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SCREEN SEMI SLANT LIGHTLIKE SUBMANIFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we introduce the notion of screen semi slant lightlike submanifold of golden semi-Riemannian manifolds and providing characterization theorem with some non-trivial examples of such submanifolds. We find necessary and sufficient conditions for integrability and totally geodesic foliation of distributions RadTM, D_1 and D_2.

1. Introduction

A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric on it is degenerate i.e, there exists a non zero $\xi \in \Gamma(TM)$ such that $g(\xi, Z) = 0 \; \forall \; Z \in \Gamma(TM)$. The first studied on the geometry of lightlike submanifolds are given by Duggal-Bejancu [4]. The growing importance of lightlike in general relativity, particularly in black hole theory, motivated the study of lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures. The geometry of invariant, screen real, slant and screen-slant lightlike submanifolds have been studied in [6]. Many authors have studied on lightlike submanifolds in various spaces ([6], [14], [15]). In [13], S.S.Shukla and Akhilesh Yadav introduced the notion of screen semi slant lightlike submanifold of indefinite Sasakian manifolds. In [13], they also found some equivalent conditions for integrability of distributions and investigate the geometry of the leaves of distributions.

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Golden proportion ψ is the real positive root of the equation $x^2 - x - 1 = 0$ (thus $\psi = \frac{1+\sqrt{5}}{2} \approx 1.618...$). Inspired by the Golden proportion, Crasmareanu and Hretcanu defined golden structure $\tilde{P}$ which is a tensor field satisfying $\tilde{P}^2 - \tilde{P} - I = 0$ on $\overline{M}$ [3]. Golden structure was inspired by the Golden proportion, which was described by Kepler (1571-1630). A Riemannian manifold $\overline{M}$ with a golden structure $\tilde{P}$ is called a golden Riemannian manifold ([8]). In [8], authors studied invariant submanifolds of a golden Riemannian manifold. Submanifolds of golden manifolds in semi-Riemannian geometry were studied by Poyraz and Yasar [11]. In [11], they proved that there is no radical anti-invariant lightlike hypersurface of a golden semi-Riemannian manifold and also studied screen semi-invariant and screen conformal screen semi-invariant lightlike hypersurfaces of a golden semi-Riemannian manifold.

In [12], authors proved that there is no radical anti-invariant lightlike submanifold of a golden semi-Riemannian manifolds. In [1], author introduced the notion of screen pseudo slant of a golden semi-Riemannian manifold and also found some equivalent conditions for integrability of distributions and investigate the geometry of the leaves of distributions. In [10], N. Onen Poyraz introduced the notion of golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds and they found equivalent conditions for integrability and totally geodesic foliation of distributions. The purpose of this paper is to study screen semi slant lightlike submanifold of golden semi-Riemannian manifolds. The paper is arranged as follows. In section 2, some definitions and basic results about lightlike submanifolds and golden semi-Riemannian manifolds are given. In section 3, we study screen semi slant lightlike submanifold of a golden semi-Riemannian manifold, giving some non-trivial examples and obtain necessary and sufficient conditions for integrability of distributions. In section 4, we also study mixed geodesic screen semi slant lightlike submanifold and investigate the geometry of the leaves of distributions.

2. Preliminaries

Let $\overline{M}$ be a C^∞-differentiable manifold. If a (1, 1) type tensor field $\tilde{P}$ on $\overline{M}$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.1}$$

then $\tilde{P}$ is called a golden structure on $\overline{M}$, where I is the identity transformation [7]. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\overline{M}$. If $\tilde{P}$ satisfies the following equation

$$\overline{g}(\tilde{P}U, W) = \overline{g}(U, \tilde{P}W), \tag{2.2}$$

then $(\overline{M}, \overline{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [9], also, if $\tilde{P}$ is integrable, we have [3]

$$\overline{\nabla}_U \tilde{P}W = \tilde{P}\overline{\nabla}_U W. \tag{2.3}$$

Now, from (2.2), we get

$$\overline{g}(\tilde{P}U, \tilde{P}W) = \overline{g}(\tilde{P}U, W) + \overline{g}(U, W), \tag{2.4}$$

for all $U, W \in \Gamma(T\overline{M})$.

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [4] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM), \tag{2.5}$$

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.6}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.7}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.8}$$

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.9}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.10}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X,Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.9) and (2.10), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.11}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.12}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.13}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$ called the lightlike connection and screen transversal connection on M, respectively.

Also by using (2.9), (2.11)-(2.13) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.14}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.15}$$

Now denote the projection of TM on $S(TM)$ by S. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$\nabla_X SY = \nabla^*_X SY + h^*(X, SY), \tag{2.16}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi. \tag{2.17}$$

By using above equations, we obtain

$$\overline{g}(h^l(X,SY),\xi) = g(A^*_\xi X, SY), \tag{2.18}$$

$$\overline{g}(h^*(X,SY),N) = g(A_N X, SY), \tag{2.19}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.20}$$

It is important to note that in general ∇ is not a metric connection on M. Since $\overline{\nabla}$ is metric connection, by using (2.11), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y), \tag{2.21}$$

for all $X, Y, Z \in \Gamma(TM)$.

Definition 2.1. [2] An equivalence relation on an n-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ in which the equivalence classes are connected, immersed submanifolds (called the leaves of the foliation) of a common dimension k, $0 < k \leq n$ is called a foliation on $\overline{M}$. If each leaf of a foliation F on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is totally geodesic submanifold of $\overline{M}$, we say that F is a totally geodesic foliation.

Definition 2.2. [5] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for any $X, Y \in \Gamma(TM)$,

$$h(X,Y) = g(X,Y)H. \tag{2.22}$$

In case $H = 0$, M is called totally geodesic. Using (2.11) and (2.22), we conclude that M is totally umbilical if and only if there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

$$h^l(X,Y) = g(X,Y)H^l, \ h^s(X,Y) = g(X,Y)H^s \ \textit{and} \ D^l(X,W) = 0, \tag{2.23}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$.

3. Screen Semi Slant Lightlike Submanifolds

In this section, we introduce the notion of screen semi slant lightlike submanifold of a golden semi-Rimannian manifolds. At first, we state the following Lemma for later use:

Lemma 3.1. Let M be a $2q$-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.

Proof. The proof of above Lemma is similar to Lemma 6.8.1 of [6], so we omit it. □

Definition 3.1. Let M be a $2q$- lightlike submanifolds of golden semi-Riemannian manifold $\overline{M}$ of index $2q < dim(M)$. Then we say that M is a screen semi slant lightlike submanifolds of $\overline{M}$ if the following conditions are satisfied:
(i) $RadTM$ is invariant with respect to $\tilde{P}$, i.e. $\tilde{P}(RadTM) = RadTM$,
(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2$,
(iii) the distribution D_1 is an invariant distribution, i.e. $\tilde{P}D_1 = D_1$,
(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\tilde{P}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$. This constant angle θ is called the slant angle of distribution D_2. A screen semi slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}, D_2 \neq \{0\}$ *and* $\theta \neq \frac{\pi}{2}$.
From the above definition, we have the following decomposition.

$$TM = RadTM \oplus_{orth} D_1 \oplus_{orth} D_2. \tag{3.1}$$

Now, for any vector field X tangent to M, we put

$$\tilde{P}X = PX + FX, \tag{3.2}$$

where PX and FX are tangential and transversal parts of $\tilde{P}X$, respectively. We denote the projections on $RadTM$, D_1 and D_2 in TM by P_1, P_2 and P_3 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ by Q_1 and

Q_2 respectively. Then for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X. \tag{3.3}$$

Now applying $\tilde{P}$ to (3.3), we have

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X, \tag{3.4}$$

which gives

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + PP_3X + FP_3X, \tag{3.5}$$

where PP_3X (resp. FP_3X) denotes the tangential (resp. transversal) component of $\tilde{P}P_3X$. Thus we get $\tilde{P}P_1X \in \Gamma(RadTM)$, $\tilde{P}P_2X \in \Gamma D_1$ $PP_3X \in \Gamma D_2$ and $FP_3X \in \Gamma S(TM^{\perp})$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.6}$$

Applying $\tilde{P}$ to (3.6), we obtain

$$\tilde{P}W = \tilde{P}Q_1W + \tilde{P}Q_2W, \tag{3.7}$$

which gives

$$\tilde{P}W = \tilde{P}Q_1W + BQ_2W + CQ_2W, \tag{3.8}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\tilde{P}Q_2W$. Thus we get $\tilde{P}Q_1W \in \Gamma(ltr(TM))$, $BQ_2W \in \Gamma D_2$ and $CQ_2W \in \Gamma S(TM^{\perp})$.

Example 3.1. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.
Taking, $\tilde{P}(\partial x^1, ..., \partial x^{12}) = (\psi\partial x^1, \psi\partial x^2, \psi\partial x^3, \psi\partial x^4, (1-\psi)\partial x^5, \psi\partial x^6, \psi\partial x^7, (1-\psi)\partial x^8, (1-\psi)\partial x^9, (1-\psi)\partial x^{10}, \psi\partial x^{11}, \psi\partial x^{12})$, where $\psi = \frac{1+\sqrt{5}}{2}$ *and* $(1-\psi) = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Thus, $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = u^1 \cosh\alpha$, $x^2 = u^2 \cosh\alpha$, $x^3 = u^1 \sinh\alpha - u^2$, $x^4 = u^2 \sinh\alpha + u^1$, $x^5 = x^8 = u^3$, $x^6 = x^7 = u^4$, $x^9 = \psi u^5$, $x^{10} = \psi u^6$, $x^{11} = (1-\psi)u^5$, $x^{12} = (1-\psi)u^6$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = \cosh\alpha\partial x^1 + \sinh\alpha\partial x^3 + \partial x^4$, $Z_2 = \cosh\alpha\partial x^2 - \partial x^3 + \sinh\alpha\partial x^4$, $Z_3 = \partial x^5 + \partial x^8$, $Z_4 = \partial x^6 + \partial x^7$, $Z_5 = \psi\partial x^9 + (1-\psi)\partial x^{11}$ and $Z_6 = \psi\partial x^{10} + (1-\psi)\partial x^{12}$.
Hence $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, Z_5, Z_6\}$.
Now $ltr(TM) = span\{N_1, N_2\}$, where, $N_1 = \frac{1}{2}(-\cosh\alpha\partial x^1 - \sinh\alpha\partial x^3 + \partial x^4)$, $N_2 = \frac{1}{2}(-\cosh\alpha\partial x^2 - \partial x^3 - \sinh\alpha\partial x^4)$ and $S(TM^\perp) = span\{W_1, W_2, W_3, W_4\}$, where $W_1 = \partial x^5 - \partial x^8$, $W_2 = \partial x^6 - \partial x^7$, $W_3 = (1-\psi)\partial x^9 - \psi\partial x^{11}$ and $W_4 = (1-\psi)\partial x^{10} - \psi\partial x^{12}$.
It follows that $\tilde{P}Z_1 = \psi Z_1$ and $\tilde{P}Z_2 = \psi Z_2$, which implies that $RadTM$ is invariant i.e., $\tilde{P}RadTM = RadTM$. On the otherhand, distribution $D_1 = span\{Z_3, Z_4\}$ such that $\tilde{P}Z_3 = (1-\psi)Z_3$ and $\tilde{P}Z_4 = \psi Z_4$, which implies that D_1 is invariant with respect to $\tilde{P}$, i.e. $\tilde{P}D_1 = D_1$ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle $\theta = \arccos(\frac{1}{\sqrt{6}})$. Hence M is a screen semi slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Example 3.2. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.
Taking, $\tilde{P}(\partial x^1, ..., \partial x^{12}) = (\psi\partial x^1, (1-\psi)\partial x^2, \psi\partial x^3, \psi\partial x^4, \psi\partial x^5, (1-\psi)\partial x^6, (1-\psi)\partial x^7, \psi\partial x^8, (1-\psi)\partial x^9, (1-\psi)\partial x^{10}, (1-\psi)\partial x^{11}, \psi\partial x^{12})$, where $\psi = \frac{1+\sqrt{5}}{2}$ *and* $(1-\psi) = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Thus $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = x^5 = u^1$, $x^2 = x^6 = u^2$, $x^3 = x^4 = u^3$, $x^7 = x^9 = u^4$, $x^8 = \psi u^5$, $x^{10} = (1-\psi)u^6$, $x^{11} = (1-\psi)u^5$, $x^{12} = \psi u^6$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where
$Z_1 = \partial x^1 + \partial x^5$, $Z_2 = \partial x^2 + \partial x^6$, $Z_3 = \partial x^3 + \partial x^4$, $Z_4 = \partial x^7 + \partial x^9$, $Z_5 = \psi\partial x^8 + (1-\psi)\partial x^{11}$, $Z_6 = (1-\psi)\partial x^{10} + \psi\partial x^{12}$.
Hence $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, Z_5, Z_6\}$.
Now $ltr(TM) = span\{N_1, N_2\}$, where $N_1 = -\frac{1}{2}(\partial x^1 - \partial x^5)$, $N_2 = \frac{1}{2}(\partial x^2 - \partial x^6)$ and $S(TM^\perp) = span\{W_1, W_2, W_3, W_4\}$, where $W_1 = \partial x^3 - \partial x^4$, $W_2 = \partial x^7 - \partial x^9$, $W_3 = (1-\psi)\partial x^8 - \psi\partial x^{11}$ and $W_4 = \psi\partial x^{10} - (1-\psi)\partial x^{12}$.
It follows that $\tilde{P}Z_1 = \psi Z_1$ and $\tilde{P}Z_2 = (1-\psi)Z_2$, which implies that $RadTM$ is invariant i.e., $\tilde{P}RadTM = RadTM$. On the otherhand, distribution $D_1 = span\{Z_3, Z_4\}$ such that $\tilde{P}Z_3 = \psi Z_3$ and $\tilde{P}Z_4 = (1-\psi)Z_4$, which implies $\tilde{P}D_1 = D_1$ i.e., D_1 is

invariant with respect to $\tilde{P}$ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle $\theta = \arccos(\frac{4}{\sqrt{21}})$. Hence M is a screen semi slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Theorem 3.1. Let M be a $2q$-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then M is a screen semi slant lightlike submanifold of $\overline{M}$ if and only if

(i) $ltr(TM)$ and D_1 are invariant with respect to $\tilde{P}$,

(ii) there exists a constant $\lambda \in [0, 1)$ such that $P^2X = \lambda(\tilde{P}X + X)$,

for any $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distribution on M such that $S(TM) = D_1 \oplus_{orth} D_2$. Moreover, in this case $\lambda = \cos^2\theta$ and θ is slant agle of D_2.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then distributions D_1 and $RadTM$ are invariant with respect to $\tilde{P}$. Now for any $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(S(TM))$, using (2.2) and (3.5), we obtain $\overline{g}(\tilde{P}N, X) = \overline{g}(N, \tilde{P}X) = \overline{g}(N, \tilde{P}P_2X + PP_3X + FP_3X) = 0$. Thus $\tilde{P}N$ does not belong to $\Gamma(S(TM))$. For any $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, from (2.2) and (3.8) , we have $\overline{g}(\tilde{P}N, W) = \overline{g}(N, \tilde{P}W) = \overline{g}(N, BQ_2W + CQ_2W) = 0$. Hence, we conclude that $\tilde{P}N$ does not belong to $\Gamma(S(TM^{\perp}))$. Now suppose that $\tilde{P}N \in \Gamma(RadTM)$. Then $\tilde{P}(\tilde{P}N) = \tilde{P}^2N = \tilde{P}N + N \in \Gamma(Rad(TM) + ltr(TM))$, which contradicts that RadTM is invariant. Thus $ltr(TM)$ is invariant with respect to $\tilde{P}$.

Now for any $X \in \Gamma D_2$ we have $|PX| = |\tilde{P}X| \cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\tilde{P}X|}. \tag{3.9}$$

In view of (3.9), we get $\cos^2\theta = \frac{|PX|^2}{|\tilde{P}X|^2} = \frac{g(PX,PX)}{g(\tilde{P}X,\tilde{P}X)} = \frac{g(X,P^2X)}{g(X,\tilde{P}^2X)}$, which gives

$$g(X, P^2X) = \cos^2\theta\, g(X, \tilde{P}^2X). \tag{3.10}$$

Since M is a screen semi slant lightlike submanifold, $\cos^2\theta = \lambda(constant) \in [0, 1)$ and therefore from (3.10), we get $g(X, P^2X) = \lambda g(X, \tilde{P}^2X) = g(X, \lambda\tilde{P}^2X)$, which implies

$$g(X, (P^2 - \lambda\tilde{P}^2)X) = 0. \tag{3.11}$$

Since $(P^2-\lambda\tilde{P^2})X \in \Gamma D_2$ and the induced metric $g = g|_{D_2\times D_2}$ is non-degenerate (positive definite), from (3.11), we have $(P^2-\lambda\tilde{P^2})X = 0$, which implies

$$P^2X = \lambda\tilde{P^2}X = \lambda(\tilde{P}X + X), \tag{3.12}$$

for all $X \in \Gamma D_2$. This proves (ii).

Conversely suppose that conditions (i) and (ii) are satisfied. We can show that $RadTM$ is invariant in similar way that $ltr(TM)$ is invariant.

Now $\cos\theta = \frac{g(\tilde{P}X,PX)}{|\tilde{P}X||PX|} = \frac{g(X,\tilde{P}PX)}{|\tilde{P}X||PX|} = \frac{g(X,P^2X)}{|\tilde{P}X||PX|} = \lambda\frac{g(X,\tilde{P^2}X)}{|\tilde{P}X||PX|} = \lambda\frac{g(\tilde{P}X,\tilde{P}X)}{|\tilde{P}X||PX|}$.

From above equation, we get

$$\cos\theta = \lambda\frac{|\tilde{P}X|}{|PX|}. \tag{3.13}$$

Therefore (3.9) and (3.13) give $\cos^2\theta = \lambda(constant)$. Hence M is a screen semi slant lightlike submanifold. □

Corollary 3.1. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma D_2$, we have

(i) $g(PX, PY) = \cos^2\theta\ (g(X, Y) + g(X, PY))$,

(ii) $g(FX, FY) = \sin^2\theta\ (g(X, Y) + g(PX, Y))$.

Proof. From (2.2), (3.2) and (3.12), we obtain

$g(PX, PY) = g(X, \lambda(\tilde{P}Y + Y)) = \cos^2\theta(g(X, Y) + g(X, PY))$.

Moreover, from (2.2), (3.2) and (i) part of corollary (3.1), we get

$g(FX, FY) = g(X, Y) + g(PX, Y) - g(PX, PY) = \sin^2\theta(g(X, Y) + g(PX, Y))$.

Hence, the proof is complete.

□

Theorem 3.2. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the following statements are equivalent:

(i) $Rad(TM)$ is integrable,

(ii) $\overline{g}(\nabla_X\tilde{P}Y - \nabla_Y\tilde{P}X, \tilde{P}W) = \overline{g}(\nabla_X\tilde{P}Y - \nabla_Y\tilde{P}X, W)$ and

$\overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, PZ)+\overline{g}(h^s(X, \tilde{P}Y)-h^s(Y, \tilde{P}X), FZ) = \overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, Z)$,

(iii) $\overline{g}(h^l(Y, \tilde{P}W) - h^l(Y, W), \tilde{P}X) = \overline{g}(h^l(X, \tilde{P}W) - h^l(X, W), \tilde{P}Y)$ and

$\overline{g}(h^l(Y, PZ)-h^l(Y, Z)+D^l(Y, FZ), \tilde{P}X) = \overline{g}(h^l(X, PZ)-h^l(X, Z)+D^l(X, FZ), \tilde{P}Y)$,

for all $X, Y \in \Gamma Rad(TM)$, $W \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the distribution $Rad(TM)$ is integrable if and only if

$$\overline{g}([X,Y],W) = \overline{g}([X,Y],Z) = 0, \tag{3.14}$$

for all $X, Y \in \Gamma(Rad(TM))$, $W \in \Gamma(D_1)$, and $Z \in \Gamma(D_2)$.
(i) $\Rightarrow$ (ii) From (2.4), (2.3), (2.11) and (3.14), we get

$$\overline{g}(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X, \tilde{P}W) = \overline{g}(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X, W). \tag{3.15}$$

From (2.3), (2.4), (2.11), (3.5) and (3.14), we obtain

$$\overline{g}(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X, PZ) + \overline{g}(h^s(X,\tilde{P}Y) - h^s(Y,\tilde{P}X), FZ) = \overline{g}(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X, Z). \tag{3.16}$$

(ii) $\Rightarrow$ (iii) using (2.11) in (3.15) and taking $\overline{\nabla}$ is metric connection, we get

$$-\overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}W) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y \tilde{P}W) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X W) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y W), \tag{3.17}$$

for all $X, Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(D_1)$. Using (2.11) in (3.17), we obtain

$$\overline{g}(h^l(Y,\tilde{P}W) - h^l(Y,W), \tilde{P}X) = \overline{g}(h^l(X,\tilde{P}W) - h^l(X,W), \tilde{P}Y). \tag{3.18}$$

By using (2.11), (3.5) in (3.16) and taking $\overline{\nabla}$ is metric connection, we obtain

$$-\overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y \tilde{P}Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X Z) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y Z), \tag{3.19}$$

for all $X, Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(D_2)$. From (2.11), (2.13), (3.5) and (3.19), we get

$$\overline{g}(h^l(Y,PZ) - h^l(Y,Z) + D^l(Y,FZ), \tilde{P}X) = \overline{g}(h^l(X,PZ) - h^l(X,Z) + D^l(X,FZ), \tilde{P}Y). \tag{3.20}$$

(iii) $\Rightarrow$ (i) using (2.11) in (3.18), we get (3.17) and from (2.11), (2.13) and (3.5) in (3.20), we get (3.19).
Now taking $\overline{\nabla}$ is metric connection, using (2.3) and (2.4), in (3.17) and (3.19), respectively we obtain (3.14). Hence the proof is completed. □

Theorem 3.3. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the following statements are equivalent:

(i) D_1 is integrable,

(ii) $\overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X), FZ) = \overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, Z)$ and $\overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, \tilde{P}N) = \overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, N)$,

(iii) $\overline{g}(\nabla_Y PZ-\nabla_Y Z-A_{FZ}Y, \tilde{P}X) = \overline{g}(\nabla_X PZ-\nabla_X Z-A_{FZ}X, \tilde{P}Y)$ and $\overline{g}(A_{\tilde{P}N}Y-A_N Y, \tilde{P}X) = \overline{g}(A_{\tilde{P}N}X-A_N X, \tilde{P}Y)$,

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the distribution D_1 is integrable if and only if

$$\overline{g}([X,Y],Z) = \overline{g}([X,Y],N) = 0, \tag{3.21}$$

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, and $N \in \Gamma(ltr(TM))$.

(i) ⇒ (ii) From (2.4), (2.3), (2.11), (3.5) and (3.21), we get

$$\overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X), FZ) = \overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, Z). \tag{3.22}$$

From (2.3), (2.4) and (2.11), we obtain

$$\overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, \tilde{P}N) = \overline{g}(\nabla_X\tilde{P}Y-\nabla_Y\tilde{P}X, N). \tag{3.23}$$

(ii) ⇒ (iii) using (2.11) and (3.5) in (3.22) and taking $\overline{\nabla}$ is metric connection, we get

$$-\overline{g}(\tilde{P}Y, \overline{\nabla}_X\tilde{P}Z)+\overline{g}(\tilde{P}X, \overline{\nabla}_Y\tilde{P}Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X Z)+\overline{g}(\tilde{P}X, \overline{\nabla}_Y Z), \tag{3.24}$$

for all $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. Using (2.11), (2.13), (3.5) in (3.24), we obtain

$$\overline{g}(\nabla_Y PZ-\nabla_Y Z-A_{FZ}Y, \tilde{P}X) = \overline{g}(\nabla_X PZ-\nabla_X Z-A_{FZ}X, \tilde{P}Y). \tag{3.25}$$

By using (2.11) in (3.23) and taking $\overline{\nabla}$ is metric connection, we obtain

$$-\overline{g}(\tilde{P}Y, \overline{\nabla}_X\tilde{P}N)+\overline{g}(\tilde{P}X, \overline{\nabla}_Y\tilde{P}N) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X N)+\overline{g}(\tilde{P}X, \overline{\nabla}_Y N), \tag{3.26}$$

for all $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$. Using (2.12) in (3.26), we get

$$\overline{g}(A_{\tilde{P}N}Y-A_N Y, \tilde{P}X) = \overline{g}(A_{\tilde{P}N}X-A_N X, \tilde{P}Y). \tag{3.27}$$

(iii) ⇒ (i) using (2.11), (2.13) and (3.5) in (3.25), we get (3.24) and from (2.12) in (3.27), we get (3.26).

Now taking $\overline{\nabla}$ is metric connection, using (2.3) and (2.4), in (3.24) and (3.26), respectively we obtain (3.21). Hence the proof is completed.

□

Theorem 3.4. Let M be a screen semi slant lightlike submanifolds of a golden semi-Riemannian manifold $\overline{M}$. Then the following statements are equivalent:

(i) D_2 is integrable,

(ii) $\overline{g}(\nabla_X PY - \nabla_Y PX - A_{FY}X + A_{FX}Y, \tilde{P}Z) = \overline{g}(\nabla_X PY - \nabla_Y PX - A_{FY}X + A_{FX}Y, Z)$ and $\overline{g}(\nabla_X PY - \nabla_Y PX - A_{FY}X + A_{FX}Y, \tilde{P}N) = \overline{g}(\nabla_X PY - \nabla_Y PX - A_{FY}X + A_{FX}Y, N)$,

(iii) $\overline{g}(\nabla_Y \tilde{P}Z - \nabla_Y Z, PX) + \overline{g}(\nabla_X Z - \nabla_X \tilde{P}Z, PY) = \overline{g}(h^s(Y,Z) - h^s(Y,\tilde{P}Z), FX) + \overline{g}(h^s(X,\tilde{P}Z) - h^s(X,Z), FY)$ and $\overline{g}(A_N Y - A_{\tilde{P}N}Y, PX) + \overline{g}(A_{\tilde{P}N}X - A_N X, PY) = \overline{g}(D^s(Y,N) - D^s(Y,\tilde{P}N), FX) + \overline{g}(D^S(X,\tilde{P}N) - D^s(X,N), FY)$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the distribution D_2 is integrable if and only if

$$\overline{g}([X,Y],Z) = \overline{g}([X,Y],N) = 0, \tag{3.28}$$

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.
(i) $\Rightarrow$ (ii) From (2.3), (2.4), (2.11), (2.13), (3.5) and (3.28), we get

$$\begin{aligned}\overline{g}(\nabla_X PY - \nabla_Y PX - A_{FY}X + A_{FX}Y, \tilde{P}Z) =& \overline{g}(\nabla_X PY - \nabla_Y PX \\ &- A_{FY}X + A_{FX}Y, Z).\end{aligned} \tag{3.29}$$

From (2.3), (2.4), (2.11), (2.13), (3.5) and (3.28), we obtain

$$\begin{aligned}\overline{g}(\nabla_X PY - \nabla_Y PX - A_{FY}X + A_{FX}Y, \tilde{P}N) =& \overline{g}(\nabla_X PY - \nabla_Y PX \\ &- A_{FY}X + A_{FX}Y, N).\end{aligned} \tag{3.30}$$

(ii) $\Rightarrow$ (iii) using (2.11), (2.13) and (3.5) in (3.29) and taking $\overline{\nabla}$ is metric connection, we get

$$-\overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y \tilde{P}Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X Z) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y Z), \tag{3.31}$$

for all $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$. Using (2.11), (3.5) in (3.31), we obtain

(3.32)
$$\overline{g}(\nabla_Y \tilde{P}Z - \nabla_Y Z, PX) + \overline{g}(\nabla_X Z - \nabla_X \tilde{P}Z, PY) = \overline{g}(h^s(Y, Z) - h^s(Y, \tilde{P}Z), FX) + \overline{g}(h^s(X, \tilde{P}Z) - h^s(X, Z), FY).$$

By using (2.11) and (3.5) in (3.30) and taking $\overline{\nabla}$ is metric connection, we obtain

$$-\overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}N) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y \tilde{P}N) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X N) + \overline{g}(\tilde{P}X, \overline{\nabla}_Y N), \quad (3.33)$$

for all $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$. From (2.12) and (3.5) in (3.33), we get

$$\overline{g}(A_N Y - A_{\tilde{P}N} Y, PX) + \overline{g}(A_{\tilde{P}N} X - A_N X, PY) = \overline{g}(D^s(Y, N) - D^s(Y, \tilde{P}N), FX) + \overline{g}(D^S(X, \tilde{P}N) - D^s(X, N), FY). \quad (3.34)$$

(iii) $\Rightarrow$ (i) using (2.11) and (3.5) in (3.32), we get (3.31) and from (2.12) and (3.5) in (3.34), we get (3.33).

Now taking $\overline{\nabla}$ is metric connection, using (2.3) and (2.4), in (3.31) and (3.33), respectively we obtain (3.28). Hence the proof is completed. □

Theorem 3.5. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if

(i) $\tilde{P}P_2 \nabla_X Y = 0$,

(ii) $PP_3 \nabla_X Y + BQ_2 h^s(X, Y) = 0$,

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ [4]. From (2.2), (2.3), (3.5) and (3.8), for any $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$, we have $\overline{\nabla}_X \tilde{P}Y = \tilde{P}P_1 \nabla_X Y + \tilde{P}P_2 \nabla_X Y + PP_3 \nabla_X Y + FP_3 \nabla_X Y + \tilde{P}h^l(X, Y) + BQ_2 h^s(X, Y) + CQ_2 h^s(X, Y)$. On comparing tangential components of both sides of above equation, we obtain $\nabla_X \tilde{P}Y = \tilde{P}P_1 \nabla_X Y + \tilde{P}P_2 \nabla_X Y + PP_3 \nabla_X Y + BQ_2 h^s(X, Y)$, which completes the proof. □

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a screen semi slant lightlike submanifolds of a golden semi-Riemannian manifold to be totally geodesic.

Definition 4.1. [6] A screen semi slant lightlike submanifold M of a golden semi-Riemannian manifold $\overline{M}$ is said to be a mixed geodesic if its second fundamental form h satisfies $h(X,Y) = 0$, for all $X \in \Gamma D_1$ and $Y \in \Gamma D_2$. Thus M is mixed geodesic screen semi slant lightlike submanifolds if $h^l(X,Y) = 0$ and $h^s(X,Y) = 0$, for all $X \in \Gamma D_1$ and $Y \in \Gamma D_2$.

Theorem 4.1. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then $RadTM$ defines a totally geodesic foliation if and only if $\overline{g}(h^l(X,\tilde{P}P_2Z)+h^l(X,PZ)+D^l(X,FZ),\tilde{P}Y) = \overline{g}(h^l(X,\tilde{P}P_2Z)+h^l(X,PZ)+D^l(X,FZ),Y)$, for all $X,Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution $RadTM$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(RadTM)$, for all $X,Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is a metric connection, using (2.3), (2.4),(2.11) and (3.5), for any $X,Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}P_2Z + \overline{\nabla}_X PZ + \overline{\nabla}_X FZ) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}P_2Z + \overline{\nabla}_X PZ + \overline{\nabla}_X FZ)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, h^l(X,\tilde{P}P_2Z)+h^l(X,PZ)+D^l(X,FZ)+\overline{g}(Y, h^l(X,\tilde{P}P_2Z)+h^l(X,PZ)+D^l(X,FZ))$. Thus, the theorem is completed.

□

Theorem 4.2. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if

(i) $\overline{g}(\nabla_X PZ - A_{FZ}X, \tilde{P}Y) = \overline{g}(\nabla_X PZ - A_{FZ}X, Y)$,

(ii) $\overline{g}(A_{\tilde{P}N}X, \tilde{P}Y) = \overline{g}(A_{\tilde{P}N}X, Y)$,

for all $X,Y \in \Gamma D_1$, $Z \in \Gamma D_2$ and $N \in \Gamma ltr(TM)$.

Proof. Let M be a screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma D_1$, for all $X, Y \in \Gamma D_1$. Since $\overline{\nabla}$ is metric connection, from (2.3), (2.4), (2.11) and (3.5), for any $X, Y \in \Gamma D_1$ and$Z \in \Gamma D_2$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X PZ + \overline{\nabla}_X FZ) + \overline{g}(Y, \overline{\nabla}_X PZ + \overline{\nabla}_X FZ)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \nabla_X PZ - A_{FZ}X) + \overline{g}(Y, \nabla_X PZ - A_{FZ}X)$. Now, from (2.3), (2.4), and (2.11), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}N) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}N)$, which implies $\overline{g}(\nabla_X Y, N) = \overline{g}(\tilde{P}Y, A_{\tilde{P}N}X) - \overline{g}(Y, A_{\tilde{P}N}X)$. This proves the theorem.

□

Theorem 4.3. Let M be a screen semi slant lightlike submanifolds of a golden semi-Riemannian manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if

(i) $\overline{g}(\nabla_X \tilde{P}Z, Y) - \overline{g}(PY, \nabla_X \tilde{P}Z) = \overline{g}(FY, h^s(X, \tilde{P}Z))$,

(ii) $\overline{g}(PY, A_{\tilde{P}N}X) - \overline{g}(Y, A_{\tilde{P}N}X) = \overline{g}(FY, D^s(X, \tilde{P}N))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi slant lightlike submanifolds of a golden semi-Riemannian manifolds $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma D_2$, for all $X, Y \in \Gamma D_2$. Since $\overline{\nabla}$ is metric connection, from (2.3), (2.4), (2.11) and (3.5), for any $X, Y \in \Gamma D_2$ and $Z \in \Gamma D_1$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}Z) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \nabla_X \tilde{P}Z) - \overline{g}(FY, h^s(X, \tilde{P}Z)) + \overline{g}(Y, \nabla_X \tilde{P}Z))$. In view of (2.3), (2.4), (2.11) and (3.5), for any $X, Y \in \Gamma D_2$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}N) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}N) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(PY, A_{\tilde{P}N}X) - \overline{g}(FY, D^s(X, \tilde{P}N)) - \overline{g}(Y, A_{\tilde{P}N}X)$. Hence, the proof is completed.

□

Theorem 4.4. Let M be a mixed geodesic screen semi slant lightlike submanifolds of a golden semi-Riemannian manifolds $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if

(i) $\overline{g}(PY, \nabla_X \tilde{P}Z) = \overline{g}(Y, \nabla_X \tilde{P}Z)$,

(ii) $\overline{g}(PY, A_{\tilde{P}N}X) - \overline{g}(Y, A_{\tilde{P}N}X) = \overline{g}(FY, D^s(X, \tilde{P}N))$,

for all $X, Y \in \Gamma D_2$, $Z \in \Gamma D_1$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a mixed geodesic screen semi slant lightlike submanifold of a golden semi-Riemannian manifolds $\overline{M}$, we have $h^s(X, \tilde{P}Z) = 0$, for all $X \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma D_2$, for all $X, Y \in \Gamma D_2$. Since $\overline{\nabla}$ is metric connection, from (2.3), (2.4), (2.11) and (3.5), for any $X, Y \in \Gamma D_2$ and $Z \in \Gamma D_1$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}Z) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \nabla_X \tilde{P}Z) - \overline{g}(FY, h^s(X, \tilde{P}Z)) + \overline{g}(Y, \nabla_X \tilde{P}Z))$.Now, from (2.3), (2.4), (2.11) and (3.5), for any $X, Y \in \Gamma D_2$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}N) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}N) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(PY, A_{\tilde{P}N}X) - \overline{g}(FY, D^s(X, \tilde{P}N)) - \overline{g}(Y, A_{\tilde{P}N}X)$. Hence, the proof is completed.

□

Theorem 4.5. Let M be a mixed geodesic screen semi slant lightlike submanifolds of a golden semi-Riemannian manifolds $\overline{M}$. Then the induced connection ∇ on $S(TM)$ is a metric connection if and only if

(i) $A^*_\xi X$ has no component in D_1,

(ii) $\overline{g}(PW, A^*_{\tilde{P}\xi}Z) - \overline{g}(FW, h^s(Z, \tilde{P}\xi)) = \overline{g}(W, A^*_{\tilde{P}\xi}Z)$,

for all $Z, W \in \Gamma D_2$, $X \in \Gamma D_1$ and $\xi \in \Gamma Rad(TM)$.

Proof. Let M be a mixed geodesic screen semi slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then $h^l(X, Z) = 0$, for all $X \in \Gamma D_1$ and $Z \in \Gamma D_2$. From (2.18), for any $X, Y \in \Gamma(D_1)$ and $\xi \in \Gamma(Rad(TM))$, we obtain $\overline{g}(h^l(X, Y), \xi) = \overline{g}(Y, A^*_\xi X)$. Since $\overline{\nabla}$ is metric connection, from (2.3), (2.4) and (3.5), for any $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(Rad(TM))$, we have $\overline{g}(h^l(Z, W), \xi) = -\overline{g}(PW, \overline{\nabla}_Z \tilde{P}\xi) - \overline{g}(FW, \overline{\nabla}_Z \tilde{P}\xi) + \overline{g}(W, \overline{\nabla}_Z \tilde{P}\xi)$, which gives $\overline{g}(h^l(Z, W), \xi) = \overline{g}(PW, A^*_{\tilde{P}\xi}Z) - \overline{g}(FW, h^s(Z, \tilde{P}\xi)) - \overline{g}(W, A^*_{\tilde{P}\xi}Z)$. Thus, the theorem is completed.

□

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