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TIGHT EXPONENTIAL BOUNDS FOR HYPERBOLIC TANGENT

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Abstract. In this article, we aim to obtain very tight exponential bounds for the hyperbolic tangent function. Our inequalities refine a double inequality recently proved by Zhang and Chen. In addition, graphical and numerical analysis are carried out, and a number of auxiliary lemmas may be of use on their own.

1. Introduction

The hyperbolic tangent function is the function $\tanh : \mathbb{R} \to (-1, 1)$ defined by $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$. Clearly, it is a continuous, differentiable, and bounded function that can produce negative, positive, and zero outputs. It occurs in many branches of pure and applied mathematics. In particular, in differential equations, it is at the heart of the so-called "tanh method" (see [10] and [11]), and in statistics, it is known to be one of the most important zero-centered activation function (see [9], and the references therein). The tight and tractable bounds of tanh can therefore be useful in the fields of concern, mainly to evaluate mathematical quantities involving it. Even so, very little can be found related to the bounds of this function in the literature. For instance, L. Zhu in [13] proved the following inequalities:

$$\left(\frac{r^2 - x^2}{r^2 + x^2}\right)^{\beta} \leq \frac{\tanh x}{x} \leq \left(\frac{r^2 - x^2}{r^2 + x^2}\right)^{\alpha}, \ x \in (0, r), \tag{1.1}$$

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where $\alpha \leq 0$ and $\beta \geq r^2/6$. Also, for all $x \in (0, \infty)$, the inequalities

$$\frac{1}{1 + 2\log\left(\frac{\sinh x}{x}\right)} < \frac{\tanh x}{x} < \frac{1}{1 + \log\left(\frac{\sinh x}{x}\right)} \tag{1.2}$$

appeared in [4] and [6]. It can be shown that the inequalities in (1.1) are weaker than those in (1.2). On his side, Bhayo et al. [5] obtained a tight algebraic bound for $\tanh x$ as follows:

$$\tanh x < \frac{2x}{\sqrt{4x^2 + 9} - 1}, \quad x > 0. \tag{1.3}$$

Very recently, Zhang and Chen [12] proposed an alternative by proving the following inequalities:

$$\sqrt{1 - \exp\left(-\frac{x^2}{\sqrt{x^2 + 1}}\right)} < \tanh x < \sqrt[3]{1 - \exp\left(-\frac{x^3}{\sqrt{x^3 + 1}}\right)}, \quad x > 0. \tag{1.4}$$

The bounds in (1.3) and (1.4) are tighter than the corresponding bounds in (1.1) and (1.2). It is unnecessary to say that the algebraic bounds are better than transcendental bounds due to their computational efficiency, and hence an upper bound of $\tanh x$ in (1.3) is better. However, in terms of tightness, the inequality (1.4) is the strongest of all the inequalities listed above, except in $(0, \zeta)$, $\zeta \approx 1.557$, where an upper bound of $\tanh x$ in (1.3) is the best.

In this paper, we aim to refine the inequalities in (1.3) and (1.4). The rest of the paper is organized as follows: The main theorems are presented in Section 2. The proofs of the main results are based on auxiliary results involving differentiation, series and integration methods, which are collected in Section 3. These auxiliary results give us simple algebraic bounds for $\tanh x$, which may be of independent interest. Section 4 contains the proofs of the main results. While obtaining these proofs, we also establish tight exponential bounds for the hyperbolic cosine function. As a complementary study, a discussion on the tightness of (1.3) near the point zero is finally given in Section 5. Section 6 concludes the paper.

2. Main results

Our main results are presented as the following theorems:

Theorem 2.1. *For $x > 0$, the inequalities*

$$\sqrt{1-\exp\left\{\frac{(15)^{5/7}}{2}\left[(15)^{2/7}-(7x^2+15)^{2/7}\right]\right\}} < \tanh x$$

$$< \sqrt{1-\exp\left\{\frac{2\sqrt{15}}{7}\left(\sqrt{15}-\sqrt{7x^2+15}\right)\right\}} \tag{2.1}$$

hold.

Theorem 2.2. *The inequalities*

$$\sqrt{1-\exp\left[2\left(1-\sqrt{1+x^2}\right)\right]} < \tanh x < \sqrt{1-\exp\left[3\left(1-\sqrt{1+\frac{2}{3}x^2}\right)\right]} \tag{2.2}$$

are fulfilled for $x > 0$.

We claim that the inequalities in (2.1) and (2.2) are very tight and they are clear refinements of the inequalities in (1.4). In order to support this claim,

- we point out that

$$1-\sqrt{1+x^2} = -\frac{x^2}{1+\sqrt{1+x^2}} \le -\frac{x^2}{2\sqrt{1+x^2}},$$

  implying that the lower bound in (2.2) is uniformly better than the one in (1.4).
- a graphical analysis is performed for the comparison of the involved lower and upper bounds in (1.4), (2.1) and (2.2) in Figures 1 and 2, respectively.

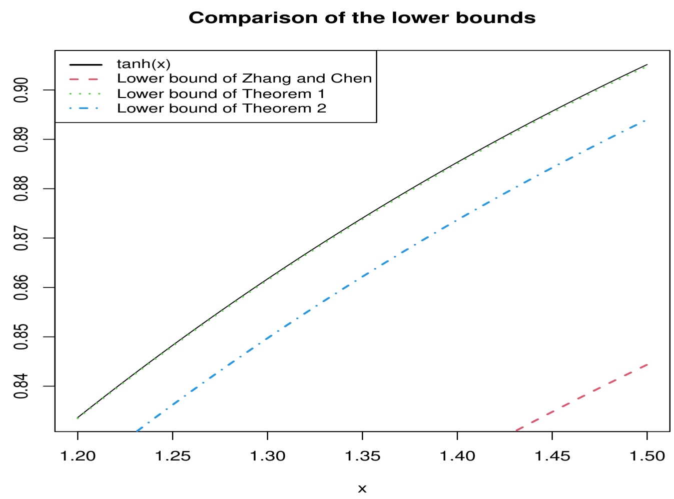

FIGURE 1. Selected plot of $\tanh(x)$ and the presented lower bounds; the one by Zhang and Chen [9] and the two candidates in Theorems 2.1 and 2.2.

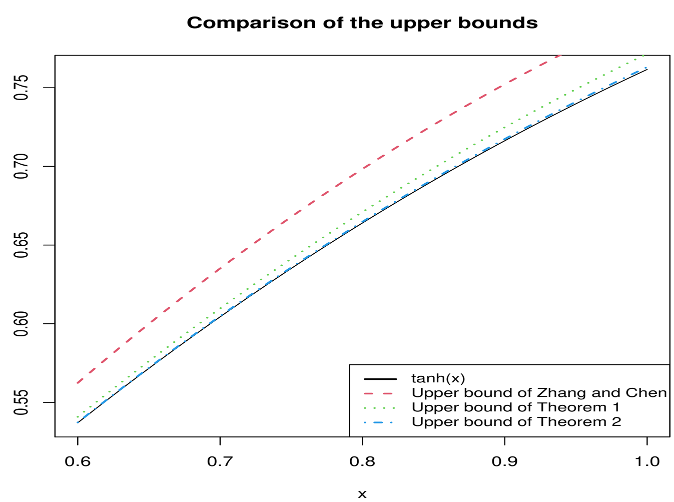

FIGURE 2. Selected plot of $\tanh(x)$ and the presented upper bounds; the one by Zhang and Chen [9] and the two candidates in Theorems 2.1 and 2.2.

For a zoom reason, in Figure 1, the mentioned lower bounds are plotted for $x \in (1.2, 1.5)$. In this setting, we can see that the black and green curves, which correspond to $\tanh x$ and the lower bound of (2.1), are almost confounded. The related upper bounds are displayed in Figure 2 for $x \in (0.6, 1)$, still for a zoom reason. In this figure, the upper bound of (2.2) is the best.

In fact, with the help of any graphing calculator, it can be found that the double inequalities in (2.2) is a refinement of those in (2.1) except in the interval $(0, \gamma)$, with $\gamma \approx 4.279$, where the lower bound of (2.1) dominates the corresponding lower bound of (2.2).

We complete the above remarks by investigating the following global $\mathbb{L}^2$ error:

$$\mathcal{G}(\phi) = \int_0^{10} [\tanh x - \phi(x)]^2 dx,$$

where $\phi(x)$ is a function considered among the lower and upper bounds in (1.4), (2.1) and (2.2). The limit of 10 is taken as arbitrary. The numerical results are collected in Table 1.

TABLE 1. Values of $\mathcal{G}(\phi)$ for the lower and upper bounds in (1.4), (2.1) and (2.2).

	Lower bound	Upper bound
(1.4)	0.005688297	0.001200189
(2.1)	7.632127×10^{-7}	0.0002079122
(2.2)	0.0001644162	1.562102×10^{-5}

Table 1 thus confirms the "ultra tightness" of the lower bound in (2.1), and that of the upper bound in (2.2).

In order to prove the theorems above, we need several Lemmas that are recalled in the next section. Simple algebraic bounds for $\tanh x$ are also established.

3. Lemmas

We begin by presenting some well-known results in Lemmas 3.1 and 3.2.

Lemma 3.1. [1, p.10] *Let $f, g : [m, n] \to \mathbb{R}$ be two continuous functions which are differentiable on (m, n) and $g'(x) \neq 0$ for $x \in (m, n)$. If $f'(x)/g'(x)$ is increasing (or decreasing) on (m, n), then the ratio functions $[f(x) - f(m)]/[g(x) - g(m)]$ and $[f(x)-f(n)]/[g(x)-g(n)]$ are also increasing (or decreasing) on (m, n). If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 3.2. *([8]) Let $A(x)=\sum_{k=0}^{\infty} a_k x^k$ and $B(x)=\sum_{k=0}^{\infty} b_k x^k$ be convergent for $x\in(-R,R)$, where a_k and b_k are real numbers for $k=0,1,2,\cdots$ such that $b_k>0$. If the sequence a_k/b_k is strictly increasing(or decreasing), then the function $A(x)/B(x)$ is also strictly increasing(or decreasing) on $(0,R)$.*

Besides, we state and prove the following important lemmas.

Lemma 3.3. *For $a>0$, define the function*

$$f_a(x)=\frac{\log\left(\frac{\tanh x}{x}\right)}{\log\left(\frac{a}{a+x^2}\right)},\ x>0.$$

Then, the function $f_a(x)$ is strictly decreasing over $(0,\infty)$ if $a\geq 15/7$.

Proof. Let us decompose $f_a(x)$ as

$$f_a(x)=\frac{\log\left(\frac{\tanh x}{x}\right)}{\log\left(\frac{a}{a+x^2}\right)}:=\frac{g(x)}{g_a(x)},\ x>0,$$

where $g(x)=\log(\tanh x/x)$ and $g_a(x)=\log\left[a/(a+x^2)\right]$ with $g(0+)=0=g_a(0)$. Upon differentiation, it comes

$$\frac{g'(x)}{g_a'(x)}=\frac{(a+x^2)}{2}\frac{\tanh x-x\operatorname{sech}^2 x}{x^2\tanh x}=\frac{(a+x^2)}{2}\frac{\sinh(2x)-2x}{x^2\sinh(2x)}:=\frac{1}{2}\frac{L(t)}{M(t)},$$

where $L(t)=(4a+t^2)(\sinh t-t)$, $M(t)=t^2\sinh t$ and $t=2x$. Using the well-established series expansion of $\sinh t$, we can write

$$\begin{aligned}L(t)&=(4a+t^2)\sum_{k=1}^{\infty}\frac{t^{2k+1}}{(2k+1)!}=\sum_{k=1}^{\infty}\frac{4a\cdot t^{2k+1}}{(2k+1)!}+\sum_{k=1}^{\infty}\frac{t^{2k+3}}{(2k+1)!}\\&=-t^3+\sum_{k=1}^{\infty}\frac{4a\cdot t^{2k+1}}{(2k+1)!}+\sum_{k=0}^{\infty}\frac{t^{2k+3}}{(2k+1)!}\end{aligned}$$

and

$$M(t)=\sum_{k=0}^{\infty}\frac{t^{2k+3}}{(2k+1)!}.$$

Therefore,

$$\frac{L(t)}{M(t)}:=1+\frac{A(t)}{B(t)},$$

where

$$A(t) = -t^3 + \sum_{k=1}^{\infty} \frac{4a \cdot t^{2k+1}}{(2k+1)!} = -t^3 + \sum_{k=0}^{\infty} \frac{4a \cdot t^{2k+3}}{(2k+3)!}$$
$$= \left(\frac{2a}{3} - 1\right) t^3 + \sum_{k=1}^{\infty} \frac{4a \cdot t^{2k+3}}{(2k+3)!} := \sum_{k=0}^{\infty} a_k t^{2k+3}$$

and

$$B(t) = M(t) = \sum_{k=0}^{\infty} \frac{t^{2k+3}}{(2k+1)!} = t^3 + \sum_{k=1}^{\infty} \frac{t^{2k+3}}{(2k+1)!} := \sum_{k=0}^{\infty} b_k t^{2k+3}.$$

Here, we have $a_0 = 2a/3 - 1$, $b_0 = 1$, $a_k = 4a/(2k+3)!$, $b_k = 1/(2k+1)!$, $k \geq 1$ with $k \in \mathbb{N}$. Consider

$$\frac{a_k}{b_k} = \frac{2a}{(k+1)(2k+3)}, \ k \geq 1.$$

Clearly $\{a_k/b_k\}_{k=1}^{\infty}$ is a strictly decreasing sequence. By Lemma 3.2, $A(t)/B(t)$ and hence $L(t)/M(t)$ or $g'(x)/g_a'(x)$ will be strictly decreasing if

$$\frac{a_0}{b_0} \geq \frac{a_1}{b_1},$$

which is equivalent to $a \geq 15/7$. Therefore, by Lemma 3.1, $f_a(x)$ is strictly decreasing if $a \geq 15/7$. This completes the proof. □

Corollary 3.1. *Let $x > 0$ and $a \geq 15/7$. Then, the best possible constants such that*

$$\left(\frac{a}{a+x^2}\right)^{\alpha} < \frac{\tanh x}{x} < \left(\frac{a}{a+x^2}\right)^{\beta} \tag{3.1}$$

are $\alpha = a/3$ and $\beta = 1/2$.

Proof. Since the function $f_a(x)$ defined in Lemma 3.3 is strictly decreasing on $(0, \infty)$ for $a \geq 15/7$, we have

$$f_a(0+) > f_a(x) > f_a(\infty), \ x > 0.$$

The limits $f_a(0+) = a/3$ and $f_a(\infty) = 1/2$ give the required inequalities in (3.1). □

By putting $a = 3$ in (3.1), we get the following inequalities:

$$\frac{3}{3+x^2} < \frac{\tanh x}{x} < \left(\frac{3}{3+x^2}\right)^{1/2}, \ x > 0. \tag{3.2}$$

The left inequality of (3.2) already appeared in [2]. Similarly, by putting $a = 15/7$, we obtain

$$\left(\frac{15}{15+7x^2}\right)^{5/7} < \frac{\tanh x}{x} < \left(\frac{15}{15+7x^2}\right)^{1/2}, \quad x > 0. \tag{3.3}$$

The inequalities in (3.3) are the tightest inequalities of kind (3.1). A graphical comparison says that the lower bound in (3.3) is finer than the corresponding lower bound in (1.4) for $x \in (0, \varsigma)$, where $\varsigma \approx 2.4126$. The upper bound in (3.3) is not tight enough, so we obtain better algebraic bounds for $\tanh x$ in the next lemma.

Lemma 3.4. *The best possible constants α and β such that*

$$\frac{1}{\sqrt{1+\alpha x^2}} < \frac{\tanh x}{x} < \frac{1}{\sqrt{1+\beta x^2}}, \quad x > 0 \tag{3.4}$$

are 1 *and* 2/3*, respectively.*

Proof. Let

$$f(x) = \frac{x^2 - \tanh^2 x}{x^2 \tanh^2 x},$$

which can be written as

$$f(x) = \frac{x^2\cosh^2 x - \sinh^2 x}{x^2 \sinh^2 x} = \frac{x^2(1+\cosh(2x)) - (\cosh(2x)-1)}{x^2(\cosh(2x)-1)}$$
$$= \frac{(x^2+1) + (x^2-1)\cosh(2x)}{x^2(\cosh(2x)-1)}.$$

Using a known series expansion of $\cosh x$, we get

$$f(x) = \frac{(x^2+1) + (x^2-1) + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}(x^2-1)x^{2k}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2}}$$
$$= \frac{2x^2 + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2} - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2}}$$
$$= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2} - \sum_{k=1}^{\infty} \frac{2^{2k+2}}{(2k+2)!}x^{2k+2}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2}}$$
$$= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}\left[1 - \frac{2}{(k+1)(2k+1)}\right]x^{2k+2}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2}}$$
$$= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}\left[\frac{2k^2+3k-1}{2k^2+3k+1}\right]x^{2k+2}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}x^{2k+2}} := \frac{\sum_{k=1}^{\infty} a_k x^{2k+2}}{\sum_{k=1}^{\infty} b_k x^{2k+2}}.$$

Here, we have $a_k = (2^{2k}/(2k)!)\,[(2k^2+3k-1)/(2k^2+3k+1)]$ and $b_k = 2^{2k}/(2k)!$. Now

$$\frac{a_k}{b_k} = \frac{2k^2+3k-1}{2k^2+3k+1} := t_k.$$

Suppose that $t_k < t_{k+1}$, i.e.,

$$\frac{2k^2+3k-1}{2k^2+3k+1} < \frac{2k^2+7k+4}{2k^2+7k+6}.$$

Equivalently,

$$(2k^2+3k-1)(2k^2+7k+6) < (2k^2+3k+1)(2k^2+7k+4)$$

or after simplifying $11k-6 < 19k+4$, i.e., $8k+10>0$, which is obviously true for $k \geq 1$. Thus a sequence $\{a_k/b_k\}$ is strictly increasing for $k \geq 1$ and by Lemma 3.2, we conclude that the function $f(x)$ is also strictly increasing for $x>0$. Hence

$$f(0+) < f(x) < f(\infty).$$

Lastly, the limits $f(0+) = 2/3$ and $f(\infty) = 1$ prove the lemma. □

It should be noted that the upper bound in (3.4) is finer than the corresponding upper bound in (1.4) for $x \in (0,\epsilon)$, where $\epsilon \approx 1.2952$.

The main proofs of the article are presented in the section below.

4. Proofs of main results

This section is devoted to the technical proofs of our main results.

4.1. **Proof of theorem 2.1.** The inequalities in (3.3) can be written as

$$t\left(\frac{15}{7t^2+15}\right)^{5/7} < \tanh t < t\left(\frac{15}{7t^2+15}\right)^{1/2},\ t>0.$$

Let $t \in (0,x)$ with $x>0$. Then, an integration gives

$$\int_0^x t\left(\frac{15}{7t^2+15}\right)^{5/7} dt < \int_0^x \tanh t dt < \int_0^x t\left(\frac{15}{7t^2+15}\right)^{1/2} dt,$$

i.e.,

$$\frac{(15)^{5/7}}{14}\int_0^x (7t^2+15)^{-5/7}\cdot 14t dt < \log(\cosh x) < \frac{\sqrt{15}}{14}\int_0^x \frac{1}{\sqrt{7t^2+15}}\cdot 14t dt$$

or

$$\frac{(15)^{5/7}}{4}\left[(7x^2+15)^{2/7}-(15)^{2/7}\right]<\log(\cosh x)<\frac{\sqrt{15}}{7}\left(\sqrt{7x^2+15}-\sqrt{15}\right),$$

i.e.,

$$\exp\left\{\frac{(15)^{5/7}}{4}\left[(7x^2+15)^{2/7}-(15)^{2/7}\right]\right\}<\cosh x$$
$$<\exp\left\{\frac{\sqrt{15}}{7}\left(\sqrt{7x^2+15}-\sqrt{15}\right)\right\}.$$

By first squaring and then taking reciprocals, we have

$$\exp\left\{\frac{(15)^{5/7}}{2}\left[(7x^2+15)^{2/7}-(15)^{2/7}\right]\right\}>\operatorname{sech}^2 x$$
$$>\exp\left\{\frac{2\sqrt{15}}{7}\left(\sqrt{7x^2+15}-\sqrt{15}\right)\right\}.$$

Owing to the relation $\operatorname{sech}^2 x = 1-\tanh^2 x$, we get

$$1-\exp\left\{\frac{(15)^{5/7}}{2}\left[(15)^{2/7}-(7x^2+15)^{2/7}\right]\right\}<\tanh^2 x$$
$$<1-\exp\left\{\frac{2\sqrt{15}}{7}\left(\sqrt{15}-\sqrt{7x^2+15}\right)\right\}.$$

This gives the inequalities in (2.1). □

4.2. **Proof of Theorem 2.2.** We write the inequalities in (3.4) as

$$\frac{t}{\sqrt{1+t^2}}<\tanh t<\frac{t}{\sqrt{1+\frac{2}{3}t^2}},\quad t>0.$$

Let $t\in(0,x)$ with $x>0$. Then, through an integration, it comes

$$\int_0^x \frac{1}{2\sqrt{1+t^2}}2t\,dt<\int_0^x \tanh t\,dt<\frac{3}{4}\int_0^x \frac{\frac{4}{3}t}{\sqrt{1+\frac{2}{3}t^2}}dt,$$

i.e.,

$$\sqrt{1+x^2}-1<\log(\cosh x)<\frac{3}{2}\left(\sqrt{1+\frac{2}{3}x^2}-1\right)$$

or

$$\exp\left(\sqrt{1+x^2}-1\right)<\cosh x<\exp\left[\frac{3}{2}\left(\sqrt{1+\frac{2}{3}x^2}-1\right)\right].$$

By squaring and then taking reciprocals as in the proof of Theorem 2.1, we get the desired inequalities . □

If $r > 0$ then, for $x \in (0, r)$, the inequalities

$$\exp(\lambda x^2) < \cosh x < \exp\left(\frac{x^2}{2}\right), \tag{4.1}$$

where $\lambda = \log(\cosh r)/r^2$ and

$$\left(1 + \frac{x^2}{3}\right)^{3/2} < \cosh x < \left(1 + \frac{x^2}{3}\right)^{\delta}, \tag{4.2}$$

where $\delta = \log(\cosh r)/\log(1 + r^2/3)$ are proved in [3] and [7], respectively.

While giving proofs of Theorems 2.1 and 2.2, we in fact refined the inequalities in (4.1) and (4.2) and obtained better exponential bounds for hyperbolic cosine. For $x \in (0, r)$, with $r \to \infty$, the refined inequalities for $\cosh x$ are given as

$$\begin{aligned} \exp\left\{\frac{(15)^{5/7}}{4}\left[(7x^2+15)^{2/7} - (15)^{2/7}\right]\right\} &< \cosh x \\ &< \exp\left\{\frac{\sqrt{15}}{7}\left(\sqrt{7x^2+15} - 15\right)\right\} \end{aligned} \tag{4.3}$$

and

$$\exp\left(\sqrt{1+x^2} - 1\right) < \cosh x < \exp\left[\frac{3}{2}\left(\sqrt{1 + \frac{2}{3}x^2} - 1\right)\right]. \tag{4.4}$$

5. On the inequality (1.3)

We now complete the previous study by discussing the tightness of the inequality in (1.3) near the point zero. First of all, observe that the inequality in (1.3) can be written as

$$\frac{e^{2x} - 1}{e^{2x} + 1} \leq \frac{2x}{\sqrt{4x^2 + 9} - 1}, \quad x > 0,$$

i.e., as

$$\frac{e^{x} - 1}{e^{x} + 1} \leq \frac{x}{\sqrt{x^2 + 9} - 1}, \quad x > 0. \tag{5.1}$$

Putting $t = e^x > 1$, and observing that the mapping $t \mapsto (t-1)/(t+1)$, $t > 1$ is strictly increasing, it can be simply shown that the inequality in (5.1) is equivalent on the interval $(0,4)$ with

$$e^x \leq \frac{1+\frac{x}{\sqrt{x^2+9}-1}}{1-\frac{x}{\sqrt{x^2+9}-1}} = \frac{g(x)+x}{g(x)-x}, \quad x \in [0,4), \tag{5.2}$$

where $g(x) := \sqrt{x^2+9}-1$, $x \in [0,4)$, Our idea is to find a differentiable function $f : [0,4) \to (0,\infty)$ such that $f(0) = 2$, $f(x) \geq g(x)$ for all $x \in [0,4)$ and

$$e^x \leq \frac{f(x)+x}{f(x)-x}, \quad x \in [0,4).$$

Since

$$\frac{f(x)+x}{f(x)-x} \leq \frac{g(x)+x}{g(x)-x}, \quad x \in [0,4),$$

we immediately get the following extension of (5.2):

$$e^x \leq \frac{f(x)+x}{f(x)-x} \leq \frac{g(x)+x}{g(x)-x}, \quad x \in [0,4). \tag{5.3}$$

The first inequality in (5.3), which will be considered in what follows, is equivalent with

$$x \leq \ln\left(\frac{f(x)+x}{f(x)-x}\right), \quad x \in [0,4).$$

Now let us set

$$F(x) := x - \ln\left(\frac{f(x)+x}{f(x)-x}\right), \quad x \in [0,4).$$

Since $f(0) = 2$, we have $F(0) = 0$. Moreover

$$F'(x) = \frac{f^2(x)+2xf'(x)-x^2-2f(x)}{(f(x)+x)(f(x)-x)}, \quad x \in [0,4),$$

so that $F'(x) \leq 0$, $x \in [0,4)$ if $f^2(x)+2xf'(x)-x^2-2f(x) \leq 0$, $x \in [0,4)$. A detailed analysis of the class $\mathcal{F}$ consisting of differentiable functions $f : [0,4) \to (0,\infty)$ such that $f(0) = 2$, $f(x) \geq g(x)$ for all $x \in [0,4)$, and

$$f^2(x)+2xf'(x)-x^2-2f(x) \leq 0, \quad x \in [0,4) \tag{5.4}$$

is far from being trivial and falls out from the scope of this paper. We only note that the solution $y = f(x) = 2/(1-Cx)$ of the associated Bernoulli differential equation

$$y' - \frac{y}{x} = -\frac{y^2}{2x}$$

presents a good candidate for an element of the class $\mathcal{F}$. In the particular case $C = 1/4$, the function $y = 2/(1-(x/4))$ belongs to $\mathcal{F}$ since

$$\frac{2}{1-(x/4)} \geq \sqrt{x^2+9}-1, \quad x \in [0,4);$$

see https://www.desmos.com/calculator/2g7wpl8fri. Unfortunately, if we disregard the term x^2 in (5.4), then the solution of the associated Ricatti differential equation

$$y' - \frac{y}{x} + \frac{y^2}{2x} = 0,$$

given by $y = 6Cx/(Cx^3-1)$ does not belong to the class $\mathcal{F}$ because $y(0) = 0$ (the complete Ricatti differential equation associated to (5.4) is not solvable in quadratures). We finally note that all the established results are checked and compared at `https://www.desmos.com/calculator`.

6. Conclusion

We have established new bounds for the hyperbolic tangent function, which have the merit of being simple and tight. In particular, it extends the bounds of recent literature. To achieve this aim, we have proved several intermediary results that can be of independent interest, including new bounds for the hyperbolic cosine function. We have performed a graphical and numerical analysis to illustrate the tightness of the bounds. Possible applications of our findings are numerous, including the precise evaluation of special integral involving the hyperbolic tangent function, and the construction of new activation functions that have ordering properties with the hyperbolic tangent function, which is intensively used in this regard.

Declaration of interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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