Jordan Journal of Mathematics and Statistics (JJMS), 16(1), 2023, pp 139 - 152
DOI: https://doi.org/10.47013/16.1.9

POSITION VECTORS OF A RELATIVELY NORMAL-SLANT HELIX IN EUCLIDEAN 3-SPACE

ABDERRAZZAK EL HAIMI [(1)] AND AMINA OUAZZANI CHAHDI[(2)]

Abstract. In this paper, we give a new characterization of a relatively normal-slant helix. Thereafter, we construct a vector differential equation of the third order to determine the parametric representation of a relatively normal-slant helix according to standard frame in Euclidean 3-space. Finally, we apply this method to find the position vector of some special cases.

1. Introduction

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in E^3. So, the investigating position vectors of the curves in a classical aim to determine behavior of the particle (curve).

Helix is one of the most fascinating curves in science and nature. Scientist have long held a fascinating, sometimes bordering on mystical obsession, for helical structures in nature. We have a lot of special curves such as circular helices, general helices, slant helices, k-slant helices etc. Characterizations of these special curves are heavily studied for a long time and are still studies. We can see the applications of helical structures in nature and mechanic tools. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or design of highways. Also we can see the helix curve or helical structure in fractal geometry.

2020 *Mathematics Subject Classification.* Mathematics Subject Classification 53A04.
Key words and phrases. Darboux frame, Position vector, Relatively normal-slant, Slant helices, General helices.

Received: Feb. 17, 2022 Accepted: Sept. 5, 2022 .

In a recent paper, Dogan and Yayli [9] study isophote curves and their characterizations in Euclidean 3-space. An isophote curve is defined as a curve on a surface whose unit normal field restricted to the curve makes a constant angle with a fixed direction. They also obtain the axis of an isophote curve. In 2017, Macit and Duldul [13] have defined a relatively normal-slant helix on a surface by using the Darboux frame (T,V,U) along the curve whose vector field V makes a constant angle with a fixed direction.

The determining of the parametric representation of curves defined by a geometric property using the intrinsic equation $\kappa = \kappa(s)$ and $\tau = \tau(s)$ (where κ and τ are the curvature and the torsion of the curve, respectively) is a one of important subjects. Recently, the position vector of general helices([8, 12], slant helices, spherical slant helices and k-slant helices in Euclidean space E^3 are deduced in [1, 2, 4, 5, 6], respectively. For more details, see also [10, 11, 14, 15, 16].

In this work, we propose to determine the position vector of relatively normal-slant helix in the case whose the geodesic torsion τ_g, is non-zero (the case $\tau_g = 0$, is studied in [3]).

We first, establish a characterization of relatively normal-slant helix in the case $\tau_g \neq 0$. Thereafter, we construct a vector differential equation of third order to determine the parametric representation of relatively normal-slant helix according to standard frame e_1, e_2, e_3 in Euclidean 3-space. Finally, we apply this method to find the position vectors of some special cases.

2. Definition and characterization of relatively normal-slant helix

In this section, we give the definition and a characterization of relatively normal-slant helices.

Let M be an regular surface, and $\varphi : I \subset \mathbb{R} \longrightarrow M$ be a regular curve with arc-length parametrization. If we denote the Darboux frame along the curve φ by (T, V, U), we have the derivative formulae of the Darboux frame as follows:

$$\begin{cases} T' = \kappa_g V + \kappa_n U, \\ V' = -\kappa_g T + \tau_g U, \\ U' = -\kappa_n T - \tau_g V, \end{cases} \tag{2.1}$$

where T is the unit tangent vector of the curve φ, U is the unit normal vector of the surface restricted to the curve φ, V is the unit vector given by $V = U \times T$, and $\kappa_g, \kappa_n, \tau_g$ denote the geodesic curvature, normal curvature, geodesic torsion of the curve φ, respectively [7]

In classic differential geometry, the non-zero geodesic torsion have limitations in some theorems e.g. Gauss and Weingarten. However, assume that $\tau_g \neq 0$ and consider the new parameter $t = \int \kappa_g ds$,, we get the new Darboux equations [13] as follows :

$$\begin{cases} \frac{dT}{dt} = \sigma_1 V + \sigma_2 U, \\ \frac{dV}{dt} = -\sigma_1 T + U, \\ \frac{dU}{dt} = -\sigma_2 T - V, \end{cases} \tag{2.2}$$

where $\sigma_1 = \dfrac{\kappa_g}{\tau_g}$, and $\sigma_2 = \dfrac{\kappa_g}{\tau_g}$.

Definition 2.1. [13]Let φ be a unit speed curve lying on a regular surface and (T, V, U) be the Darboux frame along φ. The curve φ is called a relatively normal-slant helix if the vector field V of φ makes a constant angle with a fixed direction, i.e. there exists a fixed unit vector W and a constant angle θ such that

$$\langle V, W \rangle = \cos(\theta). \tag{2.3}$$

3. Position vector of a relatively normal-slant helix

We start this section by giving a characterization of normal-slant helix.

Theorem 3.1. *Let $\varphi(s)$ be a unit speed curve lying on a regular surface with $\tau_g \neq 0$. The curve φ is a relatively normal-slant helix if and only if*

$$\left(\frac{\sigma_2\sigma_1^2 + \sigma_2 + \sigma_1'}{(\sigma_1^2+1)^{\frac{3}{2}}}\right)(t) = \mp m, \tag{3.1}$$

where $t = \int \tau_g ds$, $\sigma_1 = \dfrac{\kappa_g}{\tau_g}$, $\sigma_2 = \dfrac{\kappa_g}{\tau_g}$, $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos(\theta)$, and θ is the angle between the vector V and a fixed direction.

Proof. ($\Rightarrow$) Let W be a unit fixed vector satisfying

$$\langle V, W \rangle = n. \tag{3.2}$$

Differentiating the $Eq.(5)$, with respect to the variable $t = \int \tau_g ds$, and using the derivative formulae (2), we get

$$\sigma_1 \langle T, W \rangle = \langle U, W \rangle . \tag{3.3}$$

If we put $\langle T, W \rangle = b$, we can write

$$W = bT + nV + b\sigma_1 U.$$

From the unitary of the vector W we get

$$b = \pm\sqrt{\frac{1-n^2}{1+\sigma_1^2}}. \tag{3.4}$$

Differentiating the $Eq.(6)$, we obtain

$$b\sigma_1^{'} + \sigma_1^2 n + \sigma_1\sigma_2(\sigma_1 b) = -\sigma_2 b - n.$$

Then

$$n = -\left(\frac{\sigma_2\sigma_1^2 + \sigma_2 + \sigma_1'}{1+\sigma_1^2}\right)b, \tag{3.5}$$

and by $Eq.(7)$ and (8), we get the following equation

$$\frac{\sigma_2\sigma_1^2 + \sigma_2 + \sigma_1'}{(1+\sigma_1^2)^{\frac{3}{2}}} = \mp m,$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$.
($\Leftarrow$) Suppose that

$$\frac{\sigma_2\sigma_1^2 + \sigma_2 + \sigma_1'}{(1+\sigma_1^2)^{\frac{3}{2}}} = m, \tag{3.6}$$

and let us consider the vector

$$W(t) = \frac{n}{m}\left(\pm\frac{1}{\sqrt{1+\sigma_1^2}}T + mV \pm \frac{\sigma_1}{\sqrt{1+\sigma_1^2}}U\right)(t).$$

Differentiating the vector W by using the derivative formulae (2) and $Eq.(9)$,

$$\begin{aligned}\frac{dW}{dt} &= \frac{n}{m}\left[\left(\mp\frac{\sigma_1^{'}\sigma_1 + \sigma_1\sigma_2(1+\sigma_1^2)}{(1+\sigma_1^2)^{\frac{3}{2}}} - m\sigma_1\right)T + \left(m \pm \frac{\sigma_2(1+\sigma_1^2) + \sigma_1^{'}(1+\sigma_1^2) - \sigma_1^{'}\sigma_1^2}{(1+\sigma_1^2)^{\frac{3}{2}}}\right)U\right],\\ &= \frac{n}{m}\left[\left(\mp\sigma_1\frac{\sigma_1^{'} + \sigma_2 + \sigma_2\sigma_1^2}{(1+\sigma_1^2)^{\frac{3}{2}}} - m\sigma_1\right)T + \left(m \pm \frac{\sigma_2 + \sigma_2\sigma_1^2 + \sigma_1^{'}}{(1+\sigma_1^2)^{\frac{3}{2}}}\right)U\right],\\ &= 0.\end{aligned}$$

Therefore, the vector V is constant and $\langle V, W\rangle = n$, which completes the proof. □

To determine the parametric representation of the position vector of a relatively normal-slant helix (its vector field V make a constant angle with a fixed direction), we firstly establish that for any arbitrary curve, the vector tangent T satisfies a vector differential equation as follows:

Theorem 3.2. *Let $\varphi(s)$ be a unit speed curve lying on a regular surface with $\tau_g \neq 0$ Then, the vector T satisfies a vector differential equation as follows :*

$$T'(t) - \sigma_1(t)\sigma_2(t)T(t) = \sigma_1(t)V(t) + \sigma_2(t)V'(t), \tag{3.7}$$

where $t = \int \tau_g ds$, $\sigma_1 = \dfrac{\kappa_g}{\tau_g}$, and $\sigma_2 = \dfrac{\kappa_n}{\tau_g}$.

Proof. Let $\varphi(s)$ be a curve lying on a regular surface. If we differentiate the first equation of the new Darboux $Eq.(2)$, we get:

$$T' = \sigma_1 V + \sigma_2 U.$$

We multiply the second equations of (2) by σ_2, we obtain:

$$\sigma_2 V' = -\sigma_2\sigma_1 T + \sigma_2 U.$$

Therefore

$$T' - \sigma_2 V' = \sigma_1 V + \sigma_2\sigma_1 T.$$

□

Following theorem 3.1, if we know the vector V of an arbitrary curve lying on a regular surface and satisfying $\tau_g \neq 0$, we can determine the tangent vector of this curve and hence its parametric representation. Hence the theorem:

Theorem 3.3. *Let $\varphi(s)$ be a unit speed curve lying on a regular surface with $\tau_g \neq 0$. The position vector of this curve can be determined as follows*

$$\varphi(t) = \int \frac{1}{\tau_g}[(\int (\sigma_1 V + \sigma_2 V')e^{-\int \sigma_2\sigma_1 dt}dt)e^{\int \sigma_2\sigma_1 dt} + ke^{\int \sigma_2\sigma_1 dt}]dt. \tag{3.8}$$

where $t = \int \tau_g ds$, $\sigma_1 = \dfrac{\kappa_g}{\tau_g}$, and $\sigma_2 = \dfrac{\kappa_n}{\tau_g}$. and k as a constant.

Proof. Let $\varphi = \varphi(s)$ be a curve lying on a regular surface. We construct the solution of $Eq.$(10) by the method of variation of a constant. Consider the corresponding homogeneous equation:

$$T_h'(t) - \sigma_1(t)\sigma_2(t)T_h(t) = 0$$

and find its general solution

$$T_h(t) = ke_2^{\sigma}\sigma_1 dt, \ \ where\ k \in \mathbb{R}.$$

Next, we suppose that k is a function of t and substitute the solution $T_p(t) = k(t)e^{\sigma_2\sigma_1 dt}$ into the initial nonhomogeneous equation (10). We can write

$$(k^{'}e^{\int \sigma_2\sigma_1 dt} + k\sigma_2\sigma_1 e^{\int \sigma_2\sigma_1 dt}) - \sigma_2\sigma_1 e^{\int \sigma_2\sigma_1 dt} = \sigma_1 V + \sigma_2 V^{'},$$

hence

$$k = \int (\sigma_1 V + \sigma_2 V^{'})e^{-\int \sigma_2\sigma_1 dt}dt.$$

As a result, the general solution of the differential equation (10) is given by:

$$T(t) = \left(\int (\sigma_1 V + \sigma_2 V^{'})e^{-\int \sigma_2\sigma_1 dt}dt\right)e^{\int \sigma_2\sigma_1 dt} + ke^{\int \sigma_2\sigma_1 dt},$$

where $k \in \mathbb{R}$.

On the other hand, $\varphi(t) = \int \dfrac{1}{\tau_g}Tdt$, which completes the proof.

□

Theorem 3.4. *Let $\varphi = \varphi(s)$ be a unit speed curve lying on a regular surface with $\tau_g \neq 0$. If φ is a relatively normal-slant helix, then the vector V satisfies a vector differential equation of the third order as follows:*

(3.9)

$$\left(\frac{-(V^{'} + \sigma_1\sigma_2 V + (\sigma_1^2 + 1)V)}{(\sigma_1^{'} + \sigma_2 + \sigma_2\sigma_1^2)}\right)^{'}(t) + \left(\frac{\sigma_2\sigma_1(V^{'} + \sigma_2\sigma_1 V + (\sigma_1^2 + 1)V)}{(\sigma_1^{'} + \sigma_2 + \sigma_2\sigma_1^2)}\right)(t) = (\sigma_1 V + \sigma_2 V')(t)$$

where $t = \int \tau_g ds$, $\sigma_1 = \dfrac{\kappa_g}{\tau_g}$. and $\sigma_2 = \dfrac{\kappa_n}{\tau_g}$.

Proof. Differentiating the second equation of the new Darboux formulae (2) and using the first and the third equations, we have:

$$-V^{''}(\sigma_1^2 + 1)V = (\sigma_1^{'} + \sigma_2)T + \sigma_1\sigma_2 U, \tag{3.10}$$

from the second equation of (2), we have

$$\sigma_1\sigma_2V^{'} = -\sigma_2\sigma_1^2T + \sigma_1\sigma_2U,$$

then

$$\sigma_1\sigma_2V^{'} = -\sigma_2\sigma_1^2T + \sigma_1\sigma_2U + (\sigma_1^{'} + \sigma_2)T - (\sigma_1^{'} + \sigma_2)T \tag{3.11}$$

Substituting $Eq.$(14) in $Eq.$(13), we obtain

$$V^{''} + \sigma_1\sigma_2V^{'} + (\sigma_1^2 + 1)V = -(\sigma_1^{'} + \sigma_2 + \sigma_2\sigma_1^2)T.$$

Since φ is a relatively normal-slant helix, we get

$$T = \frac{-1}{(\sigma_1^{'} + \sigma_2 + \sigma_2\sigma_1^2)}(V^{''} + \sigma_1\sigma_2V^{'} + (\sigma_1^2 + 1)V), \tag{3.12}$$

and

$$\sigma_2\sigma_1T = \frac{-\sigma_2\sigma_1}{(\sigma_1^{'} + \sigma_2 + \sigma_2\sigma_1^2)}(V^{''} + \sigma_1\sigma_2V^{'} + (\sigma_1^2 + 1)V). \tag{3.13}$$

Substituting $Eq.$(16) and the derivative of $Eq.$(15) in $Eq.$(10), we get the formula as desired. □

Theorem 3.5. *Let $\varphi = \varphi(s)$ be a unit speed curve lying on a regular surface with $\tau_g \neq$. If φ is a relatively normal-slant helix, then the vector V satisfies a vector differential equation of the third order as follows:*

$$V^{'''}(t) + A(t)V^{''}(t) + B(t)V^{'}(t) = 0 \tag{3.14}$$

with

$$\begin{cases} A = -3\sigma_1^{'}\sigma_1(\sigma_1^2 + 1)^{-1}. \\ B = \sigma_1\sigma_2^{'} + 2\sigma_1^{'}\sigma_2 - 3\sigma_1^{'}\sigma_1^2\sigma_2(\sigma_1^2 + 1)^{-1} + \sigma_1^2 + \sigma_2^2 + 1, \end{cases}$$

where $t = \int \tau_g ds$, $\sigma_1 = \dfrac{\kappa_g}{\tau_g}$ and $\sigma_2 = \dfrac{\kappa_n}{\tau_g}$.

Proof. The curve φ is a relatively normal-slant helix lying on a regular surface, *i.e.* satisfied $Eq.$(4), substituting this equation in $Eq.$(12), we get

$$V^{'''} - \frac{3\sigma_1^{'}\sigma_1}{(\sigma_1^2 + 1)}V^{''} + \sigma_1\sigma_2V^{''} + \sigma_1\sigma_2^{'}V^{'} + \sigma_1^{'}\sigma_2V^{'} - \frac{3\sigma_1^{'}\sigma_1\sigma_1\sigma_2}{(\sigma_1^2 + 1)}V^{'} + (\sigma_1^2 + 1)V^{'}$$
$$-\sigma_1^{'}\sigma_1V - \sigma_2\sigma_1V^{''} - \sigma_1^2\sigma_2^2V^{'} - \sigma_2\sigma_1(\sigma_1^2 + 1)V + m(\sigma_1^2 + 1)^{\frac{3}{2}}\sigma_1V + m(\sigma_1^2 + 1)^{\frac{3}{2}}\sigma_2V^{'} = 0.$$

The equation becomes

$$V^{'''}(t) + A(t)V^{''}(t) + B(t)V^{'} + C(t)V = 0,$$

where

$$A = \frac{-3\sigma_1^{'}\sigma_1}{(\sigma_1^2+1)},$$

for B, we have

$$B = \sigma_1\sigma_2^{'} + \sigma_1^{'}\sigma_2 - \frac{3\sigma_1^{'}\sigma_2^2\sigma_2}{(\sigma_1^2+1)} + \sigma_1^2 + 1 - \sigma_1^2\sigma_2^2 + m(\sigma_1^2+1)^{\frac{3}{2}}\sigma_2,$$

then

$$B = \sigma_1\sigma_2^{'} + 2\sigma_1^{'}\sigma_2 - \frac{3\sigma_1^{'}\sigma_2^2\sigma_2}{(\sigma_1^2+1)} + \sigma_1^2 + \sigma_2^2 + 1,$$

and

$$C = 0.$$

□

Theorem 3.6. *Let $\varphi = \varphi(s)$ be a unit speed curve lying on a regular surface. The position vector $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ of a relatively normal-slant helix lying on a regular surface satisfying $\tau_g \neq 0$, with respect to standard frame (e_1, e_2, e_3), is computed in the parametric form as follows:*

$$\begin{cases} \varphi_1(t) = \frac{n}{m}\int \frac{e^{\beta}}{\tau_g}\left[\int(\sigma_1\cos(\int \alpha.dt) - \sigma_2\alpha\sin(\int\alpha.dt))e^{-\beta}dt\right]dt, \\ \varphi_2(t) = \frac{n}{m}\int \frac{e^{\beta}}{\tau_g}\left[\int(\sigma_1\sin(\int \alpha.dt) + \sigma_2\alpha\cos(\int\alpha.dt))e^{-\beta}dt\right]dt, \\ \varphi_3(t) = n\int \frac{e^{\beta}}{\tau_g}\left[\int \sigma_1 e^{-\beta}dt\right]dt, \end{cases}$$

where $t = \int \tau_g ds$, $\sigma_1 = \frac{\kappa_g}{\tau_g}$, $\sigma_2 = \frac{\kappa_n}{\tau_g}$, $\beta = \int \sigma_2\sigma_1 dt$, $\alpha = \left(B - \frac{2A^2}{9} - \frac{A^{'}}{3}\right)^{\frac{1}{2}}$, $m = \frac{n}{\sqrt{1-n^2}}$, where $n = \cos(\theta)$ and θ is the angle between e_3 (axis of relatively normal-slant helix) and the vector V.

Proof. The curve φ is a relatively normal-slant helix lying on a regular surface, *i.e.* the vector V makes a constant angle with a fixed direction called axis of helix. Then the vector V satisfies a vector differential equation:

$$V^{'''}(t) + A(t)V^{''}(t) + B(t)V^{'}(t) = 0. \tag{3.15}$$

So, without loss of generality, we can take the axis of the relatively normal-slant helix parallel to e_3, where (e_1, e_2, e_3) is an orthonormal frame in E^3, then

$$V = V_1 e_1 + V_2 e_2 + n e_3. \tag{3.16}$$

From the unitary of the vector V, we get

$$V_1^2 + V_2^2 = 1 - n^2 = \frac{n^2}{m^2}, \tag{3.17}$$

The solution of $Eq.(20)$ is given as follows:

$$\begin{cases} V_1(t) = \dfrac{n}{m}\cos(\lambda(t)), \\ \\ V_2(t) = \dfrac{n}{m}\sin(\lambda(t)), \end{cases}$$

where λ is an arbitrary function of t. Every component of the vector V satisfies the $Eq.(18)$. So, substituting the components $V_1(t)$ and $V_2(t)$ in the $Eq.(19)$, we have the following differential equations of the function $\lambda(t)$

$$\begin{aligned} &\left(\frac{-n}{m}(\lambda^{'''}\sin(\lambda) + \lambda^{''}\lambda^{'}\cos(\lambda) + 2\lambda^{''}\lambda^{'}\cos(\lambda) - \lambda^{'3}\sin(\lambda))\right) \\ &+A\left(\frac{-n}{m}(\lambda^{''}\sin(\lambda) + \lambda^{'2}\cos(\lambda))) + B(\frac{-n}{m}\lambda^{'}(t)\sin(\lambda(t))\right) = 0, \end{aligned}$$

$$\begin{aligned} &\left(\frac{n}{m}(\lambda^{'''}\cos(\lambda) - \lambda^{''}\lambda^{'}\sin(\lambda) - 2\lambda^{''}\lambda^{'}\sin(\lambda) - \lambda^{'3}\cos(\lambda))\right) \\ &+A\left(\frac{n}{m}(\lambda^{''}\cos(\lambda) - \lambda^{'2}\sin(\lambda))) + B(\frac{n}{m}\lambda^{'}(t)\cos(\lambda(t))\right) = 0. \end{aligned}$$

It is easy to prove that the above two equations lead to the following two equations:

$$-3\lambda^{'}\lambda^{''} - \lambda^{'2}A = 0, \tag{3.18}$$

$$-\lambda^{'''} + \lambda^{'3} - \lambda^{''}A - \lambda^{'}B = 0. \tag{3.19}$$

$Eq.$(21), gives the following equation:

$$\lambda^{''} = -\frac{A}{3}\lambda^{'}. \tag{3.20}$$

Substituting $Eq.$(23) and its derivative in $Eq.$(22), then solving the differtial equation that we get , we obtain the following condition:

$$\lambda(t) = \int \left(B - \frac{2A^2}{9} - \frac{A^{'}}{3}\right)^{\frac{1}{2}} dt = \int \alpha.dt,$$

where $\alpha = \left(B - \frac{2A^2}{9} - \frac{A^{'}}{3}\right)^{\frac{1}{2}}$. Now, the vector V take the following form:

$$\begin{cases} V_1(t) = \frac{n}{m}\cos(\int \alpha.dt), \\ V_2(t) = \frac{n}{m}\sin(\int \alpha.dt)), \\ V_3(t) = n. \end{cases} \tag{3.21}$$

If we substitute the $Eq.$(24) in the $Eq.$(11), which completes the proof. □

Remark 1. *if* $B < \frac{2A^2}{9} - \frac{A^{'}}{3}$, *it means there does not exist a relatively normal-slant helix.*

4. Applications

In this section, we determine the position vectors of special curves by applying $theorem$3.5. We first recall this result.

Lemma 4.1. *Let φ be a curve lying on a regular surface M:*
If $\kappa_g = 0$, then, φ is relatively normal-slant helix on M if and only if φ is a general helix [13].
If $\kappa_n = 0$, then, φ is relatively normal-slant helix on M if and only if φ is a slant helix [13].

Example 4.1. *The position vector of a relatively normal-slant helix with $\kappa_g = 0$ (general helix) and $\tau_g \neq 2$, is expressed in the natural representation form, with* *respect to standard frame* (e_1, e_2, e_3) *by :*

$$\begin{cases} \varphi_1(s) = \mp n \int \cos(\sqrt{1+m^2} \int \tau_g ds) ds, \\ \varphi_2(s) = \mp n \int \sin(\sqrt{1+m^2} \int \tau_g ds) ds, \\ \varphi_3(s) = ncs, \end{cases}$$

or in the parametric form :

$$\begin{cases} \varphi_1(t) = \mp n \int \frac{1}{\tau_g} \cos\left(\sqrt{1+m^2} t\right) dt, \\ \varphi_2(t) = \mp n \int \frac{1}{\tau_g} \sin\left(\sqrt{1+m^2} t\right) dt, \\ \varphi_3(t) = nc \int \frac{1}{\tau_g} dt, \end{cases}$$

where $t = \int \tau_g ds$ *and* c *as constant.*

Example 4.2. *The position vector of a relatively normal-slant helix with* $\kappa_n = 0$ *(slant helix) and* $\tau_g \neq$*, is expressed in the natural representation form, with respect to standard frame* (e_1, e_2, e_3) *by :*

$$\begin{cases} \varphi_1(s) = \mp n \int \left(\int \frac{\tau_g \int \tau_g ds}{\sqrt{1 - m^2 (\int \tau_g ds)^2}} \cos\left(\frac{1}{n} \arcsin(m \int \tau_g ds)\right) ds \right) ds. \\ \\ \varphi_2(s) = \mp n \int \left(\int \frac{\tau_g \int \tau_g ds}{\sqrt{1 - m^2 (\int \tau_g ds)^2}} \sin\left(\frac{1}{n} \arcsin(m \int \tau_g ds)\right) ds \right) ds. \\ \\ \varphi_3(s) = \pm \frac{n}{m} \int \sqrt{1 - m^2 \left(\int \tau_g ds \right)^2} ds. \end{cases}$$

or in the parametric form :

$$\begin{cases} \varphi_1(t) = \mp n \int \left(\frac{1}{\tau_g} \int \frac{t}{\sqrt{1-m^2 t^2}} \cos\left(\frac{1}{n} \arcsin(mt)\right) dt \right) dt. \\ \\ \varphi_2(t) = \mp n \int \left(\frac{1}{\tau_g} \int \frac{t}{\sqrt{1-m^2 t^2}} \sin\left(\frac{1}{n} \arcsin(mt)\right) dt \right) dt. \\ \\ \varphi_3(t) = \pm \frac{n}{m} \int \frac{1}{\tau_g} \sqrt{1-m^2 t^2} dt, \end{cases}$$

where $t = \tau_g ds$.

Example 4.3. *The position vector of a relatively normal-slant helix with* $\kappa_g = \kappa_n = \tau_g \neq$, *is expressed in the natural representation form, with respect to standard frame* (e_1, e_2, e_3) *by :*

$$\begin{cases} \varphi_1(s) = \frac{1}{\sqrt{6}} \int (\sqrt{3}\sin(\sqrt{3}\int \tau_g ds) + \cos(\sqrt{3}\int \tau_g ds))ds. \\ \varphi_2(s) = \frac{1}{\sqrt{6}} \int (\sin(\sqrt{3}\int \tau_g ds) - \sqrt{3}\cos(\sqrt{3}\int \tau_g ds))ds. \\ \varphi_3(s) = -ns, \end{cases}$$

or in the parametric form :

$$\begin{cases} \varphi_1(t) = \frac{1}{\sqrt{6}} \int \dfrac{1}{\tau_g}(\sqrt{3}\sin(\sqrt{3}t) + \cos(\sqrt{3}t))dt. \\ \varphi_2(t) = \frac{1}{\sqrt{6}} \int \dfrac{1}{\tau_g}(\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t))dt. \\ \varphi_3(t) = -n \int \dfrac{1}{\tau_g} dt, \end{cases}$$

where $t = \tau_g ds$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request. The articles used to support the findings of this study are included within the article and are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgement

We would like to thank the editor and the referees :

References

[1] A. Elhaimi, M. Izid and A. Ouazzani Chahdi, Parametric equations for space curves whose spherical images are slant Helices, *Journal of Mathematics Research*, (2019), 82–88.

[2] A. Elhaimi, M. Izid and A. Ouazzani Chahdi, Position vectors of curves generalizing general Helices and slant Helices in Eeuclidean 3-space, *Tamkan Journal of mathematics* **52** (2021).

[3] A. Elhaimi and A. Ouazzani Chahdi, Parametric equations of special curves lying on a regular surface in Euclidean 3-space, *Nonlinear Functional Analysis and Application (NFAA).* **25(4)**, (2020).

[4] A. T. Ali, Position vector of general helices in Euclidean space, *Bull. Math. Anal. Appl.* (2011) 1-8.

[5] A. T. Ali, Position vector of slant helices in Euclidean space, *Journal of Egyptian Mathematical Society*, (2012).

[6] A. T. Ali, Generalization of general helices and slant helices, *Journal of mahani mathematical research center* **6**, (2017) 25–41.

[7] B. O'Neill, Elementary differential geometry, Academic Press, (1996).

[8] D.J. Struik, In Lectures in Classical Differential Geometry, Addison-Wesley, Reading, Ma, (1961).

[9] F. Dogan and Y. Yayl, On isophote curves and their characterizations, *Turkish Journal of Mathematics*, **3 (39)**(2015), 650–664.

[10] Hussien, R. A., T. Ali and A. A., Study on the differential properties of a robot end-effector motion using the curvature theory, *J. Math. Comput. Sci.*, **6(5)**(2016), 875–893.

[11] L. Cui and J. S. Dai, A Darboux-frame-based formulation of spin-rolling motion of rigid objects with point contact, *IEEE Transactions on Robotics*, **26(2)**(2010), 383–388. doi: 10.1109/TRO.2010.2040201.

[12] M. A. Lancret, Mémoire sur les courbes à double courbure, *Mémoires présentés à l'Institut*, (1806) 416–454.

[13] N. Macit and M. Duldul, Relatively normal-slant helices lying on a surface and their characterizations, *Hacettepe Journal of Mathematics and Statistics*, (2017) 397–408.

[14] Seyed .A. Tafrishi, M. Svinin and M. Yamamoto, Darboux-frame-based parametrization for a spin-rolling sphere on a plane: A nonlinear transformation of underactuated system to fully-actuated model. *Mechanism and Machine Theory*, **164**(2021), p.104415.

[15] Seyed .A. Tafrishi, M. Svinin, M. Yamamoto and Y. Hirata, Path planning of a spin-rolling sphere on a plane, 2021 arXiv preprint arXiv:2110.12397.

[16] S. Izumiya and N. Takeuchi, New special curves and developable surfaces, *Turk J Math*, **28** (2004), 153–163.

(1) Laboratory of Analysis, Algebra and applications (L3A), Faculty of Sciences Ben M'sik,Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco

Email address: abderrazzakelhaimi@gmail.com

(2) Laboratory of Analysis, Algebra and applications (L3A), Faculty of Sciences Ben M'sik, Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco

Email address: ouazzaniamina@hotmail.com