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A CLASSIFICATION OF KENMOTSU MANIFOLD ADMITTING $*$-EINSTEIN SOLITON

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Abstract. In this paper, we initiate the study of $*$-Einstein soliton on Kenmotsu manifold, whose potential vector field is torse-forming. Here, we have shown the nature of the soliton and find the scalar curvature when the manifold admitting $*$-Einstein soliton on Kenmotsu manifold. Next, we have evolved the characterization of the vector field when the manifold satisfies $*$-Einstein soliton. We have embellished some applications of vector field as torse-forming in terms of $*$-Einstein soliton on Kenmotsu manifold. Also, we have studied infinitesimal CL-transformation and Schouten-Van Kampen connection on Kenmotsu manifold, whose metric is $*$-Einstein soliton. We have developed an example of $*$-Einstein soliton on 3-dimensional Kenmotsu manifold to prove our findings.

1. Introduction

In 2016, G. Catino and L. Mazzieri [3] introduced the Einstein soliton, which generates self-similar solutions to Einstein flow, given by

$$\frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g), \tag{1.1}$$

where S is Ricci tensor, g is Riemannian metric and r is the scalar curvature. The equation of the Einstein soliton [2] is given by ([14, 15])

$$\pounds_V g + 2S + (2\lambda - r)g = 0,$$

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where $\pounds_V$ is the Lie derivative along the vector field V, S is the Ricci tensor, r is the scalar curvature of the Riemannian metric g, and λ is a real constant. Also it is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

The notion of $*$-Ricci tensor on almost Hermitian manifolds and $*$-Ricci tensor of real hypersurfaces in non-flat complex space were introduced by Tachibana [22] and Hamada [6] respectively where the $*$-Ricci tensor is defined by

$$S^*(X,Y) = \frac{1}{2}(\mathrm{Tr}\{\varphi \circ R(X, \varphi Y)\}), \tag{1.2}$$

for all vector fields X, Y on M^n, φ is a (1,1)-tensor field and Tr denotes Trace.

If $S^*(X,Y) = \Lambda g(X,Y) + \nu\eta(X)\eta(Y)$ for all vector fields X, Y and Λ, ν are smooth functions, then the manifold is called $*$-η-Einstein manifold. Further if $\nu = 0$ i.e $S^*(X,Y) = \Lambda g(X,Y)$ for all vector fields X, Y then the manifold becomes $*$-Einstein.

In 2014, Kaimakamis and Panagiotidou [8] introduced the notion of $*$-Ricci soliton which can be defined as [13, 16, 17, 18],

$$\pounds_V g + 2S^* + 2\lambda g = 0,$$

for λ being a constant.

Motivating from the above notions, we develop $*$-Einstein soliton as follows:

Definition 1.1. A Riemannian metric on an almost contact metric manifold M is said to $*$-*Einstein soliton*, if

$$\pounds_V g + 2S^* + (2\lambda - r^*)g = 0, \tag{1.3}$$

where $\pounds_V$ is the Lie derivative along the vector field V, S^* is the $*$-Ricci tensor, $r^* = \mathrm{Tr}(S^*)$ is the $*$- scalar curvature, and λ is a real constant. Also it is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

On the other hand, a nowhere vanishing vector field τ on a Riemannian or pseudo-Riemannian manifold (M, g) is called torse-forming [29] if

$$\nabla_X \tau = \psi X + \omega(X)\tau, \tag{1.4}$$

where ∇ is the Levi-Civita connection of g, ψ is a smooth function and ω is a 1-form. Moreover,

- τ is called *concircular* ([4], [28]), if $\omega = 0$,
- τ is called *concurrent* ([19], [27]), if $\omega = 0$ and $\psi = 1$,
- τ is called *recurrent*, if $\psi = 0$,
- τ is *parallel*, if $\psi = \omega = 0$.

In 2017, Chen [5] introduced a new vector field called *torqued vector field* τ, which satisfies (1.4) with $\omega(\tau) = 0$. Also in this case, ψ is known as the torqued function and the 1-form ω is the torqued form of τ.

The outline of the article goes as follows: In Section 2, after a brief introduction, we have discussed some needful results, which will be used in the later section. In Section 3, we have proved some results where $*$-Einstein soliton admitting Kenmotsu manifold and obtained the nature of soliton, Laplacian of the smooth function. We have also investigated that the manifold is η-Einstein when the manifold satisfies $*$-Einstein soliton and the vector field is conformal Killing. In section 4, we have demonstrated some properties of vector fields on $*$-Einstein soliton. Section 5 and Section 6 deals with the CL-transformation and Schouten-Van Kampen connection respectively on Kenmotsu manifold, whose metric is $*$-Einstein soliton. In section 7, we have clarified geometrical and physical motivation of $*$-Einstein soliton respectively. In Section 8, we have constructed an example to illustrate the existence of $*$-Einstein soliton on 3-dimensional Kenmotsu manifold, which is shrinking.

2. Preliminaries

The methods of contact geometry make a major contribution in modern mathematics,. Contact geometry has evolved from the mathematical formalism of classical mechanics. in 1969, S. Tanno [24] classified the connected almost contact metric manifolds whose automorphism groups have maximal dimensions as follows:

(a) Homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if $k(\xi, X) > 0$;

(b) Global Riemannian product of a line or a circle and a Kählerian manifold with constant holomorphic sectional curvature if $k(\xi, X) = 0$;

(c) A warped product space $\mathbb{R} \times_\lambda \mathbb{C}^n$ if $k(\xi, X) < 0$; where $k(\xi, X)$ denotes the sectional curvature of the plane section containing the characteristic vector field ξ and an arbitrary vector field X.

In 1972, K. Kenmotsu [9] obtained some tensor equations to characterize the manifolds of the third class. Since then the manifolds of the third class were called Kenmotsu manifolds.

Let M be a $(2n+1)$-dimensional connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric manifold is said to be a Kenmotsu manifold [9] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g. In a Kenmotsu manifold the following relations hold ([1, 15]):

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.11}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.12}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Now we know

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \tag{2.13}$$

for all vector fields $X, Y \in \chi(M)$. Then using (2.6) and (2.13), we get

$$(\pounds_\xi g)(X,Y) = 2[g(X,Y) - \eta(X)\eta(Y)]. \tag{2.14}$$

Proposition 2.1. [26] *On a $(2n+1)$-dimensional Kenmotsu manifold, the $*$-Ricci tensor is given by,*

$$S^*(X,Y) = S(X,Y) + (2n-1)g(X,Y) + \eta(X)\eta(Y). \tag{2.15}$$

Also taking $X = e_i$, $Y = e_i$ in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, \cdots, (2n+1)$, we get

$$r^* = r + 4n^2, \tag{2.16}$$

where r^* is the $*$-scalar curvature of M.

3. Main Results and proofs

Let M be a $(2n+1)$-dimensional Kenmotsu manifold. Consider $V = \xi$ in the equation of $*$-Einstein soliton (1.3) on M, we obtain:

$$(\pounds_\xi g)(X,Y) + 2S^*(X,Y) + (2\lambda - r^*)g(X,Y) = 0 \tag{3.1}$$

for all vector fields $X, Y \in \chi(M)$.

Using (2.14), (2.15) and (2.16), the above equation takes the form,

$$\Big[\lambda + 2n - \frac{r + 4n^2}{2}\Big] g(X,Y) + S(X,Y) = 0. \tag{3.2}$$

Taking $Y = \xi$ in the above equation and from (2.10), we get

$$\lambda = \frac{r + 4n^2}{2}, \tag{3.3}$$

since $\eta(X) \neq 0$. This leads to the following:

Theorem 3.1. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfies the $*$-Einstein soliton (g, ξ, λ), where ξ is the Reeb vector field, then the soliton is expanding, steady, shrinking according as $(r + 4n^2) \gtreqless 0$.*

Also we see that if the manifold M becomes flat i.e $r = 0$ then (3.3) gives $\lambda = 2n^2 > 0$. So we can state,

Corollary 3.1. *If the metric g of a (2n+1)-dimensional flat Kenmotsu manifold satisfies the $*$-Einstein soliton (g, ξ, λ), where ξ is the Reeb vector field, then the soliton is expanding.*

Now consider a $*$-Einstein soliton (g, V, λ) on M as:

$$(\pounds_V g)(X, Y) + 2S^*(X, Y) + (2\lambda - r^*)g(X, Y) = 0. \tag{3.4}$$

for all vector fields $X, Y \in \chi(M)$.

Taking $X = e_i$, $Y = e_i$, in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, ..., (2n + 1)$ and using (2.16), we get,

$$divV - \frac{(r + 4n^2)(2n - 1)}{2} + \lambda(2n + 1) = 0. \tag{3.5}$$

If we take the vector field V is of gradient type i.e $V = grad(f)$, for f is a smooth function on M, then the equation (3.5) becomes,

$$\Delta(f) = \frac{(r + 4n^2)(2n - 1)}{2} - \lambda(2n + 1), \tag{3.6}$$

where $\Delta(f)$ is the Laplacian equation satisfied by f. So we can state the following theorem.

Theorem 3.2. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfies the $*$-Einstein soliton (g, V, λ), where V is the gradient of a smooth function f, then the Laplacian equation satisfied by f is,*

$$\Delta(f) = \frac{(r + 4n^2)(2n - 1)}{2} - \lambda(2n + 1).$$

Also if we consider the vector field V as solenoidal i.e $divV = 0$, then (3.5) gives,

$$\lambda = \frac{(r + 4n^2)(2n - 1)}{2(2n + 1)}. \tag{3.7}$$

Again if λ takes the form of (3.7), then (3.5) becomes,

$$divV = 0, \tag{3.8}$$

i.e. V is solenoidal. This leads to the following:

Theorem 3.3. *Let the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfy the ∗-Einstein soliton (g, V, λ). Then the vector field V is solenoidal if and only if* $\lambda = \frac{(r+4n^2)(2n-1)}{2(2n+1)}$.

A vector field V is said to be a conformal Killing vector field iff the following relation holds:

$$(\pounds_V g)(X, Y) = 2\Omega g(X, Y), \tag{3.9}$$

where Ω is some function of the co-ordinates(conformal scalar). Moreover if Ω is not constant the conformal Killing vector field V is said to be proper. Also when Ω is constant, V is called homothetic vector field and when the constant Ω becomes non zero, V is said to be proper homothetic vector field. If $\Omega = 0$ in the above equation, then V is called Killing vector field.

Let (g, V, λ) be a ∗-Einstein soliton on a $(2n{+}1)$ dimensional Kenmotsu manifold M, where V is a conformal Killing vector field. Then from (1.3), (2.15) and (3.9), we have,

$$S(X, Y) = -[\lambda + \Omega + 2n - \frac{r^*}{2} - 1]g(X, Y) - \eta(X)\eta(Y). \tag{3.10}$$

which leads to the fact that the manifold is η-Einstein. This leads to the following,

Theorem 3.4. *If the metric g of a (2n+1) dimensional Kenmotsu manifold satisfies the ∗-Einstein soliton (g, V, λ), where V is a conformal Killing vector field, then the manifold becomes η-Einstein.*

Taking $Y = \xi$ in (3.10) and using (2.10), (2.16) we get

$$\Omega = \frac{r + 4n^2}{2} - \lambda, \tag{3.11}$$

since $\eta(X) \neq 0$.

Hence we can state,

Theorem 3.5. *Let the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfy the $*$-Einstein soliton (g, V, λ), where V is a conformal Killing vector field. Then V is one of the following cases:*

(i) *proper vector field if* $\frac{r+4n^2}{2} - \lambda$ *is not constant.*
(ii) *homothetic vector field if* $\frac{r+4n^2}{2} - \lambda$ *is constant.*
(iii) *proper homothetic vector field if* $\frac{r+4n^2}{2} - \lambda$ *is non-zero constant.*
(iv) *Killing vector field if* $\lambda = \frac{r+4n^2}{2}$.

Using the property of Lie derivative we can write,

$$(\pounds_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X), \tag{3.12}$$

for any vector fields X, Y. Then from (2.10), (2.15), (2.16) and (3.12), (1.3) takes the form,

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + [2\lambda - r - 4n^2 - 2]g(X, Y) + 2\eta(X)\eta(Y) = 0. \tag{3.13}$$

Suppose θ is a 1-form, which is metrically equivalent to V and is given by $\theta(X) = g(X, V)$ for an arbitrary vector field X. Then the exterior derivative $d\theta$ of θ can be written as

$$2(d\theta)(X, Y) = g(\nabla_X V, Y) - g(\nabla_Y V, X). \tag{3.14}$$

As $d\theta$ is skew-symmetric, so if we define a tensor field F of type (1,1) by

$$(d\theta)(X, Y) = g(X, FY), \tag{3.15}$$

then F is skew self-adjoint i.e. $g(X, FY) = -g(FX, Y)$. So (3.15) can be written as:

$$(d\theta)(X, Y) = -g(FX, Y) \tag{3.16}$$

Using (3.16), (3.14) becomes

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = -2g(FX, Y). \tag{3.17}$$

Adding (3.17) and (3.13) side by side and factoring out Y, we get

$$\nabla_X V = -FX - \Big[\lambda - \frac{r}{2} - 2n^2 - 1\Big]X - \eta(X)\xi. \tag{3.18}$$

Substituting the above equation in $R(X,Y)V = \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V$, we have

$$R(X,Y)V = (\nabla_Y F)X - (\nabla_X F)Y - [\eta(Y)X + \eta(X)Y - 2\eta(X)\eta(Y)\xi]. \tag{3.19}$$

Noting that $d\theta$ is closed, we obtain

$$g(X,(\nabla_Z F)Y) + g(Y,(\nabla_X F)Z) + g(Z,(\nabla_Y F)X) = 0. \tag{3.20}$$

Making inner product of (3.19) with respect to Z, we get

$$\begin{aligned} g(R(X,Y)V,Z) = \ & g((\nabla_Y F)X,Z) - g((\nabla_X F)Y,Z) - [\eta(Y)g(X,Z) \\ & +\eta(X)g(Y,Z) - 2\eta(X)\eta(Y)\eta(Z)]. \end{aligned} \tag{3.21}$$

As F is skew self-adjiont, then $\nabla_X F$ is also skew self-adjiont. Then using (3.20), (3.21) takes the form

$$\begin{aligned} g(R(X,Y)V,Z) = \ & -[\eta(Y)g(X,Z) + \eta(X)g(Y,Z) - 2\eta(X)\eta(Y)\eta(Z)] \\ & -g(X,(\nabla_Z F)Y). \end{aligned} \tag{3.22}$$

Putting $X = Z = e_i$ in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1,2,3,..,(2n+1)$, we obtain

$$S(Y,V) = -2n\eta(Y) - (divF)Y \tag{3.23}$$

where $divF$ is the divergence of the tensor field F.

Using (2.10), the above equation becomes

$$(divF)Y = 2n[g(Y,V) - \eta(Y)]. \tag{3.24}$$

Now, using (3.18) we compute the covariant derivative of the squared g-norm of V as follows:

$$\begin{aligned} \nabla_X |V|^2 &= 2g(\nabla_X V, V) \\ &= -2g(FX,V) - [2\lambda - r - 4n^2 - 2]g(X,V) \\ & - 2\eta(X)\eta(V). \end{aligned} \tag{3.25}$$

Again using (1.3), (2.15), (2.10) and (2.16), we have

$$(\pounds_V g)(X,V) = -[2\lambda - r - 4n^2 - 2]g(X,V) - 2\eta(X)\eta(V). \tag{3.26}$$

Then using (3.26), (3.25) becomes

$$\nabla_X |V|^2 + 2g(FX, V) - (\pounds_V g)(X, V) = 0. \tag{3.27}$$

So we can state,

Theorem 3.6. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfies the $*$-Einstein soliton (g, V, λ) then the vector V and its metric dual 1-form θ satisfies the relations*

$$(divF)Y = 2n[g(Y, V) - \eta(Y)]$$

and

$$\nabla_X |V|^2 + 2g(FX, V) - (\pounds_V g)(X, V) = 0.$$

4. Application of torse forming vector field on Kenmotsu manifold admitting $*$-Einstein soliton

Let (g, τ, λ) be a $*$-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold M, where τ is a torse-forming vector field. Then from (1.3), (2.15) and (2.16), we have

$$(\pounds_\tau g)(X, Y) + 2[S(X, Y) + (2n - 1)g(X, Y) + \eta(X)\eta(Y)] + [2\lambda - (r + 4n^2)]g(X, Y) = 0 \tag{4.1}$$

where $\pounds_\tau g$ denotes the Lie derivative of the metric g along the vector field τ.

Now using(1.4), we obtain,

$$\begin{aligned} (\pounds_\tau g)(X, Y) &= g(\nabla_X \tau, Y) + g(X, \nabla_Y \tau) \\ &= 2\psi g(X, Y) + \omega(X)g(\tau, Y) + \omega(Y)g(\tau, X), \end{aligned} \tag{4.2}$$

for all $X, Y \in \chi(M)$.

Then from (4.2) and (4.1), we get,

$$\Big[\frac{(r + 4n^2)}{2} - \lambda - \psi - 2n + 1\Big]g(X, Y) - S(X, Y) - \eta(X)\eta(Y) = \frac{1}{2}\Big[\omega(X)g(\tau, Y) + \omega(Y)g(\tau, X)\Big]. \tag{4.3}$$

Taking contraction of (4.3) over X and Y, we have,

$$\Big[\frac{(r+4n^2)}{2}-\lambda-\psi-2n+1\Big](2n+1)-r-1=\omega(\tau), \tag{4.4}$$

which leads to,

$$\lambda=\frac{(r+4n^2)}{2}-\psi-2n+1-\frac{r+\omega(\tau)+1}{(2n+1)}. \tag{4.5}$$

So we can state the following theorem,

Theorem 4.1. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfies the $*$-Einstein soliton (g,τ,λ), where τ is a torse-forming vector field, then*

$$\lambda=\frac{(r+4n^2)}{2}-\psi-2n+1-\frac{r+\omega(\tau)+1}{(2n+1)}$$

and the soliton is expanding, steady, shrinking according as $\frac{(r+4n^2)}{2}-\psi-2n+1-\frac{r+\omega(\tau)+1}{(2n+1)}\gtreqless 0$.

Now in (4.5), if the 1-form ω vanishes identically then,

$$\lambda=\frac{(r+4n^2)}{2}-\psi-2n+1-\frac{r+1}{(2n+1)}.$$

If the 1-form ω vanishes identically and the function $\psi=1$ in (4.5), then,

$$\lambda=\frac{(r+4n^2)}{2}-2n-\frac{r+1}{(2n+1)}.$$

In (4.5), if the function $\psi=0$, then,

$$\lambda=\frac{(r+4n^2)}{2}-2n+1-\frac{r+\omega(\tau)+1}{(2n+1)}.$$

If $\psi=\omega=0$ in (4.5), then,

$$\lambda=\frac{(r+4n^2)}{2}-2n+1-\frac{r+1}{(2n+1)}.$$

Finally in (4.5), if $\omega(\tau)=0$, then,

$$\lambda=\frac{(r+4n^2)}{2}-\psi-2n+1-\frac{r+1}{(2n+1)}.$$

Then we have,

Corollary 4.1. *Let the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfy the $*$-Einstein soliton (g,τ,λ), where τ is a torse-forming vector field, then if τ is one of the following cases:*

(i) *concircular, then* $\lambda = \frac{(r+4n^2)}{2} - \psi - 2n + 1 - \frac{r+1}{(2n+1)}$ *and the soliton is expanding, steady, shrinking according as* $\frac{(r+4n^2)}{2} - \psi - 2n + 1 - \frac{r+1}{(2n+1)} \gtreqless 0$.

(ii) *concurrent, then* $\lambda = \frac{(r+4n^2)}{2} - 2n - \frac{r+1}{(2n+1)}$ *and the soliton is expanding, steady, shrinking according as* $\frac{(r+4n^2)}{2} - 2n - \frac{r+1}{(2n+1)} \gtreqless 0$.

(iii) *recurrent, then* $\lambda = \frac{(r+4n^2)}{2} - 2n + 1 - \frac{r+\omega(\tau)+1}{(2n+1)}$ *and the soliton is expanding, steady, shrinking according as* $\frac{(r+4n^2)}{2} - 2n + 1 - \frac{r+\omega(\tau)+1}{(2n+1)} \gtreqless 0$.

(iv) *parallel, then* $\lambda = \frac{(r+4n^2)}{2} - 2n + 1 - \frac{r+1}{(2n+1)}$ *and the soliton is expanding, steady, shrinking according as* $\frac{(r+4n^2)}{2} - 2n + 1 - \frac{r+1}{(2n+1)} \gtreqless 0$.

(v) *torqued, then* $\lambda = \frac{(r+4n^2)}{2} - \psi - 2n + 1 - \frac{r+1}{(2n+1)}$ *and the soliton is expanding, steady, shrinking according as* $\frac{(r+4n^2)}{2} - \psi - 2n + 1 - \frac{r+1}{(2n+1)} \gtreqless 0$.

5. Infinitesimal CL-transformations and $*$-Einstein soliton

In this section we discuss about infinitesimal CL-transformation, which was introduced by Tashiro and Tachibana [25] in 1963, on a Kenmotsu manifold, admitting $*$-Einstein soliton.

Definition 5.1. A vector field V on a Kenmotsu manifold M is said to be an infinitesimal CL-transformation ([21], [23]) if it satisfies,

$$\pounds_V \begin{Bmatrix} h \\ ij \end{Bmatrix} = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha(\eta_j \phi_i^h + \eta_i \phi_j^h), \tag{5.1}$$

for a certain constant α, where ρ_i are the components of the 1-form ρ, $\pounds_V$ denotes the Lie derivative with respect to V and $\begin{Bmatrix} h \\ ij \end{Bmatrix}$ is the Christoffel symbol of the Riemannian metric g.

For an infinitesimal CL-transformation on Kenmotsu manifold, we have the followings:

Proposition 5.1. [21] *If V is an infinitesimal CL-transformation on a Kenmotsu manifold M, then the 1-form ρ is closed.*

Theorem 5.1. [21] *If V is an infinitesimal CL-transformation on a Kenmotsu manifold M, then the relation,*

$$(\pounds_V g)(X, Y) = (\nabla_X \rho)(Y) - \alpha g(X, \phi Y), \tag{5.2}$$

holds for any vector fields X, Y on M.

Definition 5.2. [10] A transformation f on an $(2n+1)$-dimensional Kenmotsu manifold M with structure (ϕ, ξ, η, g) is said to be a CL-transformation if the Levi-Civita connection ∇ and a symmetric affine connection ∇^f , called CL-connection, induced from ∇ by f are related by,

$$\nabla^f_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X + \alpha\{\eta(X)\phi Y + \eta(Y)\phi X\}; \tag{5.3}$$

where ρ is a 1-form and α is a constant.

If S and S^f are the Ricci tensor of an $(2n+1)$-dimensional Kenmotsu manifold M with respect to Levi-Civita connection ∇ and CL-connection ∇^f respectively, then we have [7],

$$S^f(X, Y) = S(X, Y) - 2nB(X, Y); \tag{5.4}$$

for all vector fields X, Y on M, where the symmetric tensor field B is given by,

$$B(X, Y) = (\nabla_X\rho)(Y) - \rho(X)\rho(Y) - \alpha^2\eta(X)\eta(Y) - \alpha[\eta(X)\rho(\phi Y) + \eta(Y)\rho(\phi X)]. \tag{5.5}$$

From (5.4), we have,

$$r^f = r - 2n \operatorname{Tr} B, \tag{5.6}$$

where r^f and r are the scalar curvatures of Kenmotsu manifold M with respect to the CL-connection ∇^f and Levi-Civita connection ∇ respectively.
Let us take a ∗-Einstein soliton (g, V, λ) on an $(2n+1)$-dimensional Kenmotsu manifold M. Then from (1.3), (2.15) and (2.16), we get,

$$(\pounds_V g)(X, Y) + 2S(X, Y) + [2\lambda - r - 4n^2 + 4n - 2]g(X, Y) + 2\eta(X)\eta(Y) = 0. \tag{5.7}$$

Now we consider the soliton (g, V, λ) on M with respect to CL-connection ∇^f. Then we have from (5.7),

$$(\pounds^f_V g)(X, Y) + 2S^f(X, Y) + [2\lambda - r^f - 4n^2 + 4n - 2]g(X, Y) + 2\eta(X)\eta(Y) = 0. \tag{5.8}$$

where $\pounds^f_V$ is the Lie derivative along the vector field V on M with respect to CL-connection ∇^f.

By the virtue of (5.3), we have,

$$\begin{aligned}(\pounds^f_V g)(X,Y) &= g(\nabla^f_X V, Y) + g(X, \nabla^f_Y V)\\ &= (\pounds_V g)(X,Y) + \rho(X)g(V,Y) + \rho(Y)g(X,V)\\ &+ 2\rho(V)g(X,Y) + \alpha[\eta(X)g(\phi V, Y) + \eta(Y)g(X,\phi V)].\end{aligned} \tag{5.9}$$

Using (5.4),(5.6) and (5.9), (5.8) becomes,

$$\begin{aligned}&(\pounds_V g)(X,Y) + 2S(X,Y) + [2\lambda - r - 4n^2 + 4n - 2]g(X,Y) + 2\eta(X)\eta(Y)\\ &+ \{\rho(X)g(V,Y) + \rho(Y)g(X,V) + 2\rho(V)g(X,Y) + \alpha[\eta(X)g(\phi V,Y) + \eta(Y)g(X,\phi V)]\\ &\qquad - 4nB(X,Y) + 2n \operatorname{Tr} B \quad g(X,Y)\} = 0.\end{aligned} \tag{5.10}$$

If (g, V, λ) is a $*$-Einstein soliton on M with respect to Levi-Civita connection, then (5.7) holds. Then from (5.7) and (5.10), we can state the following,

Theorem 5.2. *A $*$-Einstein soliton (g, V, λ) on an $(2n+1)$-dimensional Kenmotsu manifold is invariant under CL-connection if and only if the relation,*

$$\begin{aligned}&\rho(X)g(V,Y) + \rho(Y)g(X,V) + 2\rho(V)g(X,Y) + \alpha[\eta(X)g(\phi V,Y) + \eta(Y)g(X,\phi V)]\\ &\qquad - 4nB(X,Y) + 2n \operatorname{Tr} B \quad g(X,Y) = 0\end{aligned}$$

holds for arbitrary vector fields X and Y.

Now, let (g, ξ, λ) be a $*$-Einstein soliton on the $(2n+1)$-dimensional Kenmotsu manifold M with respect to CL-connection. Then from (5.7) we have,

$$(\pounds^f_\xi g)(X,Y) + 2S^f(X,Y) + [2\lambda - r^f - 4n^2 + 4n - 2]g(X,Y) + 2\eta(X)\eta(Y) = 0. \tag{5.11}$$

From (2.1),(2.3),(2.6) and (5.3), we get,

$$\begin{aligned}(\pounds^f_\xi g)(X,Y) &= g(\nabla^f_X \xi, Y) + g(\nabla^f_X \xi, Y)\\ &= 2[\{1 + \rho(\xi)\}g(X,Y) - \eta(X)\eta(Y)] + \rho(X)\eta(Y) + \rho(Y)\eta(X).\end{aligned} \tag{5.12}$$

In view of (5.4), (5.6) and (5.12), (5.11) becomes,

$$\Big[\lambda-\frac{r}{2}+n\operatorname{Tr}B-2n^2+2n+\rho(\xi)\Big]g(X,Y)+S(X,Y)-2nB(X,Y)+\frac{1}{2}[\rho(X)\eta(Y)+\rho(Y)\eta(X)] = 0. \tag{5.13}$$

This leads to the following,

Theorem 5.3. *If (g,ξ,λ) is a $*$-Einstein soliton on an $(2n+1)$-dimensional Kenmotsu manifold with respect to CL-connection, then (5.13) holds, for arbitrary vector fields X and Y.*

Putting $X=Y=\xi$ in (5.13) and using (2.1),(2.10), we obtain,

$$\lambda=\frac{r}{2}-n\operatorname{Tr}B+2n^2-2\rho(\xi)+2nB(\xi,\xi). \tag{5.14}$$

Again taking $X=Y=\xi$ in (5.5) and using (2.1), we get,

$$B(\xi,\xi)=(\nabla_\xi\rho)(\xi)-(\rho(\xi))^2-\alpha^2. \tag{5.15}$$

As ρ is a 1-form, let us consider $\rho(X)=g(X,\kappa)$, for any vector field X, where κ is a vector field.
Then using (2.1), we have,

$$(\nabla_\xi\rho)(\xi)=\eta(\nabla_\xi\kappa). \tag{5.16}$$

Hence using (5.16), (5.15), (5.14) takes the form,

$$\lambda=\frac{r}{2}-n\operatorname{Tr}B+2n^2-2\rho(\xi)+2n\eta(\nabla_\xi\kappa)-2n(\rho(\xi))^2-2n\alpha^2. \tag{5.17}$$

Thus we can state,

Corollary 5.1. *A $*$-Einstein soliton (g,ξ,λ) on an $(2n+1)$-dimensional Kenmotsu manifold with respect to CL-connection is expanding, steady and shrinking according as $r \gtreqless 2n\operatorname{Tr}B-4n^2+4\rho(\xi)-4n\eta(\nabla_\xi\kappa)+4n(\rho(\xi))^2+4n\alpha^2$ respectively.*

6. Schouten-Van Kampen connection and $*$-Einstein soliton

The Schouten-Van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a smooth manifold endowed with an Affine connection [20].
The Schouten-Van Kampen connection $\tilde{\nabla}$ and Levi-Civita connection ∇ on an $(2n+1)$-dimensional Kenmotsu manifold M are related by [20],

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X. \tag{6.1}$$

Now on a 3-dimensional Kenmotsu manifold M, if $\tilde{R}, \tilde{S}$ and $\tilde{r}$ are the curvature tensor, Ricci tensor and scalar curvature respectively with respect to Schouten-Van Kampen connection, then we have [7],

$$\tilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y, \tag{6.2}$$

$$\tilde{S}(X,Y) = S(X,Y) + 2g(X,Y), \tag{6.3}$$

$$\tilde{r} = r + 6. \tag{6.4}$$

for all vector fields $X, Y, Z \in \chi(M)$.
Let us (g, V, λ) be a $*$-Einstein soliton on a 3-dimensional Kenmotsu manifold M with respect to Schouten-Van Kampen connection $\tilde{\nabla}$. Then we have from (5.7),

$$(\tilde{\pounds}_V g)(X,Y) + 2\tilde{S}(X,Y) + [2\lambda - \tilde{r} - 2]g(X,Y) + 2\eta(X)\eta(Y) = 0. \tag{6.5}$$

Using (6.1), we have,

$$\begin{aligned}(\tilde{\pounds}_V g)(X,Y) &= g(\tilde{\nabla}_X V, Y) + g(X, \tilde{\nabla}_Y V)\\ &= (\pounds_V g)(X,Y) + g(X,V)\eta(Y) + g(Y,V)\eta(X) - 2\eta(V)g(X,Y).\end{aligned} \tag{6.6}$$

In view of (6.6),(6.3), (6.4), (6.5) becomes,

$$\begin{aligned}&(\pounds_V g)(X,Y) + 2S(X,Y) + [2\lambda - r - 2]g(X,Y) + 2\eta(X)\eta(Y)\\ &\quad + \{g(X,V)\eta(Y) + g(Y,V)\eta(X) - 2\eta(V)g(X,Y) - 2g(X,Y)\} = 0.\end{aligned} \tag{6.7}$$

If (g, V, λ) is a $*$-Einstein soliton on the 3-dimensional Kenmotsu manifold M with respect to Levi-Civita connection, then (5.7) holds and it takes the form $(\pounds_V g)(X,Y)+$

$2S(X,Y)+[2\lambda-r-2]g(X,Y)+2\eta(X)\eta(Y)=0$. Using this equation and (6.7), we can state the following,

Theorem 6.1. *A $*$-Einstein soliton (g,V,λ) on an 3-dimensional Kenmotsu manifold is invariant under Schouten-Van Kampen connection if and only if the relation,*

$$g(X,V)\eta(Y)+g(Y,V)\eta(X)-2\eta(V)g(X,Y)-2g(X,Y)=0,$$

holds for arbitrary vector fields X and Y.

7. Geometrical and physical motivation of $*$-Einstein soliton

The notion of $*$-Einstein soliton is replaced by Einstein soliton as a kinematic solution of Einstein flow, whose profile develop a characterization of spaces of constant sectional curvature along with the locally symmetric spaces. Also, geometric phenomenon of $*$-Einstein solitons can evolve an aqueduct between a sectional curvature inheritance symmetry of space time and class of Einstein solitons. As an application to relativity there are some physical models of perfect fluid Einstein soliton space times which generates a curvature inheritance symmetry. Here, we can find some physical and geometrical models of perfect $*$-Einstein soliton space time and that will give the physical significance, the concept of $*$-Einstein soliton.

As an application to cosmology and general relativity by investigating the kinetic and potential nature of relativistic space time, we present a physical models of 3-class namely, shrinking, steady and expanding of perfect and dust fluid solution of $*$-Einstein soliton space time. The first case shrinking ($\lambda<0$) which exists on a minimal time interval $-1<t<b$ where $b<1$, steady ($\lambda=0$) that exists for all time or expanding ($\lambda>0$) which exists on maximal time interval $a<t<1$, $a>-1$. These three classes give an example on ancient, eternal and immortal solution respectively.

8. Example of a 3-dimensional Kenmotsu m Manifold Admitting $*$-Einstein Soliton

We consider the three-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3,(x,y,z)\neq(0,0,0)\}$, where (x,y,z) are standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1=z\frac{\partial}{\partial x},\quad e_2=z\frac{\partial}{\partial y},\quad e_3=-z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$,where $\chi(M)$ is the set of all differentiable vector fields on M and ϕ be the (1, 1)-tensor field defined by,

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we have,

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The connection ∇ of the metric g is given by,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.

Using Koszul's formula, we can easily calculate,

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above it follows that the manifold satisfies $\nabla_X \xi = X - \eta(X)\xi$ for $\xi = e_3$. Hence the manifold is a Kenmotsu Manifold.

Also, the Riemannian curvature tensor R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Hence

$$R(e_1,e_2)e_2=-e_1,\quad R(e_1,e_3)e_3=-e_1,\quad R(e_2,e_1)e_1=-e_2,$$

$$R(e_2,e_3)e_3=-e_2,\quad R(e_3,e_1)e_1=-e_3,\quad R(e_3,e_2)e_2=-e_3,$$

$$R(e_1,e_2)e_3=0,\quad R(e_2,e_3)e_1=0,\quad R(e_3,e_1)e_2=0.$$

Then, the Ricci tensor S is given by

$$S(e_1,e_1)=-2,\quad S(e_2,e_2)=-2,\quad S(e_3,e_3)=-2. \tag{8.1}$$

Hence the manifold satisfies $S(X,Y)=-2g(X,Y)$, for all vector fields X,Y.

Also the scalar curvature becomes

$$r=\sum_{i=1}^{3}S(e_i,e_i)=-6. \tag{8.2}$$

Using (2.15) and (8.1), we have

$$S^*(e_1,e_1)=-1,\quad S^*(e_2,e_2)=-1,\quad S^*(e_3,e_3)=0. \tag{8.3}$$

Hence,

$$r^*=Tr(S^*)=-2. \tag{8.4}$$

From (8.3), we can see that the manifold satisfies

$$S^*(X,Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{8.5}$$

for all vector fields X,Y.

Let us take the the potential vector field as

$$V=2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

Then $(\pounds_V g)(e_1,e_1)=-2g(\pounds_V e_1,e_1)=2$. Similarly,

$$(\pounds_V g)(e_2,e_2)=2,\quad (\pounds_V g)(e_3,e_3)=0.$$

Hence one can easily justify that,

$$(\pounds_V g)(X,Y)=2[g(X,Y)-\eta(X)\eta(Y)], \tag{8.6}$$

for all vector fields X,Y.

From (8.5) and (8.6), we obtain g is a $*$-Einstein soliton i.e. (1.3) holds with the potential vector field V, as defined above, on the 3-dimensional Kenmotsu manifold M and $\lambda = -1$, which gives, the soliton is shrinking.

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