 = \int_0^1 \left(t - \tfrac{3}{8}\right) f'\left((1-t)a + t\tfrac{2a+b}{3}\right) dt,$$

$$I_2 = \int_0^1 \left(t - \tfrac{1}{2}\right) f'\left((1-t)\tfrac{2a+b}{3} + t\tfrac{a+2b}{3}\right) dt,$$

$$I_3 = \int_0^1 \left(t - \tfrac{5}{8}\right) f'\left((1-t)\tfrac{a+2b}{3} + tb\right) dt.$$

Integrating by parts I_1, we obtain

$$\begin{aligned}I_1 =& \tfrac{3}{b-a}\left(t - \tfrac{3}{8}\right) f\left((1-t)a + t\tfrac{2a+b}{3}\right)\Big|_{t=0}^{t=1}\\
&- \tfrac{3}{b-a}\int_0^1 f\left((1-t)a + t\tfrac{2a+b}{3}\right) dt\end{aligned}$$

$$=\frac{15}{8(b-a)}f\left(\frac{2a+b}{3}\right)+\frac{9}{8(b-a)}f(a)-\frac{9}{(b-a)^2}\int_a^{\frac{2a+b}{3}}f(u)\,du. \tag{2.1}$$

Similarly, we obtain

$$\begin{aligned}I_2=&\frac{3}{b-a}\left(t-\frac{1}{2}\right)f\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\Big|_{t=0}^{t=1}\\&-\frac{3}{b-a}\int_0^1 f\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)dt\end{aligned}$$

$$=\frac{3}{2(b-a)}f\left(\frac{a+2b}{3}\right)+\frac{3}{2(b-a)}f\left(\frac{2a+b}{3}\right)-\frac{9}{(b-a)^2}\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}f(u)\,du \tag{2.2}$$

$$\begin{aligned}I_3=&\frac{3}{b-a}\left(t-\frac{5}{8}\right)f\left((1-t)\frac{a+2b}{3}+tb\right)\Big|_{t=0}^{t=1}\\&-\frac{3}{b-a}\int_0^1 f\left((1-t)\frac{a+2b}{3}+tb\right)dt\end{aligned}$$

$$=\frac{9}{8(b-a)}f(b)+\frac{15}{8(b-a)}f\left(\frac{a+2b}{3}\right)-\frac{9}{(b-a)^2}\int_{\frac{a+2b}{3}}^{b}f(u)\,du. \tag{2.3}$$

Summing (2.1)-(2.3), and then multiplying the resulting equality by $\frac{b-a}{9}$, we get the desired result. □

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If $|f'|$ is s-convex in the second sense for some fixed $s\in(0,1]$, then we have*

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\\ \le&\frac{b-a}{9(s+1)(s+2)}\left(\left(2\left(\frac{5}{8}\right)^{s+2}+\frac{3s-2}{8}\right)\left(|f'(a)|+|f'(b)|\right)\right.\\&\left.+\left(\left(1+\left(\frac{3}{4}\right)^{s+2}\right)\left(\frac{1}{2}\right)^{s+1}+\frac{9s+2}{8}\right)\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right).\end{aligned}$$

Proof. From Lemma 2.1, properties of modulus and s-convexity in the second sense of $|f'|$, we have

$$\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq\tfrac{b-a}{9}\left(\int_0^1\left|t-\tfrac{3}{8}\right|\left|f'\left((1-t)a+t\tfrac{2a+b}{3}\right)\right|dt\right.$$

$$+\int_0^1\left|t-\tfrac{1}{2}\right|\left|f'\left((1-t)\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)\right|dt$$

$$\left.+\int_0^1\left|t-\tfrac{5}{8}\right|\left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)\right|dt\right)$$

$$\leq\tfrac{b-a}{9}\left(\int_0^{\frac{3}{8}}\left(\tfrac{3}{8}-t\right)\left((1-t)^s|f'(a)|+t^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|\right)dt\right.$$

$$+\int_{\frac{3}{8}}^1\left(t-\tfrac{3}{8}\right)\left((1-t)^s|f'(a)|+t^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|\right)dt$$

$$+\int_0^{\frac{1}{2}}\left(\tfrac{1}{2}-t\right)\left((1-t)^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|+t^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)dt$$

$$+\int_{\frac{1}{2}}^1\left(t-\tfrac{1}{2}\right)\left((1-t)^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|+t^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)dt$$

$$+\int_0^{\frac{5}{8}}\left(\tfrac{5}{8}-t\right)\left((1-t)^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|+t^s|f'(b)|\right)dt$$

$$\left.+\int_{\frac{5}{8}}^1\left(t-\tfrac{5}{8}\right)\left((1-t)^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|+t^s|f'(b)|\right)dt\right)$$

$$\leq\tfrac{b-a}{9(s+1)(s+2)}\left(\left(2\left(\tfrac{5}{8}\right)^{s+2}+\tfrac{3s-2}{8}\right)(|f'(a)|+|f'(b)|)\right.$$

$$\left.+\left(\left(1+\left(\tfrac{3}{4}\right)^{s+2}\right)\left(\tfrac{1}{2}\right)^{s+1}+\tfrac{9s+2}{8}\right)\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|+\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)\right),$$

where we have used the fact that

$$\int_0^{\frac{3}{8}} \left(\tfrac{3}{8}-t\right)(1-t)^s\,dt = \int_{\frac{5}{8}}^{1} \left(t-\tfrac{5}{8}\right) t^s dt = \tfrac{1}{(s+1)(s+2)} \left(\left(\tfrac{5}{8}\right)^{s+2} + \tfrac{3s-2}{8}\right), \tag{2.4}$$

$$\int_0^{\frac{3}{8}} \left(\tfrac{3}{8}-t\right) t^s dt = \int_{\frac{5}{8}}^{1} \left(t-\tfrac{5}{8}\right)(1-t)^s\,dt = \tfrac{1}{(s+1)(s+2)} \left(\tfrac{3}{8}\right)^{s+2}, \tag{2.5}$$

$$\int_{\frac{3}{8}}^{1} \left(t-\tfrac{3}{8}\right)(1-t)^s\,dt = \int_0^{\frac{5}{8}} \left(\tfrac{5}{8}-t\right) t^s dt = \tfrac{1}{(s+1)(s+2)} \left(\tfrac{5}{8}\right)^{s+2}, \tag{2.6}$$

$$\int_{\frac{3}{8}}^{1} \left(t-\tfrac{3}{8}\right) t^s dt = \int_0^{\frac{5}{8}} \left(\tfrac{5}{8}-t\right)(1-t)^s\,dt = \tfrac{1}{(s+1)(s+2)} \left(\left(\tfrac{3}{8}\right)^{s+2} + \tfrac{5s+2}{8}\right), \tag{2.7}$$

$$\int_0^{\frac{1}{2}} \left(\tfrac{1}{2}-t\right)(1-t)^s\,dt = \int_{\frac{1}{2}}^{1} \left(t-\tfrac{1}{2}\right) t^s dt = \tfrac{1}{(s+1)(s+2)} \left(\tfrac{s}{2} + \left(\tfrac{1}{2}\right)^{s+2}\right) \tag{2.8}$$

$$\int_{\frac{1}{2}}^{1} \left(t-\tfrac{1}{2}\right)(1-t)^s\,dt = \int_0^{\frac{1}{2}} \left(\tfrac{1}{2}-t\right) t^s dt = \tfrac{1}{(s+1)(s+2)} \left(\tfrac{1}{2}\right)^{s+2}. \tag{2.9}$$

The proof is completed. □

Corollary 2.1. *In Theorem 2.1, if we take $s=1$, then we get*

$$\left| \tfrac{1}{8}\left(f(a) + 3f\left(\tfrac{2a+b}{3}\right) + 3f\left(\tfrac{a+2b}{3}\right) + f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\,du \right|$$
$$\leq \tfrac{b-a}{13824}\left(157\left(|f'(a)| + |f'(b)|\right) + 443\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right| + \left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)\right).$$

Corollary 2.2. *In Theorem 2.1, if we assume that $|f'(x)| \leq M = \|f\|_\infty$, then we obtain*

$$\left| \tfrac{1}{8}\left(f(a) + 3f\left(\tfrac{2a+b}{3}\right) + 3f\left(\tfrac{a+2b}{3}\right) + f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{25(b-a)}{288}\|f\|_\infty .$$

Remark 1. *The result of Corollary 2.2 is the same result obtained in Corollary 4 from [8].*

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\ &\leq\tfrac{b-a}{9(p+1)^{\frac{1}{p}}}\left(\left(\tfrac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\tfrac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.\\ &\left.+\tfrac{1}{2}\left(\tfrac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\tfrac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. From Lemma 2.1, properties of modulus, Hölder's inequality and s-convexity in the second sense of $|f'|$, we have

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\ &\leq\tfrac{b-a}{9}\left(\left(\int_0^1\left|t-\tfrac{3}{8}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)a+t\tfrac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &+\left(\int_0^1\left|t-\tfrac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\left.+\left(\int_0^1\left|t-\tfrac{5}{8}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)\\ &\leq\tfrac{b-a}{9(p+1)^{\frac{1}{p}}}\left(\left(\tfrac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s|f'(a)|^q+t^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\ &+\tfrac{1}{2}\left(\int_0^1\left((1-t)^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q+t^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\end{aligned}$$

$$+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{b-a}{9(p+1)^{\frac{1}{p}}}\left(\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{1}{2}\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

The proof is completed. □

Remark 2. *Theorem 2.2 will be reduced to Corollary 3.5 from [20], if we take $s=1$.*

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q\ge 1$, then we have*

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\le\frac{b-a}{9}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\frac{5}{8}\right)^{s+2}+\frac{3s-2}{16}\right)|f'(a)|^q\right.\right.$$

$$\left.+\left(\left(\frac{3}{8}\right)^{s+2}+\frac{5s+2}{16}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{s}{4}+\left(\frac{1}{2}\right)^{s+2}\right)^{\frac{1}{q}}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\frac{3}{8}\right)^{s+2}+\frac{5s+2}{16}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$

$$\left.\left.+\left(\left(\frac{5}{8}\right)^{s+2}+\frac{3s-2}{16}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 2.1, properties of modulus, power mean inequality and s-convexity in the second sense of $|f'|$, we have

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\le\frac{b-a}{9}\left(\left(\int_0^1\left|t-\frac{3}{8}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t-\frac{3}{8}\right|\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 \left|t-\tfrac{1}{2}\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|t-\tfrac{1}{2}\right| \left|f'\left((1-t)\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1 \left|t-\tfrac{5}{8}\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|t-\tfrac{5}{8}\right| \left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq \tfrac{b-a}{9}\left(\left(\tfrac{17}{64}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t-\tfrac{3}{8}\right| \left((1-t)^s |f'(a)|^q + t^s \left|f'\left(\tfrac{2a+b}{3}\right)\right|^q\right) dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{1}{4}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t-\tfrac{1}{2}\right| \left((1-t)^s \left|f'\left(\tfrac{2a+b}{3}\right)\right|^q + t^s \left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\right) dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\tfrac{17}{64}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t-\tfrac{5}{8}\right| \left((1-t)^s \left|f'\left(\tfrac{a+2b}{3}\right)\right|^q + t^s |f'(b)|^q\right) dt\right)^{\frac{1}{q}}\right)$$

$$= \tfrac{b-a}{9}\left(\left(\tfrac{17}{64}\right)^{1-\frac{1}{q}}\left(|f'(a)|^q \int_0^{\frac{3}{8}} \left(\tfrac{3}{8}-t\right)(1-t)^s dt + \left|f'\left(\tfrac{2a+b}{3}\right)\right|^q \int_0^{\frac{3}{8}} \left(\tfrac{3}{8}-t\right) t^s dt\right.\right.$$

$$\left.+|f'(a)|^q \int_{\frac{3}{8}}^1 \left(t-\tfrac{3}{8}\right)(1-t)^s dt + \left|f'\left(\tfrac{2a+b}{3}\right)\right|^q \int_{\frac{3}{8}}^1 \left(t-\tfrac{3}{8}\right) t^s dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{4}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q \int_0^{\frac{1}{2}} \left(\tfrac{1}{2}-t\right)(1-t)^s dt + \left|f'\left(\tfrac{a+2b}{3}\right)\right|^q \int_0^{\frac{1}{2}} \left(\tfrac{1}{2}-t\right) t^s dt\right.$$

$$\left.+\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q \int_{\frac{1}{2}}^1 \left(t-\tfrac{1}{2}\right)(1-t)^s dt + \left|f'\left(\tfrac{a+2b}{3}\right)\right|^q \int_{\frac{1}{2}}^1 \left(t-\tfrac{1}{2}\right) t^s dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{17}{64}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q \int_0^{\frac{5}{8}} \left(\tfrac{5}{8}-t\right)(1-t)^s dt + |f'(b)|^q \int_0^{\frac{5}{8}} \left(\tfrac{5}{8}-t\right) t^s dt\right.$$

$$+\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\int_{\frac{5}{8}}^{1}\left(t-\tfrac{5}{8}\right)(1-t)^s\,dt+\left|f'(b)\right|^q\int_{\frac{5}{8}}^{1}\left(t-\tfrac{5}{8}\right)t^s dt\Bigg)^{\frac{1}{q}}\Bigg)$$

$$=\tfrac{b-a}{9}\left(\tfrac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\left(\tfrac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\tfrac{5}{8}\right)^{s+2}+\tfrac{3s-2}{16}\right)\left|f'(a)\right|^q\right.\right.$$

$$+\left.\left(\left(\tfrac{3}{8}\right)^{s+2}+\tfrac{5s+2}{16}\right)\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{4}\right)^{1-\frac{1}{q}}\left(\tfrac{s}{4}+\left(\tfrac{1}{2}\right)^{s+2}\right)^{\frac{1}{q}}\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q+\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\tfrac{3}{8}\right)^{s+2}+\tfrac{5s+2}{16}\right)\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\right.$$

$$\left.\left.+\left(\left(\tfrac{5}{8}\right)^{s+2}+\tfrac{3s-2}{16}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used (2.4)-(2.9),

$$\int_0^1\left|t-\tfrac{3}{8}\right|dt=\int_0^1\left|t-\tfrac{5}{8}\right|dt=\tfrac{17}{64}\text{ and }\int_0^1\left|t-\tfrac{1}{2}\right|dt=\tfrac{1}{4}. \tag{2.10}$$

The proof is achieved. □

Corollary 2.3. *In Theorem 2.3, if we take $s=1$, then we get*

$$\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq\tfrac{b-a}{36}\left(\tfrac{17}{16}\left(\tfrac{157|f'(a)|^q+251\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{408}\right)^{\frac{1}{q}}+\tfrac{1}{4}\left(\tfrac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\tfrac{17}{16}\left(\tfrac{251\left|f'\left(\frac{a+2b}{3}\right)\right|^q+157|f'(b)|^q}{408}\right)^{\frac{1}{q}}\right).$$

3. Further results

This section is not connected to the previous one. Here, we will discuss the case where the first derivative is bounded as well as the case where it satisfy the Lipschitz condition in a general way.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If there exist constants $-\infty<m<M<+\infty$ such that*

$m \leq f'(x) \leq M$ for all $x \in [a,b]$, then we have

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{25}{576}(b-a)(M-m).$$

Proof. From Lemma 2.1, we have

$$\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du$$
$$=\frac{b-a}{9}\left(\int_0^1\left(t-\frac{3}{8}\right)f'\left((1-t)a+t\frac{2a+b}{3}\right)dt\right.$$
$$+\int_0^1\left(t-\frac{1}{2}\right)f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)dt$$
$$\left.+\int_0^1\left(t-\frac{5}{8}\right)f'\left((1-t)\frac{a+2b}{3}+tb\right)dt\right)$$
$$=\frac{b-a}{9}\left(\int_0^1\left(t-\frac{3}{8}\right)\left(f'\left((1-t)a+t\frac{2a+b}{3}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dt\right.$$
$$+\int_0^1\left(t-\frac{1}{2}\right)\left(f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dt$$
$$\left.+\int_0^1\left(t-\frac{5}{8}\right)\left(f'\left((1-t)\frac{a+2b}{3}+tb\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dt\right)$$
$$=\frac{b-a}{9}\left(\int_0^1\left(t-\frac{3}{8}\right)\left(f'\left((1-t)a+t\frac{2a+b}{3}\right)-\frac{m+M}{2}\right)dt\right.$$
$$+\int_0^1\left(t-\frac{1}{2}\right)\left(f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)-\frac{m+M}{2}\right)dt$$
$$\left.+\int_0^1\left(t-\frac{5}{8}\right)\left(f'\left((1-t)\frac{a+2b}{3}+tb\right)-\frac{m+M}{2}\right)dt\right), \tag{3.1}$$

where we have taken into account that:

$$\int_0^1 \left(t-\tfrac{3}{8}\right)dt+\int_0^1 \left(t-\tfrac{1}{2}\right)dt+\int_0^1 \left(t-\tfrac{5}{8}\right)dt=\int_0^1 \left(3t-\tfrac{3}{2}\right)dt=0. \tag{3.2}$$

Applying the absolute value to both sides of (3.1), we get

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\
\leq&\tfrac{b-a}{9}\left(\int_0^1 \left|t-\tfrac{3}{8}\right|\left|f'\left((1-t)a+t\tfrac{2a+b}{3}\right)-\tfrac{m+M}{2}\right|dt\right.\\
&+\int_0^1 \left|t-\tfrac{1}{2}\right|\left|f'\left((1-t)a+t\tfrac{2a+b}{3}\right)-\tfrac{m+M}{2}\right|dt\\
&\left.+\int_0^1 \left|t-\tfrac{5}{8}\right|\left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)-\tfrac{m+M}{2}\right|dt\right).
\end{aligned}\tag{3.3}$$

Since $m\leq f'(x)\leq M$ for all $x\in[a,b]$, we have

$$\left|f'\left((1-t)a+t\tfrac{2a+b}{3}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}, \tag{3.4}$$

$$\left|f'\left((1-t)a+t\tfrac{2a+b}{3}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2} \tag{3.5}$$

and

$$\left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}. \tag{3.6}$$

Using (3.4)-(3.6) in (3.3), we get

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\
\leq&\tfrac{(b-a)(M-m)}{18}\left(\int_0^1 \left|t-\tfrac{3}{8}\right|dt+\int_0^1 \left|t-\tfrac{1}{2}\right|dt+\int_0^1 \left|t-\tfrac{5}{8}\right|dt\right)\\
=&\tfrac{25}{576}(b-a)(M-m),
\end{aligned}$$

where we have used (2.10). The proof is completed. □

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If f' is L-Lipschitzian function on $[a,b]$, then we have*

$$\left|\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\le\tfrac{41(b-a)^2}{1728}L.$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\tfrac{1}{8}\left(f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\\
=&\tfrac{b-a}{9}\left(\int_0^1\left(t-\tfrac{3}{8}\right)f'\left((1-t)a+t\tfrac{2a+b}{3}\right)dt\right.\\
&+\int_0^1\left(t-\tfrac{1}{2}\right)f'\left((1-t)\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)dt\\
&\left.+\int_0^1\left(t-\tfrac{5}{8}\right)f'\left((1-t)\tfrac{a+2b}{3}+tb\right)dt\right)\\
=&\tfrac{b-a}{9}\left(\int_0^1\left(t-\tfrac{3}{8}\right)\left(f'\left((1-t)a+t\tfrac{2a+b}{3}\right)-f'(a)+f'(a)\right)dt\right.\\
&+\int_0^1\left(t-\tfrac{1}{2}\right)\left(f'\left((1-t)\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)-f'\left(\tfrac{2a+b}{3}\right)+f'\left(\tfrac{2a+b}{3}\right)\right)dt\\
&\left.+\int_0^1\left(t-\tfrac{5}{8}\right)\left(f'\left((1-t)\tfrac{a+2b}{3}+tb\right)-f'\left(\tfrac{a+2b}{3}\right)+f'\left(\tfrac{a+2b}{3}\right)\right)dt\right)\\
=&\tfrac{b-a}{9}\left(\int_0^1\left(t-\tfrac{3}{8}\right)\left(f'\left((1-t)a+t\tfrac{2a+b}{3}\right)-f'(a)\right)dt\right.\\
&+\int_0^1\left(t-\tfrac{1}{2}\right)\left(f'\left((1-t)\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)-f'\left(\tfrac{2a+b}{3}\right)\right)dt\\
&+\int_0^1\left(t-\tfrac{5}{8}\right)\left(f'\left((1-t)\tfrac{a+2b}{3}+tb\right)-f'\left(\tfrac{a+2b}{3}\right)\right)dt\\
&\left.+f'(a)\int_0^1\left(t-\tfrac{3}{8}\right)dt+f'\left(\tfrac{2a+b}{3}\right)\int_0^1\left(t-\tfrac{1}{2}\right)dt+f'\left(\tfrac{a+2b}{3}\right)\int_0^1\left(t-\tfrac{5}{8}\right)dt\right)
\end{aligned}$$

$$=\frac{b-a}{9}\left(\int_0^1\left(t-\tfrac{3}{8}\right)\left(f'\left((1-t)\,a+t\tfrac{2a+b}{3}\right)-f'\left(\tfrac{2a+b}{3}\right)\right)dt\right.$$

$$+\int_0^1\left(t-\tfrac{1}{2}\right)\left(f'\left((1-t)\,\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)-f'\left(\tfrac{2a+b}{3}\right)\right)dt$$

$$\left.+\int_0^1\left(t-\tfrac{5}{8}\right)\left(f'\left((1-t)\,\tfrac{a+2b}{3}+tb\right)-f'\left(\tfrac{a+2b}{3}\right)\right)dt+\tfrac{1}{8}f'\left(a\right)-\tfrac{1}{8}f'\left(\tfrac{a+2b}{3}\right)\right). \tag{3.7}$$

Applying the absolute value in both sides of (3.7), and by using the fact that f' is L-Lipschitzian on $[a,b]$, we obtain

$$\left|\tfrac{1}{8}\left(f\left(a\right)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f\left(b\right)\right)-\tfrac{1}{b-a}\int_a^b f\left(u\right)du\right|$$

$$\leq\frac{b-a}{9}\left(\int_0^1\left|t-\tfrac{3}{8}\right|\left|f'\left((1-t)\,a+t\tfrac{2a+b}{3}\right)-f'\left(a\right)\right|dt\right.$$

$$+\int_0^1\left|t-\tfrac{1}{2}\right|\left|f'\left((1-t)\,\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}\right)-f'\left(\tfrac{2a+b}{3}\right)\right|dt$$

$$\left.+\int_0^1\left|t-\tfrac{5}{8}\right|\left|f'\left((1-t)\,\tfrac{a+2b}{3}+tb\right)-f'\left(\tfrac{a+2b}{3}\right)\right|dt+\tfrac{1}{8}\left|f'\left(a\right)-f'\left(\tfrac{a+2b}{3}\right)\right|\right)$$

$$\leq\frac{b-a}{9}L\left(\int_0^1\left|t-\tfrac{3}{8}\right|\left|(1-t)\,a+t\tfrac{2a+b}{3}-a\right|dt\right.$$

$$+\int_0^1\left|t-\tfrac{1}{2}\right|\left|(1-t)\,\tfrac{2a+b}{3}+t\tfrac{a+2b}{3}-\tfrac{2a+b}{3}\right|dt$$

$$\left.+\int_0^1\left|t-\tfrac{5}{8}\right|\left|(1-t)\,\tfrac{a+2b}{3}+tb-\tfrac{a+2b}{3}\right|dt+\tfrac{1}{8}\left|a-\tfrac{a+2b}{3}\right|\right)$$

$$=\frac{b-a}{9}L\left(\int_0^1\left|t-\tfrac{3}{8}\right|t\tfrac{b-a}{3}dt+\int_0^1\left|t-\tfrac{1}{2}\right|t\tfrac{b-a}{3}dt+\int_0^1\left|t-\tfrac{5}{8}\right|t\tfrac{b-a}{3}dt+\tfrac{1}{4}\tfrac{(b-a)}{3}\right)$$

$$=\frac{(b-a)^2}{27}L\left(\int_0^1\left|t-\frac{3}{8}\right|tdt+\int_0^1\left|t-\frac{1}{2}\right|tdt+\int_0^1\left|t-\frac{5}{8}\right|tdt+\frac{1}{4}\right)$$
$$=\frac{41(b-a)^2}{1728}L,$$

where we have used

$$\int_0^1\left|t-\frac{3}{8}\right|tdt=\frac{251}{1536},\int_0^1\left|t-\frac{1}{2}\right|tdt=\frac{1}{8}$$

and

$$\int_0^1\left|t-\frac{5}{8}\right|tdt=\frac{157}{1536}.$$

The proof is completed. □

4. Applications

4.1. **Second Simpson's quadrature formula.** Let Υ be the partition of the points $a=x_0<x_1<...<x_n=b$ of the interval $[a,b]$, and consider the quadrature formula

$$\int_a^b f(u)\,du=\lambda(f,\Upsilon)+R(f,\Upsilon),$$

where

$$\lambda(f,\Upsilon)=\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{8}\left(f(x_i)+3f\left(\frac{2x_i+x_{i+1}}{3}\right)+3f\left(\frac{x_i+2x_{i+1}}{3}\right)+f(x_{i+1})\right)$$

and $R(f,\Upsilon)$ denotes the associated approximation error.

Proposition 4.1. *Let $n\in\mathbb{N}$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $0\leq a<b$ and $f'\in L^1[a,b]$. If $|f'|$ is s-convex function in the second sense for some fixed $s\in(0,1]$, we have*

$$|R(f,\Upsilon)|\leq\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{9(s+1)(s+2)}\left(\left(2\left(\frac{5}{8}\right)^{s+2}+\frac{3s-2}{8}\right)\left(|f'(x_i)|+|f'(x_{i+1})|\right)\right.$$
$$\left.+\left(\left(1+\left(\frac{3}{4}\right)^{s+2}\right)\left(\frac{1}{2}\right)^{s+1}+\frac{9s+2}{8}\right)\left(\left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+\left|f'\left(\frac{x_i+2x_{i+1}}{3}\right)\right|\right)\right).$$

Proof. Applying Theorem 2.1 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Υ, we get

$$\left|\frac{1}{8}\left(f(x_i) + 3f\left(\frac{2x_i+x_{i+1}}{3}\right) + 3f\left(\frac{x_i+2x_{i+1}}{3}\right) + f(x_{i+1})\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(u)\,du\right|$$
$$\leq \frac{x_{i+1}-x_i}{9(s+1)(s+2)}\left(\left(2\left(\frac{5}{8}\right)^{s+2} + \frac{3s-2}{8}\right)(|f'(x_i)| + |f'(x_{i+1})|)\right.$$
$$\left. + \left(\left(1+\left(\frac{3}{4}\right)^{s+2}\right)\left(\frac{1}{2}\right)^{s+1} + \frac{9s+2}{8}\right)\left(\left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right| + \left|f'\left(\frac{x_i+2x_{i+1}}{3}\right)\right|\right)\right). \tag{4.1}$$

Multiplying both sides of (4.1) by $(x_{i+1} - x_i)$, then summing the obtained inequalities for all $i = 0, 1, ..., n-1$ and using the triangular inequality, we get the desired result. □

4.2. **Application to special means.** For arbitrary real numbers $a, a_1, a_2, ..., a_n, b$ we have:

The Arithmetic mean: $A(a_1, a_2, ..., a_n) = \frac{a_1+a_2+...+a_n}{n}$.

The Geometric mean: $G(a, b) = \sqrt{ab}$, $a, b > 0$.

The p-Logarithmic mean: $L_p(a, b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^2, b^2\right) + 3A^2(a, a, b) + 3A^2(a, b, b) - 8L_2^2(a, b)\right|$$
$$\leq \frac{b-a}{36}\left(17\left(\frac{157}{408}a^q + \frac{251}{408}A^q(a, a, b)\right)^{\frac{1}{q}} + 2^{4-\frac{1}{q}}\left(A^q(a, a, b) + A^q(a, b, b)\right)^{\frac{1}{q}}\right.$$
$$\left. + 17\left(\frac{251}{408}A^q(a, b, b) + \frac{157}{408}b^q\right)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from Corollary 2.3, applied to the function $f(x) = x^2$. □

Proposition 4.3. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^3, b^3\right) + 3A\left(\left(\frac{2a+b}{3}\right)^3, \left(\frac{a+2b}{3}\right)^3\right) - 4L_3^3(a, b)\right|$$
$$\leq \frac{25}{24}\left(A\left(a^3, b^3\right) - G^2(a, b)\,A(a, b)\right).$$

Proof. The assertion follows from Theorem 3.1, applied to the function $f(x) = x^3$. □

5. Conclusion

In this study, we considered Simpson's second formula. The main results of the article can be summarized as follows:

(1) A new four-point integral identity is introduced.
(2) New inequalities of 3/8-Simpson type inequalities for functions whose first derivatives are s-convex, bounded as well as Lipschitzian are established.
(3) Some special cases are derived.
(4) Applications of the acquired results are provided.

This paper's findings can inspire additional research in this fascinating topic, as well as generalizations in other types of calculations, such as time scale calculus, multiplicative calculus, and quantum calculus.

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