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HYERS-ULAM-RASSIAS INSTABILITY FOR BERNOULLI'S AND NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we have obtained integral sufficient conditions under which the zero solution of nonlinear differential equations of first order with zero initial condition is unstable in Hyers-Ulam-Rassias sense. We also have proved the Hyers-Ulam-Rassias instability of Bernoulli's differential equation with zero initial condition. To illustrate the results we have given three examples.

1. Introduction

In 1940, Ulam [33] posed an important problem before the Mathematics Club of the University of Wisconsin concerning the stability of group homomorphisms. A significant breakthrough came in 1941, when Hyers [7] gave an answer to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians [e.g. 3, 5,7-9,17,21-24,30-31]. More than twenty five years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, y(t), y'(t), ..., y^{(n)}(t) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left|F(t, y(t), y'(t), ..., y^{(n)}(t))\right| \leq \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$|y(t) - y_0(t)| \leq K(\varepsilon)$$

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and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$.

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza [19,20]. Thereafter, Alsina and Ger [1] have studied the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The Hyers-Ulam stability problems of linear differential equation of first order with constant coefficients were studied by some authors [11,34] by using the method of integral factors. The results given in [11,16,31] have been generalized by Popa and Rus [25-26] for the linear differential equations of nth order with constant coefficients.

In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order [e.g 4,14,15,27,28]. Many mathematicians had considered the wide scope of this same problem for fractional equations of different types. Such problem may be found in [6,12–13] and other papers. Mathematical models of dynamical systems are sometimes prone to instability.

Instability is a serious issue in applied mathematics in that it poses problems for dynamical system models to predict the future behaviour of systems . In [2] Brillouët-Belluot indicated that there are only few outcomes of which we could say that they concern nonstability of functional equations. However in [29] Qarawani investigate the Hyers-Ulam instability of linear and nonlinear differential equations of second order. Thus our study is a continuation of preceding contributions to Hyers-Ulam stability theory of differential equations. Motivation for this study comes from the work of Qarawani [28], where Hyers-Ulam stability was obtained for Bernoulli's differential equations. Much less work has been devoted to the study of Hyers-Ulam instability for differential equations.

This paper investigates the Hyers-Ulam-Rassias instability of the following nonlinear differential equation of order one:

$$y' + P(t)y = \sum_{k=1}^{n} p_k(t)y^k(t), \tag{1.1}$$

and the initial condition

$$y(t_0) = 0 \tag{1.2}$$

where $y \in C^1(I)$, $I = [t_0, t]$, $0 \le t_0 < t \le \infty$.

We also consider a nonlinear differential equation of order one

$$y' + p(t)y = G(t, y) \tag{1.3}$$

with initial condition

$$y(t_0) = 0, \tag{1.4}$$

where $G(t, y)$ is continuous function that satisfies the condition

$$|G(t, y)| \sim \ \gamma(t)\, |y(t)|^{\beta}, \tag{1.5}$$

and $G(t, 0) = 0, \beta \in [0, \infty), \gamma(t) : I \ \to [0, \infty)$ is a positive bounded function.

Moreover we establish the Hyers-Ulam-Rassias stability for Bernoulli's equation

$$y' + p(t)y = q(t)y^{\alpha}, \ \alpha \neq 1 \tag{1.6}$$

with initial condition

$$y(t_0) = 0. \tag{1.7}$$

2. Preliminaries

Definition 2.1. We say that the equation (1.1) has the Hyers –Ulam-Rassias (HUR) stability with respect to $\varphi : I \to [0, \infty)$ if there exists a positive constant $K > 0$ with the following property: For each $y \in C^1(I)$, if

$$|y' + P(t)y - \sum_{k=1}^{n} p_k(t)y^k(t)| \ \leq \varphi(t) \tag{2.1}$$

and $y(0) = 0$, then there exists $z_0 \equiv 0$ satisfying the equation (1.1) such that $|y(t) - z_0(t)| \leq K\varphi$.

Definition 2.2. We say that equation (1.3) with initial condition (1.4) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\varphi : I \to [0, \infty)$ if there exists a positive constant $K > 0$ with the following property: For each $y \in C^1(I)$, if

$$|y' + p(t)y - G(t,y)| \ \le \varphi(t) \tag{2.2}$$

then there exists $z_0 \equiv 0$ satisfying the equation (1.1) *with (1.2)*, such that $|y(t) - z_0(t)| \le K\varphi$.

Definition 2.3. We say that equation (1.6) with initial condition (1.7) has the Hyers-Ulam-Rassias(HUR) stability with respect to $\varphi : I \to [0,\infty)$ if there exists a positive constant $K > 0$ with the following property: For each $y \in C^1(I)$, if

$$|y' + p(t)y - q(t)y^{\alpha}| \ \le \varphi(t) \tag{2.3}$$

then there exists $z_0 \equiv 0$ satisfying the equation (1.1) *with (1.2)*, such that $|y(t) - z_0(t)| \le K\varphi$.

3. On Hyers-Ulam-Rassias stability of solutions

Theorem 3.1. *Suppose that* $y \in C^1(I)$, $P(t)$ *and* $p_k(t) \ge 0, 1 \le k \le n$ *are continuous functions on* I. *If the following conditions are satisfied*

a) $\sup\limits_{0 \le t \ \le \infty} \int\limits_{t_0}^{t} \varphi(s) \exp\left(-\int\limits_{s}^{t} P(r)dr\right) ds < \infty$;

b) $\sup\limits_{0 \le t \ \le \infty} \int\limits_{t_0}^{t} e^{-\int\limits_{s}^{t} P(r)dr} \sum\limits_{k=1}^{n} p_k(s) y^k(s) ds = \infty$.

Then the initial value problem (1.1), (1.2) is unstable in the sense of HUR.

Proof. Suppose that $y \in C^1(I)$ satisfies the inequality (2.1) and the initial condition $y(0) = 0$.We will show that zero solution $z_0(y) \equiv 0$ of the equation (1.1) will satisfy the inequality $|y(y) - z_0(y)| > k\varphi$. On the contrary, let us assume that there exists $\varphi > 0$ such that $\sup\limits_{x \ge x_0} |y(t) - z_0(t)| \le k\varphi$. Then we can find a constant $M > 0$ such that $M = \sup\limits_{t \ \ge t_0} |y(t)|$.

Multiply (2.1) by integrating factor $\exp\left(\int\limits_{t_0}^{t} P(s)ds\right)$ and then integrate it to get

$$-\int\limits_{t_0}^{t} \varphi(s) e^{\int\limits_{t_0}^{s} P(r)dr} ds \le \int\limits_{t_0}^{t} \left(e^{\int\limits_{t_0}^{s} P(r)dr} y\right)' ds - \int\limits_{t_0}^{t} e^{\int\limits_{t_0}^{s} P(r)dr} \sum_{k=1}^{n} p_k(s) y^k(s) ds \le \int\limits_{t_0}^{t} \varphi(s) e^{\int\limits_{t_0}^{s} P(r)dr} ds$$

from which it implies that

(3.1)
$$-\int_{t_0}^{t}\varphi(s)e^{\int_{t_0}^{s}P(r)dr}ds \le e^{\int_{t_0}^{t}P(r)dr}y(t)-\int_{t_0}^{t}e^{\int_{t_0}^{s}P(r)dr}\sum_{k=1}^{n}p_k(s)y^k(s)ds \le \int_{t_0}^{t}\varphi(s)e^{\int_{t_0}^{s}P(r)dr}ds.$$

Multiplying (3.1) by $e^{-\int_{t_0}^{t}P(r)dr}$, we obtain

(3.2)
$$-\int_{t_0}^{t}\varphi(s)e^{-\int_{s}^{t}P(r)dr}ds \le y(t)-\int_{t_0}^{t}e^{-\int_{s}^{t}P(r)dr}\sum_{k=1}^{n}p_k(s)y^k(s)ds \le \int_{t_0}^{t}\varphi(s)e^{-\int_{s}^{t}P(r)dr}ds.$$

By virtue of (3.2) and by applying the mean value theorem for second integral, we have

$$\begin{aligned} y(t) &\ge \left|\int_{t_0}^{t}e^{-\int_{s}^{t}P(r)dr}\sum_{k=1}^{n}p_k(s)y^k(s)ds\right| - \left|\int_{t_0}^{t}\varphi(s)e^{-\int_{s}^{t}P(r)dr}ds\right| \\ &\ge |\widehat{y}(s^*)|\int_{t_0}^{t}e^{-\int_{s}^{t}P(r)dr}\sum_{k=1}^{n}p_k(s)ds - \int_{t_0}^{t}\varphi(s)e^{-\int_{s}^{t}P(r)dr}ds, \end{aligned}$$

where $|\widehat{y}(s^*)| = \inf\{|y(s^*)|, ..., |y^k(s^*)|\}$, for $s^* \in [t_0, t]$.

Therefore, in view of (a) and (b), we find that $\sup_{t \ge t_0}|y(t)|$ becomes infinite as $t \to \infty$. The contradiction completes the proof of Theorem 3.1. □

Now we give an example illustrating the Theorem 3.1.

Example 3.2. Consider the equation

$$y' + P(t)\ y = p_1(t)y^3 + p_2(t)y^5,\ \ y(0) = 0, t \ge 0 \tag{3.3}$$

with $P(t) = 2t$, $p_1(t) = t$, $p_2(t) = t^3$ and $\varphi(t) = t$.

Suppose that $y(t)$ is a solution of the inequality

$$\left|y' + 2t\ y - ty^3 - t^3y^5\right| \le t \tag{3.4}$$

Using the same argument used above, we obtain

$$\left| y - e^{-t^2} \int_0^t \left(sy^3 + s^3y^5\right) e^{s^2} ds \right| \leq \frac{1}{2}\left(1 - e^{-t^2}\right). \tag{3.5}$$

Using the triangle inequality and applying the mean value theorem to the integral in the following inequality, we obtain

$$\begin{aligned} |y(t)| &\geq \left| e^{-t^2} \int_0^t \left(sy^3 + s^3y^5\right) e^{s^2} ds \right| - \left| \frac{1}{2}\left(1 - e^{-t^2}\right) \right| \\ &\geq e^{-t^2} \left|\widehat{y}(s^*)\right| \left(\int_0^t \left(s + s^3\right) e^{s^2} ds \right) - \frac{1}{2}\left(1 - e^{-t^2}\right) \\ &= \left|\widehat{y}(s^*)\right| e^{-t^2} \left(\frac{1}{2} t^2 e^{t^2}\right) - \frac{1}{2}\left(1 - e^{-t^2}\right), \end{aligned} \tag{3.6}$$

where $|\widehat{y}(s^*)| = \inf\{|y^3(s^*)|, |y^5(s^*)|\}$, for $s^* \in [t_0, t]$.

Since $\lim\limits_{t\to\infty} \left[\frac{1}{2}\left(1 - e^{-t^2}\right)\right] = \frac{1}{2}$ and $\lim\limits_{t\to\infty} \frac{1}{2}t^2 = \infty$ then from (3.6) we get $y(t) \to \infty$, as $t \to \infty$. The contradiction proves the instability of equation (3.3).

Now we will investigate the HUR instability for Bernoulli's differential equation. First we establish HUR instability of differential equation (1.3).

Theorem 3.3. *Let $y \in C^1(I)$ and $p(t)$ be continuous functions on I. Assume that*

a) $\sup\limits_{0 \leq t \leq \infty} \int_{t_0}^{t} \varphi(s) \exp\left(-\int_s^t p(r)dr\right) ds < \infty$;

b) $\sup\limits_{0 \leq t \leq \infty} \int_{t_0}^{t} \exp\left(-\int_s^t p(r)dr\right) ds = \infty$.

Then the initial value problem (1.3), (1.4) is unstable in the sense of HUR.

Proof. Suppose that $y \in C^1(I)$ satisfies the inequality (2.2) and the initial condition $y(0) = 0$.We will show that zero solution $z_0(t) \equiv 0$ of the equation (1.3) will satisfy the inequality $|y(t) - z_0(t)| > k\varphi$. On the contrary, let us assume that there exists $\varphi > 0$ such that $\sup\limits_{t \geq t_0} |y(t) - z_0(t)| \leq k\varphi$.Then we can find a constant $M > 0$ such that $M = \sup\limits_{t \geq t_0} |y(t)|$.

Consider the inequality

$$-\varphi(t) \leq y' + p(t)y - G(t, y(t) \leq \varphi(t). \tag{3.7}$$

Multiply (3.7) by integrating factor $\exp\left(\int\limits_{t_0}^{t} p(s)ds\right)$ and then integrate with respect to t, to get

$$-\int\limits_{t_0}^{t}\varphi(s)e^{\int\limits_{t_0}^{s}p(r)dr}ds\le\int\limits_{t_0}^{t}\left(e^{\int\limits_{t_0}^{s}p(r)dr}y\right)'ds-\int\limits_{t_0}^{t}e^{\int\limits_{t_0}^{s}p(r)dr}G(s,y(s))ds\le\int\limits_{t_0}^{t}\varphi(s)e^{\int\limits_{t_0}^{s}p(r)dr}ds$$

from which it implies that

(3.8)
$$-\int\limits_{t_0}^{t}\varphi(s)e^{\int\limits_{t_0}^{s}p(r)dr}ds\le e^{\int\limits_{t_0}^{t}p(r)dr}y(t)-\int\limits_{t_0}^{t}e^{\int\limits_{t_0}^{s}p(r)dr}G(s,y(s))ds\le\int\limits_{t_0}^{t}\varphi(s)e^{\int\limits_{t_0}^{s}p(r)dr}ds.$$

Multiplying (3.8) by $e^{-\int\limits_{t_0}^{t}p(r)dr}$, we obtain

$$-\int\limits_{t_0}^{t}\varphi(s)e^{-\int\limits_{s}^{t}p(r)dr}ds\le y(t)-\int\limits_{t_0}^{t}e^{-\int\limits_{s}^{t}p(r)dr}G(s,y(s))ds\le\int\limits_{t_0}^{t}\varphi(s)e^{-\int\limits_{s}^{t}p(r)dr}ds. \tag{3.9}$$

By virtue of (1.5) and by applying the mean value theorem to second integral in (3.9), we get

$$\begin{aligned}|y(t)|&\ge\left|\int\limits_{t_0}^{t}e^{-\int\limits_{s}^{t}p(r)dr}G(s,y(s))ds\right|-\left|\int\limits_{t_0}^{t}\varphi(s)e^{-\int\limits_{s}^{t}p(r)dr}ds\right|\\&=\gamma(s^*)\left|\widehat{y}(s^*)\right|^{\beta}\int\limits_{t_0}^{t}e^{-\int\limits_{s}^{t}p(r)dr}ds-\int\limits_{t_0}^{t}\varphi(s)e^{-\int\limits_{s}^{t}p(r)dr}ds,\end{aligned}$$

where $s^*\in[t_0,t]$.

By boundedness assumption on the solution $y(t)$,$\left|\widehat{y}(s^*)\right|^{\beta}$, $\beta>0$ will be a constant. Therefore, in view of (a) and (b), we find that $\sup\limits_{t\ \geq t_0}|y(t)|$ becomes infinite as $t\to\infty$, which gives a contradiction to boundedness. Therefore the proof of instability is complete. □

Theorem 3.4. *Let $y\in C^1(I)$, $p(t),q(t)$ be continuous functions on I and α be a positive real number not equal to even-degree root. If the following conditions are all satisfied:*

a) $\sup\limits_{0\leq t\ \leq\infty} \int\limits_{t_0}^{t} \varphi(s)\exp\left(-\int\limits_{s}^{t} p(r)dr\right) ds < \infty;$

b) $\sup\limits_{0\leq t\ \leq\infty} \int\limits_{t_0}^{t} q(s)\exp\left(-\int\limits_{s}^{t} p(r)dr\right) ds = \infty.$

Then the initial value problem (1.6), (1.7) is unstable in the sense of HUR.

Proof. By setting in Theorem 3.1 $G(t,y) = q(t)y^{\alpha}(t)$,where α a positive real number not equal to even-degree root, then the proof can be carried out in the way similar to proof of Theorem 3.1 and will therefore be omitted. □

Example 3.5. Consider the equation

$$y' + p(t)\ y = p_1(t)y^2,\ y(0) = 0, t \geq 0, \tag{3.10}$$

with $p(t) = 4t$, $q(t) = 5te^{t^2}$, and $\varphi(t) = te^{-t^2}$

$$\left| y' + 4ty - 5te^{t^2}y^2 \right| \leq te^{-t^2}. \tag{3.11}$$

Using the same argument used above, we obtain

$$\left| y - e^{-2t^2}\left(5\int\limits_0^t sy^2e^{3s^2}ds\right)\right| \leq \frac{1}{2}\left(e^{t^2} - 1\right)e^{-2t^2}. \tag{3.12}$$

Applying the mean value theorem for integral in (3.12), we obtain the estimate

$$\begin{aligned} |y(t)| &\geq \left| e^{-2t^2}\left(5\int\limits_0^t sy^2e^{3s^2}ds\right)\right| - \left|\frac{1}{2}\left(e^{t^2} - 1\right)e^{-2t^2}\right| \\ &\geq e^{-2t^2}\left|\widehat{y}(s^*)\right|^2\left(5\int\limits_0^t se^{3s^2}ds\right) - \frac{1}{2}(e^{-t^2} - e^{-2t^2}) \\ &= \frac{5}{6}\left|\widehat{y}(s^*)\right|^2(e^{t^2} - e^{-2t^2}) - \frac{1}{2}\left(1 - e^{-t^2}\right) \end{aligned} \tag{3.13}$$

Now, since $\lim\limits_{t\to\infty}\left[\frac{1}{2}\left(1 - e^{-t^2}\right)\right] = \frac{1}{2}$ and $\lim\limits_{t\to\infty} e^{t^2}(1 - e^{-3t^2}) = \infty$ then from (3.13) we get $y(t) \to \infty$, as $t \to \infty$. The contradiction proves the instability of equation (3.10).

Theorem 3.6. *Let $y \in C^1(I)$, $p(t), q(t)$ be continuous functions on I and let α be a negative real number not equal to even-degree root. If the following conditions are all satisfied:*

a) $\sup\limits_{0\leq t\ \leq\infty} \int\limits_{t_0}^{t} \varphi(s) \exp\left(-\int\limits_{s}^{t} p(r)dr\right) ds < \infty;$

b) $\sup\limits_{0\leq t\ \leq\infty} \int\limits_{t_0}^{t} \exp\left(-\int\limits_{s}^{t} p(r)dr\right) ds = \infty;$

c) $\inf\{|y(t)| : t \geq t_0\} \geq m > 0.$

Then the initial value problem (1.6), (1.7) is unstable in the sense of HUR.

Proof. Suppose that $y \in C^1(I)$ satisfies the inequality (2.3) and the initial condition $y(0) = 0$.We will show that zero solution $z_0(t) \equiv 0$ of the equation (1.6) will satisfy the inequality $|y(t) - z_0(t)| > k\varphi$. On the contrary, let us assume that there exists $\varphi > 0$ such that $\sup\limits_{t\geq t_0} |y(t) - z_0(t)| \leq k\varphi$.Then we can find a constant $M > 0$ such that $M = \sup\limits_{t\ \geq t_0} |y(t)|$.

Consider the inequality

$$-\varphi(t) \leq y' + p(t)y - q(t)y^\alpha(t) \leq \varphi(t). \tag{3.14}$$

Multiply (3.14) by integrating factor $\exp\left(\int\limits_{t_0}^{t} p(s)ds\right)$ and then integrate with respect to t, to get

$$-\int\limits_{t_0}^{t} \varphi(s)e^{\int\limits_{t_0}^{s} p(r)dr} ds \leq \int\limits_{t_0}^{t} \left(e^{\int\limits_{t_0}^{s} p(r)dr} y\right)' ds - \int\limits_{t_0}^{t} e^{\int\limits_{t_0}^{s} p(r)dr} q(t)y^\alpha(t)ds \leq \int\limits_{t_0}^{t} \varphi(s)e^{\int\limits_{t_0}^{s} p(r)dr} ds$$

from which it implies that

$$-\int\limits_{t_0}^{t} \varphi(s)e^{\int\limits_{t_0}^{s} p(r)dr} ds \leq e^{\int\limits_{t_0}^{t} p(r)dr} y(t) - \int\limits_{t_0}^{t} e^{\int\limits_{t_0}^{s} p(r)dr} q(t)y^\alpha(t)ds \leq \int\limits_{t_0}^{t} \varphi(s)e^{\int\limits_{t_0}^{s} p(r)dr} ds. \tag{3.15}$$

Multiplying (3.15) by $e^{-\int\limits_{t_0}^{t} p(r)dr}$, we obtain

$$-\int\limits_{t_0}^{t} \varphi(s)e^{-\int\limits_{s}^{t} p(r)dr} ds \leq y(t) - \int\limits_{t_0}^{t} e^{-\int\limits_{s}^{t} p(r)dr} q(t)y^\alpha(t)ds \leq \int\limits_{t_0}^{t} \varphi(s)e^{-\int\limits_{s}^{t} p(r)dr} ds. \tag{3.16}$$

By boundedness assumption on the solution $y(t)$ and by applying the mean value theorem to the second integral in (3.16), we get an estimate

$$|y(t)| \geq \left| \int_{t_0}^{t} e^{-\int_s^t p(r)dr} q(t) y^{\alpha}(t) ds \right| - \left| \int_{t_0}^{t} \varphi(s) e^{-\int_s^t p(r)dr} ds \right|$$
$$= q(s^*) \left|y(s^*)\right|^{\alpha} \int_{t_0}^{t} e^{-\int_s^t p(r)dr} ds - \int_{t_0}^{t} \varphi(s) e^{-\int_s^t p(r)dr} ds$$
$$\geq m^{\alpha} q(s^*) \int_{t_0}^{t} e^{-\int_s^t p(r)dr} ds - \int_{t_0}^{t} \varphi(s) e^{-\int_s^t p(r)dr} ds,$$

where $0 < m \leq \inf\{|y(t)| : t \geq t_0\}$, $s^* \in [t_0, t]$, and α is a negative real number not equal to even-degree root.

From condition (c) it follows that $\inf\{|y(t)| : t \geq t_0\}$ keeps away from zero and so we can write

$$m^{-\alpha} |y(t)| \geq q(s^*) \int_{t_0}^{t} e^{-\int_s^t p(r)dr} ds - m^{-\alpha} \int_{t_0}^{t} \varphi(s) e^{-\int_s^t p(r)dr} ds, \ -\alpha > 0. \tag{3.17}$$

Then from (3.17) and by (a) and (b) we find that $\sup\limits_{t \geq t_0} |y(t)|$ becomes infinite as $t \to \infty$. We get a contradiction and instability is proved.

Now, we will consider a first order nonlinear differential equation of the form

$$(p(t)y)' = G(t, y) \tag{3.18}$$

with initial condition

$$y(t_0) = y_0 \tag{3.19}$$

where $p(t)$ is a positive continuous and $\lim\limits_{t \to \infty} p(t) \neq 0$. □

Theorem 3.7. *Let $y \in C^1(I)$ and $p(t)$ be a positive continuous functions on I. Assume that*

a) $\lim\limits_{t \to \infty} \frac{1}{p(t)} \int_{t_0}^{t} \varphi(s) ds < \infty$;

b) $\lim\limits_{t \to \infty} \frac{1}{p(t)} \int_{t_0}^{t} \gamma(s) ds = \infty$;

c) $\lim\limits_{t \to \infty} p(t) \neq 0$.

Then the initial value problem (3.18), (3.19) is unstable in the sense of HUR.

Proof. Suppose that $y \in C^1(I)$ satisfies the following inequality (3.20) and the initial condition $y(0) = 0$.We will show that zero solution $z_0(t) \equiv 0$ of the equation (3.18) will satisfy the inequality $|y(t) - z_0(t)| > k\varphi$. On the contrary, let us assume that there exists $\varphi > 0$ such that $\sup_{t \geq t_0} |y(t) - z_0(t)| \leq k\varphi$. Then we can find a constant $M > 0$ such that $M = \sup_{t \geq t_0} |y(t)|$.

Integrating the inequality with respect to t

$$-\varphi(t) \leq (p(t)y)' - G(t, y(t) \leq \varphi(t) \tag{3.20}$$

we get

$$-\int_{t_0}^{t} \varphi(s)ds \leq p(t)y(t) - p(t_0)y(t_0) - \int_{t_0}^{t} G(s, y(s))ds \leq \int_{t_0}^{t} \varphi(s)ds. \tag{3.21}$$

Dividing the inequality by $p(t) > 0$ and by applying the mean value theorem to the integral in (3.21), we get

$$\begin{aligned} |y(t)| &\geq \left| \frac{1}{p(t)} \int_{t_0}^{t} G(s, y(s))ds \right| - \left| \frac{p(t_0)y(t_0)}{p(t)} \right| - \frac{1}{p(t)} \left| \int_{t_0}^{t} \varphi(s)ds \right| \\ &\geq \frac{1}{p(t)} |y(s^*)|^{\beta} \int_{t_0}^{t} \gamma(s)ds - \frac{1}{p(t)} \int_{t_0}^{t} \varphi(s)ds - \left| \frac{p(t_0)y(t_0)}{p(t)} \right|, \end{aligned}$$

where $s^* \in [t_0, t]$.

By boundedness assumption on the solution $y(t)$,$|\widehat{y}(s^*)|^{\beta}$, $\beta > 0$ will be a constant. Therefore, in view of (a), (b) and (c), we find that $\sup_{t \geq t_0} |y(t)|$ becomes infinite as $t \to \infty$, a contradiction and instability is established. □

Example 3.8. Consider the equation

$$(p(t)y)' = t^3 y^2, \quad y(0) = y_0, \ t \geq 0 \tag{3.22}$$

with $p(t) = t^2 + 1$ and $\varphi(t) = e^{-t}$

$$\left| \left(\left(t^2 \ +1 \right) y \right)' - t^3 y^2 \right| \leq e^{-t}. \tag{3.23}$$

Using the same argument used above, we obtain

$$\left| y - \frac{y_0}{t^2 \ +1} - \frac{1}{t^2 \ +1} \left(\int_0^t s^3 y^2 ds \right) ds \right| \leq \frac{e^{-t} - 1}{t^2 \ +1} \tag{3.24}$$

Applying the mean value theorem for integral in (3.14), we obtain the estimate

$$\begin{aligned} |y(t)| &\geq \left| \frac{1}{t^2 \ +1} \left(\int_0^t s^3 y^2 ds \right) \right| - \left| \frac{y_0}{t^2 \ +1} \right| - \frac{e^{-t} - 1}{t^2 \ +1} \\ &\geq |y(s^*)|^2 \left(\frac{1}{t^2 \ +1} \int_0^t s^3 ds \right) - \left| \frac{y_0}{t^2 \ +1} \right| - \frac{e^{-t} - 1}{t^2 \ +1} \\ &= \frac{1}{4} |y(s^*)|^2 \left(\frac{t^4}{t^2 \ +1} \right) - \left| \frac{y_0}{t^2 \ +1} \right| - \frac{e^{-t} - 1}{t^2 \ +1}. \end{aligned} \tag{3.25}$$

Now, since $\lim\limits_{t \to \infty} \left(\frac{e^{-t} - 1}{t^2 \ +1} \right) = 0$ and $\lim\limits_{t \to \infty} \left(\frac{t^4}{t^2 \ +1} \right) = \infty$ then from (3.25) we get $y(t) \to \infty$, as $t \to \infty$. The contradiction proves the instability of equation (3.22).

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