Jordan Journal of Mathematics and Statistics (JJMS), 16(1), 2023, pp 153 - 163
DOI: https://doi.org/10.47013/16.1.10

A NOTE ON THE BOUNDS OF ZEROS OF POLYNOMIALS AND CERTAIN CLASS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract. In the paper we wish to find bounds of zeros of a polynomial. Our result in some special case sharpen some very well known results obtained for this purpose. Also, we obtain lower bound for a certain class of transcendental entire functions by restricting the coefficients of its Taylor's series expansions to some conditions.

1. Introduction

Fundamental theorem of algebra asserts that every non constant polynomial of degree n with complex coefficients has exactly n zeros but gives no information about location of zeros of a polynomial. All zeros of a polynomial of degree less than or equal to 4 can be derived algebraically for all possible values of its coefficients. But, difficulty arises when degree of polynomial is greater than or equal to 5. So, it is desirable to know bounds of zeros of a polynomial. Problem of finding bounds for zeros of a polynomial is classical one which is essential in various disciplines such as controling engineering problems, eigenvalue problems in mathematical physics and digital audio signal processing problems {cf. [18]}. Gauss and Cauchy were first contributors in this area [16]. To find bounds for the moduli of the zeros of a polynomial, Cauchy {cf. [16]} introduced the following classical result:

***Theorem* A** [16]. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* n*, then all the zeros of* $P(z)$ *lie in* $|z|\leq 1+max_{0\leq j\leq (n-1)}|\frac{a_j}{a_n}|$.

Theorem A was improved in several ways by many researchers {cf. [6], [13], [15] &

2020 *Mathematics Subject Classification.* 30D20, 30C10, 30C15, 30D10.

Key words and phrases. Polynomial, transcendental entire function, coefficients, zeros, bounds.

Received: Feb. 18, 2022 Accepted: Jun. 13, 2022 .

[17] }. As an improvement of Theorem A, Joyal et al. [11] gave the following theorem:

***Theorem* B** [11]. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$ $(a_n=1)$ *is a polynomial of degree* n *and* $\beta=\max_{0\leq j<n-1}|a_j|$, *then all the zeros of* $P(z)$ *lie in*

$$|z|\leq\frac{1}{2}\left\{1+|a_{n-1}|+\sqrt{(1-|a_{n-1}|)^2+4\beta}\right\}.$$

Theorem B gives no improvement of Theorem A if $\beta=|a_{n-1}|$. Datt and Govil [5] established following result which is an improvement of Theorem A even if $\beta=|a_{n-1}|$.

***Theorem* C** [5]. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$ $(a_n=1)$ *is a polynomial of degree* n *and if* $A=\max_{0\leq j\leq n-1}|a_j|$, *then* $P(z)$ *has all its zeros in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}.(nA+1)}\leq|z|\leq 1+\left(1-\frac{1}{(1+A)^n}\right)A.$$

Again, in a different direction G. Enström and S. Kakeya [9] established following result known as Enström-Kakeya theorem.

***Theorem* D** [9]. *If* $P(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* n *with real coefficients satisfying* $0\leq a_0\leq a_1\leq...\leq a_n$, *then all the zeros of* $P(z)$ *lie in* $|z|\leq 1$.

Many improvements and generalizations of Theorem D for polynomials and analytic functions are seen in the existing literature {cf. [1]- [4],[8],[9],[11], [12] }.

We recall that an entire function f of one complex variable z is a function analytic in the finite complex plane $\mathbb{C}$ and therefore it can be represented by an everywhere convergent power series like $f(z)=c_0+c_1z+...+c_nz^n+...$ where $c_i, i=0,1,...,n,...$ are real or complex constants. Thus entire functions can be thought of as the natural generalization of polynomials.

The prime concern of this paper is to improve Theorem A as well as Theorem B and Theorem C in some special case and also derive lower bound for zeros of a certain class of transcendental entire functions with restricted coefficients. We do not explain the standard theories, notations and definitions of entire functions as those are available in [19].

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [10] *Let* $\{f_n(z)\}, n = 1, 2, ...$ *be a sequence of functions that are analytic in a region* D *and that converges uniformly to a function* $f(z)$ *in every closed sub region of* D*. Let* z_0 *be an interior point of* D*. If* z_0 *is a limit point of the zeros of* $f_n(z)$*, then* z_0 *is a zero of* $f(z)$*. Conversely, if* z_0 *is an m-fold zero of* $f(z)$*, every sufficiently small neighborhood of* z_0 *contains exactly* m *zeros (counted with their multiplicities) of each* f_n *with* $n > N$ *for a sufficiently large integer* N*.*

Lemma 2.1 is known as Hurwitz theorem in $\mathbb{C}$.

3. Theorems

Theorem 3.1. *Let* $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree* $n > 1$ *with* $M_1 = \max\{|a_2 - a_1|, |a_3 - a_2|, ..., |a_n - a_{n-1}|,\ |a_n|\}$ *and* $M_2 = \max\{|a_0|, |a_1 - a_0|, ..., |a_{n-1} - a_{n-2}|\}$*.Then all the zeros of* $P(z)$ *are contained in the ring shaped region* $R_1 \leq |z| \leq R_2$ *where*

$$R_1 = \frac{2|a_0|}{|a_0| + |a_1 - a_0| + \sqrt{(|a_0| - |a_1 - a_0|)^2 + 4|a_0|M_1}},$$
$$R_2 = \frac{1}{2|a_n|}\left\{|a_n| + |a_n - a_{n-1}| + \sqrt{(|a_n| - |a_n - a_{n-1}|)^2 + 4|a_n|.M_2}\right\}.$$

Proof. Let $Q_1(z) = (z-1)z^n P(\frac{1}{z})$

i.e, $Q_1(z) = a_0 z^{n+1} + (a_1 - a_0)z^n + (a_2 - a_1)z^{n-1} + ... + (a_n - a_{n-1})z - a_n$.

Then,

$$\begin{aligned} (3.1)\quad |Q_1(z)| &\geq |a_0||z|^{n+1} - |(a_1 - a_0)z^n + (a_2 - a_1)z^{n-1} + ... + (a_n - a_{n-1})z - a_n| \\ &\geq (|a_0||z| - |a_1 - a_0|)|z|^n - |(a_2 - a_1)z^{n-1} + ... + (a_n - a_{n-1})z - a_n|. \end{aligned}$$

Now for $|z| = r(> 1)$, it follows that

$$\begin{aligned}
&|(a_2 - a_1)z^{n-1} + (a_3 - a_2)z^{n-2} + ... + (a_n - a_{n-1})z - a_n| \\
\leq &|a_2 - a_1|r^{n-1} + |a_3 - a_2|r^{n-2} + ... + |a_n - a_{n-1}|r + |a_n| \\
\leq &M_1.r^n \left\{ \frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^n} \right\} \\
&\qquad \text{where } M_1 = \max\{|a_2 - a_1|, |a_3 - a_2|, ..., |a_n - a_{n-1}|, |a_n|\} \\
\leq &M_1.r^n \sum_{j=1}^{\infty} \frac{1}{r^j} \\
= &M_1.r^n \frac{1}{r-1}.
\end{aligned}$$

Hence from (3.1), we get for $|z| = r(> 1)$ that

$$\begin{aligned}
|Q_1(z)| &\geq (|a_0|r - |a_1 - a_0|)r^n - M_1.r^n \frac{1}{r-1} \\
&= \frac{r^n}{r-1} \left\{ |a_0|r^2 - (|a_0| + |a_1 - a_0|)r + |a_1 - a_0| - M_1 \right\} > 0
\end{aligned}$$

$$\text{if } f(r) = |a_0|r^2 - (|a_0| + |a_1 - a_0|)r + |a_1 - a_0| - M_1 > 0.$$

Clearly, $f(r) = 0$ has only real roots. In view of Descartes' rule of signs $f(r) = 0$ has two positive roots if $|a_1 - a_0| > M_1$ and only one positive root if $|a_1 - a_0| < M_1$. For each case, positive root of $f(r) = 0$ which lies in $1 < r < \infty$ is

$$\frac{1}{2|a_0|}\{|a_0| + |a_1 - a_0| + \sqrt{(|a_0| - |a_1 - a_0|)^2 + 4|a_0|M_1}\}.$$

Since $f(+\infty) > 0$, obviously $f(r) > 0$ i.e $|Q_1(z)| > 0$ if

$$|z| = r > \frac{1}{2|a_0|}\{|a_0| + |a_1 - a_0| + \sqrt{(|a_0| - |a_1 - a_0|)^2 + 4|a_0|M_1}\}.$$

Consequently, $|P(z)| > 0$ if

$$|z| < R_1 = \frac{2|a_0|}{|a_0| + |a_1 - a_0| + \sqrt{(|a_0| - |a_1 - a_0|)^2 + 4|a_0|M_1}}.$$

Hence all the zeros of $P(z)$ contain in $|z| \geq R_1$.

Further, asssuming $Q_2(z) = (1 - z)P(z)$

$$\text{i.e, } Q_2(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... +$$
$$(a_1 - a_0)z + a_0.$$

it follows that

$$|Q_2(z)| \geq |a_n||z|^{n+1} - |(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0|$$
$$\geq (|a_n||z| - |a_n - a_{n-1}|)|z|^n - |(a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0|.$$

Now calculating in a similar fashion as in $Q_1(z)$, we obtain for $|z| = r(> 1)$ that

$$|Q_2(z)| \geq (|a_n|r - |a_n - a_{n-1}|)r^n - M_2.r^n \frac{1}{r-1}$$

where $M_2 = \max\{|a_0|, |a_1 - a_0|, ..., |a_{n-1} - a_{n-2}|\}$

$$= \frac{r^n}{r-1}\left\{|a_n|r^2 - (|a_n| + |a_n - a_{n-1}|)r + |a_n - a_{n-1}| - M_2\right\} > 0$$

if $|a_n|r^2 - (|a_n| + |a_n - a_{n-1}|)r + |a_n - a_{n-1}| - M_2 > 0$

i.e, if $r > R_2 = \frac{1}{2|a_n|}\left\{|a_n| + |a_n - a_{n-1}| + \sqrt{(|a_n| - |a_n - a_{n-1}|)^2 + 4|a_n|.M_2}\right\}.$

Thus, no zero of $Q_2(z)$ outside the unit circle lies in $|z| > R_2$.
Since zeros of $P(z)$ are the zeros of $Q_2(z)$, all the zeros of $P(z)$ lie in $|z| \leq R_2$. This proves the theorem. $\square$

If $M_2 > |a_n - a_{n-1}|$, Theorem 3.1 is an improvement of Theorem A as well as Theorem B and Theorem C. Taking $P(z) = 4z^6 + 3z^5 + 2z^4 - z^2 - z - 4$, sharpness of Theorem 3.1 is followed from the following table.

Sl. no.	Theorem	Bounds for zeros of $P(z)$
1	Theorem 3.1	$0.53 \leq |z| \leq 1.69$
2	Theorem A	$|z| \leq 2$
3	Theorem B	$|z| \leq 1.88$
4	Theorem C	$0.0045 \leq |z| \leq 1.98$

An entire function of one complex variable other than a polynomial is known as transcendental entire function. Zeros of any transcendental entire function, if exist, may be finite or countably infinite in $\mathbb{C}$. Considering the polynomial $z^n P(\frac{1}{z})$ and applying Theorem D, it is observed that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \geq a_1 \geq ... \geq a_{n-1} \geq a_n > 0$ then $P(z)$ does not vanish in $|z| < 1$. Analogous to this result, Datta et al. [7] introduced the following result:

***Theorem* E [7].** *Let* $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ *be a transcendental entire function with* $f(0) = a_0 \neq 0$ *such that* $|a_1| \geq |a_2| \geq ...$. *Then* $f(z)$ *does not vanish in* $|z| < \ln(1 + |\frac{a_0}{a_1}|)$.

Apparently, it is observed that Theorem E is not applicable for trascendental entire functions having Taylor's series expansion with some missing terms. In this regard, we develop folloing result.

Theorem 3.2. Let $f(z) = a_0 + a_{n_1} z^{n_1} + ... + a_{n_p} z^{n_p} + a_{n_q} z^{n_q} + ...$ be an entire function with $f(0) = a_0 \neq 0$ and $n_1, n_2, ..., n_p, n_q, ...$ are positive integers such that $1 \leq n_1 < ... < n_p < n_q < ....$ Also let for some positive integer p, $|a_{n_p}| \geq |a_{n_q}| \geq ...$. Then $f(z)$ does not vanish in

$$|z| < \frac{|a_0|}{|a_0| + M}$$

where $M = \max\{|a_{n_1}|, |a_{n_2}|, ..., |a_{n_p}|\}$.

Proof. Let $f_k(z) = a_0 + a_{n_1} z^{n_1} + ... + a_{n_l} z^{n_l} + a_{n_m} z^{n_m} + ... + a_{n_k} z^{n_k}$. Also, let

$$F(z) = z^{n_k} f_k(\frac{1}{z})$$

$$\text{i.e, } F(z) = a_0 z^{n_k} + a_{n_1} z^{n_k - n_1} + ... + a_{n_l} z^{n_k - n_l} + a_{n_m} z^{n_k - n_m} + ... + a_{n_k} \ .$$

Now for $|z| = r(> 1)$, we get that

$$|a_{n_1} z^{n_k - n_1} + a_{n_2} z^{n_k - n_2} + ... + a_{n_l} z^{n_k - n_l} + a_{n_m} z^{n_k - n_m} + ... + a_{n_k}|$$

$$\leq |a_{n_1}| r^{n_k - n_1} + |a_{n_2}| r^{n_k - n_2} + ... + |a_{n_l}| r^{n_k - n_l} + |a_{n_m}| r^{n_k - n_m} + ... + |a_{n_k}|$$

$$\leq M r^{n_k} \left\{ \frac{1}{r^{n_1}} + \frac{1}{r^{n_2}} + ... + \frac{1}{r^{n_l}} + \frac{1}{r^{n_m}} + ... + \frac{1}{r^{n_k}} \right\}$$

$$\text{where } M = \max\{|a_{n_1}|, |a_{n_2}|, ..., |a_{n_p}|\}$$

$$\leq M r^{n_k} \left\{ \frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^l} + \frac{1}{r^m} + ... + \frac{1}{r^k} \right\}$$

$$\text{since } n_i \geq i \ \& \ \frac{1}{r^{n_i}} \leq \frac{1}{r^i} \text{ for } i = 1, \ 2, \ 3, ..., \ k$$

$$\leq M r^{n_k} \sum_{j=1}^{\infty} \frac{1}{r^j}$$

$$= M.r^{n_k} \frac{1}{r-1} \ .$$

Hence for $|z| = r(>1)$, it follows that

$$|F(z)| \geq |a_0| r^{n_k} - M r^{n_k} \frac{1}{r-1} > 0 \text{ if } r > \frac{|a_0| + M}{|a_0|} .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_0| + M}{|a_0|} .$$

Consequently,

$$|f_k(z)| > 0 \text{ if } |z| < \frac{|a_0|}{|a_0| + M} .$$

Thus, no zero of the partial sum $f_k(z)$ is contained in $|z| < \frac{|a_0|}{|a_0|+M}$. Hence by Lemma 2.1, $f(z)$ does not vanish in $|z| < \frac{|a_0|}{|a_0|+M}$.

Thus the theorem is established. □

The following example ensures the validity of Theorem 3.2 .

Example 3.1. *Let* $f(z) = z \sin z^2 + 4z^3 - 3z^2 + z + 4$.

Then the Taylor's series expansion of $f(z)$ *is*

$f(z) = 4 + z - 3z^2 + 5z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \frac{z^{15}}{7!} + \ldots$.

Here, $a_0 = 4$ *&* $M = 5$.

Hence by Theorem 3.2, $f(z)$ *has no zero in* $|z| < 0.44$.

The above can be validated by using well known Rouche's theorem also.

Apparently, Theorem 3.2 can not be used for transcendental entire functions having Taylor's series expansion with coefficients in different monotonicity conditions. In view of this, we develop following result imposing monotonicity conditions on even indexed and odd indexed coefficients separately.

Theorem 3.3. *Let* $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ *be an entire function with* $a_0 \neq 0$. *Also, let for some positive integers* $l,\ m$

$$|a_{2l}| \geq |a_{2l+2}| \geq |a_{2l+4}| \geq \ldots$$

and

$$|a_{2m+1}| \geq |a_{2m+3}| \geq |a_{2m+5}| \geq \ldots .$$

Then $f(z)$ *does not vanish in* $|z| < \frac{|a_0|}{|a_0|+2M}$ *where* $M = \max\limits_{\substack{0 \leq i \leq l \\ 0 \leq j \leq m}} \{|a_{2i}|,\ |a_{2j+1}|\}$.

Proof. Let $f_n(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$

and $F(z) = z^n f_n(\frac{1}{z})$.

Again, let

(3.2) $Q(z) = (z^2 - 1)F(z)$

$$\text{i.e, } Q(z) = (z^2-1)(a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_{2l} z^{n-2l} + a_{2l+1} z^{n-2l-1} + ... + a_{2m} z^{n-2m} + a_{2m+1} z^{n-2m-1} + ... + a_n)$$

$$\text{i.e, } Q(z) = a_0 z^{n+2} + a_1 z^{n+1} + (a_2 - a_0) z^n + (a_3 - a_1) z^{n-1} + ... + (a_{2l+1} - a_{2l-1}) z^{n-2l+1} + (a_{2l+2} - a_{2l}) z^{n-2l} + ... + (a_{2m+1} - a_{2m-1}) z^{n-2m+1} + (a_{2m+2} - a_{2m}) z^{n-2m} + ... - a_{n-1} z - a_n$$

$$\text{i.e, } Q(z) = a_0 z^{n+2} + P(z) \ .$$

Now for $|z| = r(> 1)$, we get that

$$\begin{aligned}
|P(z)| =& |a_1 z^{n+1} + (a_2 - a_0) z^n + (a_3 - a_1) z^{n-1} + ... + (a_{2l+1} - a_{2l-1}) z^{n-2l+1} + \\
& (a_{2l+2} - a_{2l}) z^{n-2l} + ... + (a_{2m+1} - a_{2m-1}) z^{n-2m+1} + (a_{2m+2} - a_{2m}) z^{n-2m} + \\
& ... - a_{n-1} z - a_n| \\
\leq & |a_1| r^{n+1} + (|a_2| + |a_0|) r^n + (|a_3| + |a_1|) r^{n-1} + ... + (|a_{2l+1}| + |a_{2l-1}|) r^{n-2l+1} + \\
& (|a_{2l+2}| + |a_{2l}|) r^{n-2l} + ... + (|a_{2m+1}| + |a_{2m-1}|) r^{n-2m+1} + (|a_{2m+2}| + |a_{2m}|) r^{n-2m} \\
& + ... + |a_{n-1}| r + |a_n| \\
\leq & \ 2M r^{n+2} \left\{ \frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^{n+2}} \right\} \text{ where } M = \max_{\substack{0 \leq i \leq l \\ 0 \leq j \leq m}} \{|a_{2i}|, \ |a_{2j+1}|\} \\
\leq & \ 2M r^{n+2} \sum_{k=1}^{\infty} \frac{1}{r^k} \\
= & \ 2M r^{n+2} \frac{1}{r-1} \ .
\end{aligned}$$

Hence for $|z| = r(> 1)$, it follows from (3.2) that

$$|Q(z)| \geq |a_0| r^{n+2} - 2M r^{n+2} . \frac{1}{r-1} > 0 \text{ if } r > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |Q(z)| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|} \ .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |f_n(\frac{1}{z})| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |f_n(z)| > 0 \text{ if } |z| < \frac{|a_0|}{|a_0| + 2M} \ .$$

Thus, it follows by Lemma 2.1 that

$$|f(z)| > 0 \text{ if } |z| < \frac{|a_0|}{|a_0| + 2M} \ .$$

This completes the proof of the theorem. □

Following example justifies difference between Theorem 3.2 & Theorem 3.3 as well as their sharpness.

Example 3.2. *Let* $f(z) = z^2 \sin 2z + \cos z$*.*
Now the Taylor's series expansion of $f(z)$ *is*

$$f(z) = 1 - \frac{z^2}{2} + 2z^3 + \frac{z^4}{24} - \frac{4z^5}{3} - \ldots \ .$$

Here, it follows that the coefficients of the series of $f(z)$ *do not satisfy the monotonicity conditions of Theorem 3.2 but satisfy the following conditions.*

$$|a_0| \geq |a_2| \geq |a_4| \geq \ldots$$

and

$$|a_3| \geq |a_5| \geq |a_7| \geq \ldots \ .$$

Thus, Theorem 3.2 is not applicable. However, by Theorem 3.3, no zero of $f(z)$ *lies in*

$$|z| < 0.2 \ .$$

Again, let $g(z) = \sin z + \cos z$*. Then its Taylor's series expansion is*

$$g(z) = 1 + z - \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} - \ldots \ .$$

Clearly, monotonicity conditions of both Theorem 3.2 & Theorem 3.3 are satisfied by the coefficients of the series of $g(z)$*.*

It is observed that no zero of $g(z)$ by Theorem 3.2 contain in $|z| < 0.5$ whereas by Theorem 3.3 in $|z| < 0.33$.

Acknowledgement

The authors are very much thankful to the editor and the anonymous referees for their valuable suggestions to improve the paper.

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