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CHARACTERIZATION OF RANK OF A MATRIX OVER THE SYMMETRIZED MAX-PLUS ALGEBRA

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Abstract. In this paper, we characterize the rank of a matrix over the symmetrized max-plus algebra. This characterization is based on linearly independence of columns or rows of the matrix in balance sense. We show that the rank of such a matrix can be determined using maximum number of rows or columns which are linearly independent in balance sense. This completes the discussion in [1] which only uses minors to determine rank of matrix.

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers. Max-plus algebra is the set of $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ which is equipped by operations maximum (denoted by max) as addition and usual addition (denoted by +) as multiplication. There is no additive inverse for any element in $\mathbb{R}_{\max}$, except for the zero element. The symmetrization process can be carried out at $\mathbb{R}_{\max}$ to obtain a minus and balanced form of any element in $\mathbb{R}_{\max}$, which is done using the balance relation ∇. The result of symmetrization in $\mathbb{R}_{\max}$ is called the symmetrized max-plus algebra and denoted by $\mathbb{S}$.

The linearly independence of vectors over $\mathbb{S}$ has been discussed in [1][4]. When determining the solution of the linearly independence coefficient, the discussion in [1] is limited to signed solutions only, meanwhile in [4] it is not carried out. In this paper, we use definition of linearly independence as in [1]. In ordinary linear algebra, linear independence can be used to define rank of matrix. Rank of A is the number of linearly independent rows or columns [5].

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Let A be a matrix over $\mathbb{S}$. The rank of A is defined using minors of A and called max-algebraic minor rank of A [1]. The term max-algebraic minor rank will be simply refereed as minor rank. In this paper, we characterize rank of matrix over $\mathbb{S}$ using linearly independence approach as in the ordinary linear algebra. We find that the rank of a matrix over $\mathbb{S}$ can be determined by calculating the maximum number of rows or columns which are linearly independent in balance sense.

The results in this paper can be used to complete the discussion in [1] which only uses minors to determine rank of matrix over $\mathbb{S}$. This is an alternative method to determine rank other than using minors, so that we can determine rank of matrix over $\mathbb{S}$ as in the ordinary linear algebra. In the ordinary linear systems, rank of a matrix can be used to determine the criteria of reachability and observability of a linear system. If the rank of the reachability and observability matrices are full-rank, then the linear system is reachable and observable, respectively. Therefore, the rank of a matrix over $\mathbb{S}$ has the potentials to be applied in determining the criteria of reachability and observability of linear systems over $\mathbb{S}$, as in the ordinary linear system.

Section 1 gives an introduction as preliminaries of this paper. The symmetrized max-plus algebra and matrix over the symmetrized max-plus algebra discussed in Section 2 and 3, respectively. The main result is given in Section 4 which is a characterization of the rank of a matrix over S using linearly dependence of rows or columns.

2. The Symmetrized Max-Plus Algebra

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. The addition and multiplication in $\mathbb{R}_{\max}$ is defined as $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$, where $\max(a, -\infty) = a$ and $a + (-\infty) = -\infty$, for all $a, b \in \mathbb{R}_{\max}$.

Definition 2.1. [6] A semi-ring $(R, +, \cdot)$ is a nonempty set R on which we have defined operations of addition and multiplication satisfying the following axioms:

(1) $(R, +)$ is a commutative monoid with the zero element 0,

(2) $(R, \cdot)$ is a monoid with identity element $1 \neq 0$,

(3) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b \in R$,

(4) $0 \cdot a = 0 = a \cdot 0$ for all $a \in R$.

If for any $a \in R, a + a = a$ then R is called an idempotent semi-ring. A semi-ring R is called commutative semi-ring if for any $a, b \in R, a \cdot b = b \cdot a$. The mathematical system of $\mathbb{R}_{\max}$ is an idempotent commutative semi-ring where the zero element is $\varepsilon = -\infty$ and identity element is $e = 0$. There is no additive inverse for any $x \in \mathbb{R}_{\max}$, except for the zero element. Detailed discussion of max-plus algebra can be found in [7]. The symmetrization process can be carried out at $\mathbb{R}_{\max}$ to solve additive inverse problem. This process is similarly in expanding natural numbers into integers. The symmetrization of $\mathbb{R}_{\max}$ is carried out in order to obtain minus and balance element of $\mathbb{R}_{\max}$.

Let $P = \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. It is defined addition and multiplication in P as follows:

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$$
$$(a, b) \otimes (c, d) = (a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)$$

for all $(a, b), (c, d) \in P$. The mathematical system of P is an idempotent commutative semi-ring with the zero element is $(\varepsilon, \varepsilon)$ and identity element is $(0, \varepsilon)$. This semi-ring, P is called the algebra of pairs.

Definition 2.2. [3] Let $x = (a, b) \in P$. The absolute value of x is $|x|_{\oplus} = a \oplus b$, the minus of x is $\ominus x = (b, a)$ and the balance of x is $x^{\bullet} = x \oplus (\ominus x) = (|x|_{\oplus}, |x|_{\oplus})$.

Theorem 2.1. [3] For any $x, y \in P$ we have $x^{\bullet} = (\ominus x^{\bullet}) = (x^{\bullet})^{\bullet}$, $x \otimes y^{\bullet} = (x \otimes y)^{\bullet}$, $\ominus(\ominus x) = x$, $\ominus(x \oplus y) = (\ominus x) \oplus (\ominus y)$ and $\ominus(x \otimes y) = \ominus x \otimes y$.

Definition 2.3. [3] Let $x = (a, b), y = (c, d) \in P$. The balance relation ∇ in P is defined by $x \nabla y$ if and only if $a \oplus d = b \oplus c$.

Since balance relation is only reflexive and symmetric, but it is not transitive, then it is not an equivalence relation. So we can not define the quotient set of P by ∇.

Definition 2.4. [3] Let $x = (a, b), y = (c, d) \in P$. The relation $\mathcal{B}$ in P is defined by

$$x \mathcal{B} y = \begin{cases} x \nabla y & ; a \neq b \text{ and } c \neq d \\ x = y & ; a = b \text{ or } c = d. \end{cases}$$

The relation $\mathcal{B}$ is an equivalence relation, so it is possible to define the quotient set of P by $\mathcal{B}$. There are three kinds of equivalence classes generated by $\mathcal{B}$:

(1) $\overline{(a,-\infty)} = \{(a,x)|x < a\}$ is called a max-positive class,
(2) $\overline{(-\infty,a)} = \{(x,a)|x < a\}$ is called a max-negative class,
(3) $\overline{(a,a)} = \{(a,a) \in P\}$ is called a balanced class.

The quotient set of P by $\mathcal{B}$ denoted by $P/\mathcal{B}$ and it is called the symmetrized max-plus algebra. Furthermore, the symmetrized max-plus algebra denoted by $\mathbb{S}$. It is defined addition and multiplication in $\mathbb{S}$ as follows:

$$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a \oplus c, b \oplus d)}$$
$$\overline{(a,b)} \otimes \overline{(c,d)} = \overline{(a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)}.$$

Since addition and multiplication in $\mathbb{S}$ addopt addition and multiplication in P, it is guaranteed that the operations are well-defined. The mathematical system of $\mathbb{S}$ is an idempotent commutative semi-ring where the zero element is $\overline{\varepsilon} = \overline{(\varepsilon,\varepsilon)}$ and the identity element is $\overline{e} = \overline{(0,\varepsilon)}$.

Since the max-plus algebraic symmetrization has a purpose to determine the minus and balanced forms of any element of $\mathbb{R}_{\max}$, a notation is needed for this purpose. Thus, we denote the plus, minus and balanced forms in $\mathbb{S}$, which is analogous to the positive, negative and zero forms in ordinary linear algebra. In what follows, the class $\overline{(a,-\infty)}$ will be denoted as a, $\overline{(-\infty,a)}$ will be denoted as $\ominus a$ and $\overline{(a,a)}$ will be denoted as $a^{\bullet}$. Furthermore, the zero element in $\mathbb{S}$ will be denoted by ε and the identity element in $\mathbb{S}$ will be denoted by e.

The set of all max-positive or zero classes is denoted by $\mathbb{S}^{\oplus}$, the set of all max-negative or zero classes is denoted by $\mathbb{S}^{\ominus}$ and the set of all balanced classes is denoted by $\mathbb{S}^{\bullet}$. A balanced class is a class that is formed from the equivalence with the zero element $(\varepsilon,\varepsilon)$. The element of $\mathbb{S}^{\bullet}$ is called a balanced element, i.e an element in the form $a^{\bullet}$. The union of $\mathbb{S}^{\oplus}$ and $\mathbb{S}^{\ominus}$ is called the set of all signed classes, it is denoted by $\mathbb{S}^{\vee}$. Furthermore, the elements of $\mathbb{S}^{\vee}$ are called the signed elements, i.e all of elements in the form a and $\ominus a$. We have $\mathbb{S} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet} = \mathbb{S}$ and $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \{\overline{(\varepsilon,\varepsilon)}\}$. Next, $(\mathbb{S}^{\vee})_{*} = \mathbb{S}^{\vee} - \mathbb{S}^{\bullet}$ is the set of all elements that have multiplicative inverse and $\mathbb{R}_{\max}$ can be viewed as $\mathbb{S}^{\oplus}$ in the symmetrized max-plus algebraic sense.

Theorem 2.2. [3] For all $a, b, c \in \mathbb{S}, a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$. Furthermore, for all $a, b \in \mathbb{S}^{\vee}$, if $a \nabla b$ then $a = b$.

Theorem 2.3. [3] (Weak Substitution) For all $a, b, c \in \mathbb{S}$ and $x \in \mathbb{S}^{\vee}$, if $x \nabla a$ and $c \otimes x \nabla b$ then $c \otimes a \nabla b$.

3. Matrix over The Symmetrized Max-Plus Algebra

Matrices over the symmetrized max-plus algebra have importance as in the ordinary linear algebra. The basic algebraic matrix operations over $\mathbb{S}$, such as addition, multiplication and scalar multiplication are similar as in the ordinary matrix. The balance of two matrices is given in the following definition.

Definition 3.1. [1] For all $A, B \in \mathbb{S}^{m \times n}$, $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The discussion of determinant in matrix over $\mathbb{S}$ is also similar as in the case of ordinary matrices. The signature of permutation σ is

$$\text{sign}(\sigma) = \begin{cases} e & ; \sigma \text{ is even permutation} \\ \ominus e & ; \sigma \text{ is odd permutation.} \end{cases}$$

Definition 3.2. [3] Let $A = [a_{ij}] \in \mathbb{S}^{n \times n}$. Then determinant of A is

$$det(A) = \oplus_{\sigma} (\text{sign}(\sigma) \otimes_{i=1}^{n} a_{i\sigma(i)}).$$

The cofactor and transpose of A are also similar as in the ordinary algebra and denoted by $cof(A)$ and A^T, respectively. Some of properties in determinant of matrix over $\mathbb{S}$ are shown in the following theorem.

Theorem 3.1. [3] Let $A = [a_{ij}] \in \mathbb{S}^{n \times n}$. Then $det(A) = \oplus_{k=1}^{n} (a_{ik} \otimes cof_{ik}(A))$ and $det(A)^T = det(A)$.

Theorem 3.2. [3] Let $A = [a_{ij}] \in \mathbb{S}^{n \times n}$. Then $A \otimes cof(A)^T \nabla det(A) \otimes I_n$. Furthermore, if $det(A)$ is a signed element then the diagonal of $A \otimes cof(A)^T$ is also a signed element.

4. Characterization of Rank of Matrix over The Symmetrized Max-Plus Algebra Using Linearly Independence

This section discusses the main result of this paper. It contains the characterization of rank based on linearly independence of rows or columns of matrix. The definition of minor rank is given in the following theorem.

Definition 4.1. [1] Let $A = [a_{ij}] \in \mathbb{S}^{m\times n}$. The max-algebraic minor rank of A is the dimension of the largest square submatrix of A which the max-algebraic determinant is not balanced.

Consider m vectors $a_1, a_2, \ldots, a_m \in (\mathbb{R}_{\max})^n$ and m numbers $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}_{\max}$. Combination of the form $\oplus_{i=1}^{m}(\alpha_i \otimes a_i)$ is called a max-linear combination of the vectors $a_1, a_2, \ldots, a_m$.

Definition 4.2. [1] We say that a set of vectors $\{a_i \in \mathbb{S}^n | i = 1, 2, \ldots, m\}$ is max-linearly independent if the only signed solution of scalar α_i in $\oplus_{i=1}^{m}(\alpha_i \otimes a_i)\nabla\varepsilon_{n\times1}$ is $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \varepsilon$. Otherwise, we say that the vectors $a_1, a_2, \ldots, a_m$ are max-linearly dependent.

The vector $\varepsilon_{n\times1}$ is a vector in $\mathbb{S}^n$ whose each entries are ε i.e the zero element of $\mathbb{S}$. Note that the signed solution of scalar α_i has meaning that all of scalar α_i in the balance $\oplus_{i=1}^{m}(\alpha_i \otimes a_i)\nabla\varepsilon_{n\times1}$ are signed elements in $\mathbb{S}^{\vee}$. So, a set of vectors $\{a_i \in \mathbb{S}^n | i = 1, 2, \ldots, m\}$ is max-linearly independent if all of scalar α_i which satisfy $\oplus_{i=1}^{m}(\alpha_i \otimes a_i)\nabla\varepsilon_{n\times1}$ is only $\alpha_i = \varepsilon, i = 1, 2, \ldots, m$. In other words, the only signed solution of scalar α_i in $\oplus_{i=1}^{m}(\alpha_i \otimes a_i)\nabla\varepsilon_{n\times1}$ is $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \varepsilon$. The following is an example to explain Definition 4.2.

Example 4.1. Let a set of vectors $\{v_1, v_2\}$ where $v_1 = \begin{bmatrix}\varepsilon\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\\varepsilon\end{bmatrix}$. The signed solution of the balance $\alpha_1 \otimes v_1 \oplus \alpha_2 \otimes v_2\nabla\varepsilon_{2\times1}$ is only $\alpha_1 = \varepsilon$ and $\alpha_2 = \varepsilon$. No other the signed solution other than $\alpha_1 = \varepsilon$ and $\alpha_2 = \varepsilon$ which satisfy the balance $\alpha_1 \otimes v_1 \oplus \alpha_2 \otimes v_2\nabla\varepsilon_{2\times1}$. So, a set of vectors $\{v_1, v_2\}$ is max-linearly independent.$\diamond$

In the next session, max-algebraic minor rank of A and max-linearly independent are simply written as minor rank of A and linearly independent, respectively.

Theorem 4.1. [3] Let $A \in \mathbb{S}^{n\times n}$ with $det(A) \in (\mathbb{S}^{\vee})_{*}$ and $cof(A)^T \otimes b \in (\mathbb{S}^{\vee})^n$. Then there exists a unique solution of $A \otimes x \nabla b$ and it satisfies $x \nabla (cof(A)^T \otimes b) \otimes det(A)^{-1}$.

We use Definitions 4.1 and 4.2 to characterize the rank of matrix over $\mathbb{S}$. The following theorem explains the linear independence of column in matrix over $\mathbb{S}$.

Theorem 4.2. Let $A \in \mathbb{S}^{n\times n}$. The columns of A are linearly independent if and only if $det(A) \in (\mathbb{S}^{\vee})_{*}$.

Proof. ($\leftarrow$) If $det(A) \in (\mathbb{S}^{\vee})_{*}$ then there is $det(A)^{-1}$ such that $\det(A)^{-1} \otimes det(A) = e$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

and $c_1, c_2, \dots, c_n$ be columns of A. Let $k_1, k_2, \dots, k_n$ be scalars in $\mathbb{S}^n$ such that $k_1 \otimes c_1 \oplus k_2 \otimes c_2 \oplus \dots \oplus k_n \otimes c_n \nabla \varepsilon$. Then we have

$$k_1 \otimes \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \oplus k_2 \otimes \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \oplus \dots \oplus k_n \otimes \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}$$

and so

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \otimes \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}. \tag{4.1}$$

The balance in (4.1) is homogeneous linear balanced systems. Since $det(A) \in (\mathbb{S}^{\vee})_{*}$ so $det(A)$ is not balance with ε. According to Theorem 4.1, it implies that

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \nabla det(A)^{-1} \otimes cof(A)^T \otimes \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}$$

and consequently

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}$$

is a unique signed solution of the homogeneus linear balance system (4.1). So, we have $k_1 = k_2 = \ldots = k_n = \varepsilon$ and columns of A are linearly independent.

($\rightarrow$) We show that if $det(A) \notin (\mathbb{S}^{\vee})_*$ then columns of A are linearly dependent. Let $c_1, c_2, \ldots, c_n$ be columns of A. According to the balance

$$k_1 \otimes c_1 \oplus k_2 \otimes c_2 \oplus \ldots \oplus k_n \otimes c_n \nabla \varepsilon$$

we have

$$A \otimes K \nabla \varepsilon \tag{4.2}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$ and $K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$. If both of the balance sides in (4.2) are multiplied by $cof(A)^T$, then $(cof(A)^T \otimes A) \otimes K \nabla cof(A)^T \otimes \varepsilon = \varepsilon$. Since

$$cof(A)^T \otimes A = \begin{bmatrix} cof_{11}(A) & cof_{21}(A) & \ldots & cof_{n1}(A) \\ cof_{12}(A) & cof_{22}(A) & \ldots & cof_{n2}(A) \\ \vdots & \vdots & \ddots & \vdots \\ cof_{1n}(A) & cof_{2n}(A) & \ldots & cof_{nn}(A) \end{bmatrix} \otimes \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

so

$$\begin{aligned} (cof(A)^T \otimes A)_{11} &= cof(A)_{11} \otimes a_{11} \oplus cof(A)_{21} \otimes a_{21} \oplus \ldots \oplus cof(A)_{n1} \otimes a_{n1}, \\ (cof(A)^T \otimes A)_{12} &= cof(A)_{12} \otimes a_{12} \oplus cof(A)_{22} \otimes a_{22} \oplus \ldots \oplus cof(A)_{n2} \otimes a_{n2}, \\ &\vdots \\ (cof(A)^T \otimes A)_{1n} &= cof(A)_{1n} \otimes a_{1n} \oplus cof(A)_{2n} \otimes a_{2n} \oplus \ldots \oplus cof(A)_{nn} \otimes a_{nn}. \end{aligned}$$

According to Theorem 3.2, we have

$$cof(A)^T \otimes A \nabla det(A) \otimes I_n = \begin{bmatrix} det(A) & \varepsilon & \ldots & \varepsilon \\ \varepsilon & det(A) & \ldots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \ldots & det(A) \end{bmatrix}$$

and consequently

$$cof(A)^T \otimes A = \begin{bmatrix} det(A) & (\ldots)^\bullet & \ldots & (\ldots)^\bullet \\ (\ldots)^\bullet & det(A) & \ldots & (\ldots)^\bullet \\ \vdots & \vdots & \ddots & \vdots \\ (\ldots)^\bullet & (\ldots)^\bullet & \ldots & det(A) \end{bmatrix}. \tag{4.3}$$

If (4.3) is weakly substituted to (4.2) then

$$\begin{bmatrix} det(A) & (\ldots)^\bullet & \ldots & (\ldots)^\bullet \\ (\ldots)^\bullet & det(A) & \ldots & (\ldots)^\bullet \\ \vdots & \vdots & \ddots & \vdots \\ (\ldots)^\bullet & (\ldots)^\bullet & \ldots & det(A) \end{bmatrix} \otimes \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}$$

and it is obtained

$$\begin{gathered} det(A) \otimes k_1 \oplus (\ldots)^\bullet \otimes k_2 \oplus \ldots \oplus (\ldots)^\bullet \otimes k_n \nabla \varepsilon, \\ (\ldots)^\bullet \otimes k_1 \oplus det(A) \otimes k_2 \oplus \ldots \oplus (\ldots)^\bullet \otimes k_n \nabla \varepsilon, \\ \vdots \\ (\ldots)^\bullet \otimes k_1 \oplus (\ldots)^\bullet \otimes k_2 \oplus \ldots \oplus det(A) \otimes k_n \nabla \varepsilon. \end{gathered}$$

Since $det(A) \notin (\mathbb{S}^\vee)_*$ so $det(A)$ is not balance with ε. Consequently there is a non-trivial signed solution for $k_1, k_2, \ldots, k_n$. Therefore, columns of A are linearly dependent. □

As a result, we can characterize the linear dependently of the columns of matrix, as in this following corollaries.

Corollary 4.1. Let $A \in \mathbb{S}^{n \times n}$. Then the columns of A are linearly dependent if only if $det(A) \nabla \varepsilon$.

Corollary 4.2. Let $A \in \mathbb{S}^{n \times n}$. The rows of A are linearly independent if only if $det(A) \in (\mathbb{S}^\vee)_*$.

According to Corollary 4.1 and Corollary 4.2, the rows of A are linearly dependent if only if $det(A)\nabla\varepsilon$. The following theorem shows the sufficient condition of columns of A corresponding to minor rank are linearly independent.

Theorem 4.3. If minor rank of $A \in \mathbb{S}^{m\times n}$ is r, then there are r columns of A which are linearly independent.

Proof. Since the minor rank of A is r, so r is the dimension of the largest square submatrix of A which non-balanced max-algebraic determinant. Let S be the set of all submatrices of A of size $r \times r$ with non-balanced determinant. For every $M \in S$, we show that columns of A corresponding to M are linearly independent. Without loss of generality, let submatrix of A that contains columns corresponding to $M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{bmatrix}_{r\times r}$ is $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(r-1)1} & a_{(r-1)2} & \dots & a_{(r-1)r} \\ a_{r1} & a_{r2} & \dots & a_{rr} \\ a_{(r+1)1} & a_{(r+1)2} & \dots & a_{(r+1)r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix}_{m\times r}$ with

$$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(r-1)1} \\ a_{r1} \\ a_{(r+1)1} \\ \vdots \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{(r-1)2} \\ a_{r2} \\ a_{(r+1)2} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, c_r = \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{(r-1)r} \\ a_{rr} \\ a_{(r+1)r} \\ \vdots \\ a_{mr} \end{bmatrix}.$$

Consider a balance $k_1 \otimes c_1 \oplus k_2 \otimes c_2 \oplus \dots \oplus k_r \otimes c_r \nabla \varepsilon_{m\times 1}$. Then we have

$$\begin{bmatrix} M_{r\times r} \\ N_{(m-r)\times r} \end{bmatrix} \otimes K_{r\times 1} \nabla \begin{bmatrix} \varepsilon_{r\times 1} \\ \varepsilon_{(m-r)\times 1} \end{bmatrix} \tag{4.4}$$

where $K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix}$. Take note the sub-linear balance $M_{r\times r} \otimes K_{r\times 1} \nabla \varepsilon_{r\times 1}$ in (4.4). Since $det(M)$ is not balanced element then $det(M)$ is not balance with ε. According to Theorem 4.2, the columns of M are linearly independent. Consequently, the signed solution for K is unique i.e $k_1 = k_2 = \ldots = k_r = \varepsilon$. Therefore, columns of $\begin{bmatrix} M_{r\times r} \\ N_{(m-r)\times r} \end{bmatrix}$ are linearly independent and columns of A that correspond to minor rank are linearly independent. □

The following theorem shows the sufficient condition for minor rank of matrices in the symmetrized max-plus algebra.

Theorem 4.4. If the maximum number of linearly independent columns of $A \in \mathbb{S}^{m\times n}$ is r, then the minor rank of A is r.

Proof. Let $c_1, c_2, \ldots, c_r$ be the linearly independent columns of A. Then a balance

$$k_1 \otimes c_1 \oplus k_2 \otimes c_2 \oplus \ldots \oplus k_r \otimes c_r \nabla \varepsilon_{m\times 1}$$

is only satisfied by a unique signed solution $k_1 = k_2 = \ldots = k_r = \varepsilon$. Let M be the matrix which columns are $c_1, c_2, \ldots, c_r$. For the special case $m = r$, we have size of M is $r \times r$. According to Theorem 4.2, it is obtained $det(M)$ is not balance with ε and consequently, r is the maximum size of the submatrix of A which determinant is not balanced element. So, the minor rank of A is r.

Let $m > r$. According to the balance $k_1 \otimes c_1 \oplus k_2 \otimes c_2 \oplus \ldots \oplus k_r \otimes c_r \nabla \varepsilon_{m\times 1}$, we have the linear balance system

$$M \otimes K \nabla \varepsilon$$

or, in other words

$$\begin{bmatrix} S_{r\times r} \\ N_{(m-r)\times r} \end{bmatrix} \otimes K_{r\times 1} \nabla \begin{bmatrix} \varepsilon_{r\times 1} \\ \varepsilon_{(m-r)\times 1} \end{bmatrix}. \tag{4.5}$$

This is equivalent to the following two balance relations

$$S_{r\times r}\otimes K_{r\times 1}\nabla\varepsilon_{r\times 1}$$
$$N_{(m-r)\times r}\otimes K_{r\times 1}\nabla\varepsilon_{(m-r)\times 1}.$$

Since the unique signed solution for K in (4.5) is only $k_1=k_2=\ldots=k_r=\varepsilon$ so the columns of $S_{r\times r}$ are linearly independent. According to Theorem 4.2, we have $det(S_{r\times r})$ is not balance with ε. Therefore, r is size of square submatrices of A which determinant is not balanced element.

Suppose $r'>r$ is the size of the square submatrix S' which determinant is not balanced element. According to Theorem 4.3, columns of A which correspond to S' are linearly independent. Therefore, the number of columns of A that linearly independent is $r'>r$. This contradicts the assumption and hence r must be the largest size of the square submatrix of A which determinant is not balanced element. Therefore, the minor rank of A is r. □

According to Theorem 4.3 and Theorem 4.4, we can characterize rank of A using the linearly independence of columns or rows of A as in the following corollary.

Corollary 4.3. Let $A\in\mathbb{S}^{m\times n}$. The minor rank of A is r if and only if the maximum number of linearly independent columns in $A\in\mathbb{S}^{m\times n}$ is r.

Proof. In this proof, it remains only to show that if the minor rank of A is r then the maximum number of linearly independent columns of A is r. Since the minor rank of A is r, according to Theorem 4.4, we get r number columns of A that correspond to minor rank are linearly independent. Suppose that the maximum number of linearly independent columns of A is $r'>r$. According to Theorem 4.4, the minor rank of A is r'. This contradicts the minor rank of A is r. □

The following example shows characterization of rank of matrix over the symmetrized max-plus algebra using the linearly independence of columns or rows.

Example 4.2. Let $A=\begin{bmatrix}1 & 2^{\bullet} & \ominus 1^{\bullet}\\ \ominus 2 & 0 & 1\end{bmatrix}$ and columns of A are $c_1=\begin{bmatrix}1\\ \ominus 2\end{bmatrix}$, $c_2=\begin{bmatrix}2^{\bullet}\\ 0\end{bmatrix}$ and $c_3=\begin{bmatrix}\ominus 1^{\bullet}\\ 1\end{bmatrix}$. Since $0\otimes c_1\oplus 2\otimes c_2=\begin{bmatrix}4^{\bullet}\\ 2^{\bullet}\end{bmatrix}\nabla\begin{bmatrix}\varepsilon\\ \varepsilon\end{bmatrix}$, $0\otimes c_1\oplus 1\otimes c_3=\begin{bmatrix}\ominus 2^{\bullet}\\ 2^{\bullet}\end{bmatrix}\nabla\begin{bmatrix}\varepsilon\\ \varepsilon\end{bmatrix}$

and $\ominus 1 \otimes c_2 \oplus 0 \otimes c_3 = \begin{bmatrix} \ominus 3^\bullet \\ \ominus 1^\bullet \end{bmatrix} \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}$, so each of $\{c_1, c_2\}$, $\{c_1, c_3\}$ and $\{c_2, c_3\}$ are not linearly independent, respectively. Each of the balance $\alpha_1 \otimes c_1 \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}$, $\alpha_2 \otimes c_2 \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}$ and $\alpha_3 \otimes c_3 \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}$ are only satisfied by the signed solution $\alpha_1 = \varepsilon$, $\alpha_2 = \varepsilon$ and $\alpha_3 = \varepsilon$, respectively. Therefore, $\{c_1\}$, $\{c_2\}$ and $\{c_3\}$ are linearly independent, respectively. So, the maximum number of linearly independent columns of A is 1, and consequently minor rank of A is 1.

We have $A^T = \begin{bmatrix} 1 & \ominus 2 \\ 2^\bullet & 0 \\ \ominus 1^\bullet & 1 \end{bmatrix}$ and columns of A^T are $k_1 = \begin{bmatrix} 1 \\ 2^\bullet \\ \ominus 1^\bullet \end{bmatrix}$ and $k_2 = \begin{bmatrix} \ominus 2 \\ 0 \\ 1 \end{bmatrix}$. Since $3 \otimes k_1 \oplus 2 \otimes k_2 = \begin{bmatrix} 4^\bullet \\ 5^\bullet \\ \ominus 4^\bullet \end{bmatrix} \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$ then $\{k_1, k_2\}$ is not linearly independent. Every of $b_1 \otimes k_1 \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$ and $b_2 \otimes k_2 \nabla \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$ are only satistified by signed solution $b_1 = \varepsilon$ and $b_2 = \varepsilon$, respectively. Therefore, $\{k_1\}$ and $\{k_2\}$ are linearly independent, respectively. The maximum number of linearly independent columns of A^T is 1, and consequently minor rank of A^T is 1. $\diamond$

5. Conclusion

The minor rank of matrix over the symmetrized max-plus algebra is defined as dimension of the largest square submatrix which the determinant is not balanced element. This can be characterized using the linearly independence of the columns or rows of matrix, and then simply called rank. If A is matrix over the symmetrized max-plus algebra then rank of A is the maximum number of linearly independent columns or rows of A, as in the ordinary algebra. The potentials future research can be done in application of rank to linear systems over the symmetrized max-plus algebra.

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