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# GENERALIZATIONS OF THE ALEXANDER INTEGRAL OPERATOR FOR ANALYTIC MULTIVALENT FUNCTIONS

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Abstract. Let $T_{p,n}$ be a subclass of analytic multivalent functions of the form

$$f(z) = z^p + a_{p+n}z^{p+n} + a_{p+n+1}z^{p+n+1} + \dots$$

for every $z$ in the open unit disc $\mathbb{U}$. Applying the fractional calculus (fractional integral and fractional derivative), $A_{p,n}^{-\lambda}f(z)$ and $A_{p,n}^{\lambda}f(z)$ which are generalizations of the Alexander integral operator are introduced. The object of present paper is to discuss some interesting properties of $A_{p,n}^{-\lambda}f(z)$ and $A_{p,n}^{\lambda}f(z)$. Also, some simple examples of results for $A_{p,n}^{-\lambda}f(z)$ and $A_{p,n}^{\lambda}f(z)$ are shown. To give some simple examples is very important for the research of mathematics.

## 1. Introduction

Let $T_{p,n}$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad , \quad n \in \mathbb{N} = \{1, 2, 3, \dots\} \tag{1.1}$$

which are analytic multivalent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in T_{1,1}$, Alexander [1] defined the following integral operator

$$A_{1,1}f(z) = \int_0^z \frac{f(t)}{t}dt = z + \sum_{k=2}^{\infty} \frac{a_k}{k} z^k, \tag{1.2}$$

that is called as the Alexander integral operator. Applying the above the Alexander integral operator for $f(z) \in T_{p,n}$, we consider the generalization of $A_{1,1}^{-1}f(z)$ as follows:

$$A_{p,n}^{-1}f(z) = \frac{p+1}{2} z^{\frac{p-1}{2}} \int_0^z \frac{f(t)}{t^{\frac{p+1}{2}}}dt = z^p + \sum_{k=p+n}^{\infty} \frac{p+1}{2k-p+1} a_k z^k. \tag{1.3}$$

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Further, we define new operator $A^1_{p,n}f(z)$ for $T_{p,n}$ using the derivative of $f(z) \in T_{p,n}$ as follows:

$$A^1_{p,n}f(z) = \frac{2}{p+1} z^{\frac{p+1}{2}} \frac{d}{dz}\left(\frac{f(z)}{z^{\frac{p-1}{2}}}\right) = z^p + \sum_{k=p+n}^{\infty} \frac{2k-p+1}{p+1} a_k z^k. \tag{1.4}$$

With the above operators $A^{-1}_{p,n}f(z)$ and $A^1_{p,n}f(z)$, we see that

$$A^{-1}_{p,n}\left(A^1_{p,n}f(z)\right) = A^1_{p,n}\left(A^{-1}_{p,n}f(z)\right) = f(z), \tag{1.5}$$

that is, that $A^{-1}_{p,n}f(z)$ and $A^1_{p,n}f(z)$ are inverse operators each other. From the among various definitions of $f(z) \in T_{p,n}$ for fractional calculus (that is, fractional derivatives and fractional integrals) given in the literature, we would like to recall here the following definitions for fractional calculus which were used by S.Owa [8] and S.Owa and H.M.Srivastava [9].

**Definition 1.1.** The fractional integral of order $\lambda$ for $f(z) \in T_{p,n}$ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \ , \ (\lambda > 0)$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $z-t > 0$ and $\Gamma$ is the Gamma function.

With the above definition, we have that

$$D_z^{-\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)} z^{p+\lambda} + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda},$$

for $\lambda > 0$ and $f(z) \in T_{p,n}$.

**Definition 1.2.** The fractional derivative of order $\lambda$ for $f(z) \in T_{p,n}$ is defined by

$$D_z^{\lambda} f(z) = \frac{d}{dz}\left(D_z^{\lambda-1} f(z)\right) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt \ , \ (0 \leq \lambda < 1)$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.1.

**Definition 1.3.** Under the hypotheses of Definition 1.2, the fractional derivative of order $j+\lambda$ for $f(z) \in T_{p,n}$ is defined by

$$D_z^{j+\lambda} f(z) = \frac{d^j}{dz^j}\left(D_z^{\lambda} f(z)\right) \ , \ (0 \leq \lambda < 1)$$

where $j = 0, 1, 2, ..., p-1$.

In view of the definitions for fractional derivatives of $f(z) \in T_{p,n}$, we know that

$$D_z^{\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} z^{p-\lambda} + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda}$$

and

$$D_z^{j+\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-j-\lambda)} z^{p-j-\lambda} + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-j-\lambda)} a_k z^{k-j-\lambda}$$

for $f(z) \in T_{p,n}$ with $0 \leq \lambda < 1$ and $j = 0, 1, 2, ..., p-1$.

Using the fractional calculus $D_z^{-\lambda} f(z)$ and $D_z^{\lambda} f(z)$, we introduce

$$\begin{aligned} A_{p,n}^{-\lambda} f(z) &= \frac{\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma\left(\frac{p+2-\lambda}{2}\right)} z^{\frac{p-\lambda}{2}} D_z^{-\lambda} \left( \frac{f(z)}{z^{\frac{p+\lambda}{2}}} \right) \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)} a_k z^k \end{aligned} \tag{1.6}$$

for $\lambda > 0$,

$$\begin{aligned} A_{p,n}^{\lambda} f(z) &= \frac{\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma\left(\frac{p+2+\lambda}{2}\right)} z^{\frac{p+\lambda}{2}} D_z^{\lambda} \left( \frac{f(z)}{z^{\frac{p-\lambda}{2}}} \right) \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)} a_k z^k \end{aligned} \tag{1.7}$$

for $0 \leq \lambda < 1$,

$$\begin{aligned} A_{p,n}^{j+\lambda} f(z) &= \frac{\Gamma\left(\frac{p+2-j-\lambda}{2}\right)}{\Gamma\left(\frac{p+2+j+\lambda}{2}\right)} z^{\frac{p+j+\lambda}{2}} D_z^{j+\lambda} \left( \frac{f(z)}{z^{\frac{p-j-\lambda}{2}}} \right) \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p-j-\lambda}{2})\Gamma\left(\frac{p+2-j-\lambda}{2}\right)}{\Gamma(k+1-\frac{p+j+\lambda}{2})\Gamma\left(\frac{p+2+j+\lambda}{2}\right)} a_k z^k \end{aligned} \tag{1.8}$$

for $0 \leq \lambda < 1$ and $j = 0, 1, 2, ..., p-1$. If we take $\lambda = 1$ in (1.6), then

$$A_{p,n}^{-1} f(z) = \frac{p+1}{2} z^{\frac{p-1}{2}} \int_0^z \frac{f(t)}{t^{\frac{p+1}{2}}} dt,$$

which is the same as (1.3). Letting $\lambda \to 0$ in (1.6), we see

$$A_{p,n}^{0} f(z) = f(z).$$

Further, if $\lambda = 0$ in (1.7), we have

$$A_{p,n}^{0} f(z) = f(z),$$

and if $\lambda \to 1$ in (1.7), then we know that

$$A^1_{p,n}f(z) = \frac{2}{p+1} z^{\frac{p+1}{2}} \frac{d}{dz}\left(\frac{f(z)}{z^{\frac{p-1}{2}}}\right),$$

which is the same as (1.4). In view of the definition for $A^{j+\lambda}_{p,n}f(z)$ in (1.8), we say that

$$A^{\lambda}_{p,n}f(z) = \frac{\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma\left(\frac{p+2+\lambda}{2}\right)} z^{\frac{p+\lambda}{2}} D^{\lambda}_z \left(\frac{f(z)}{z^{\frac{p-\lambda}{2}}}\right) \tag{1.9}$$

for $\lambda \geq 0$. Thus by (1.6) and (1.9), we have that

$$A^{\lambda}_{p,n}\left(A^{-\lambda}_{p,n}f(z)\right) = A^{-\lambda}_{p,n}\left(A^{\lambda}_{p,n}f(z)\right) = f(z),$$

for $\lambda \geq 0$. This means that $A^{\lambda}_{p,n}f(z)$ and $A^{-\lambda}_{p,n}f(z)$ are inverse operators each other.

## 2. Dominants for the operators $A^{\lambda}_{p,1}$ and $A^{-\lambda}_{p,1}$

Let a function $g(z) \in T_{p,n}$ be given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k, \tag{2.1}$$

with $b_k \geq 0$ $(k = p+n, p+n+1, ...)$. Functions $f(z) \in T_{p,n}$ and $g(z)$ given by (2.1) satisfy

$$|a_k| \leq b_k, \quad (k = p+n, p+n+1, ...),$$

then $f(z)$ is said to be dominanted by $g(z)$ ( or $g(z)$ dominants $f(z)$), and we write

$$f(z) \ll g(z)$$

for all $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In view of (1.3) and (1.4), we have

$$A^{-1}_{p,n}f(z) \ll f(z) \ll A^1_{p,n}f(z), \quad (z \in \mathbb{U})$$

for $f(z) \in T_{p,n}$ with $a_k \geq 0$. Furthermore, by the continuty of the gamma functions for $\lambda$, we say that

$$A^{-\lambda}_{p,n}f(z) \ll f(z) \ll A^{\lambda}_{p,n}f(z), \quad (z \in \mathbb{U})$$

for $0 \leq \lambda < p$.

Now we derive the following theorem.

**Theorem 2.1.** *If* $f(z) \in T_{p,1}$ *satisfies*

$$|a_k| \leq \frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)\prod_{j=0}^{k-p-1}(2p-2\alpha-j)}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)(k-p)!}, \tag{2.2}$$

*for* $k = p+1, p+2, ...,$ *then*

$$A_{p,1}^{-\lambda} f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}}, \quad (z \in \mathbb{U}) \tag{2.3}$$

*where* $0 \leq \alpha < p$.

*If* $f(z) \in T_{p,1}$ *satisfies*

$$|a_k| \leq \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)\prod_{j=0}^{k-p-1}(2p-2\alpha-j)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)(k-p)!}, \tag{2.4}$$

*for* $k = p+1, p+2, ...,$ *then*

$$A_{p,1}^{\lambda} f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}}, \quad (z \in \mathbb{U}) \tag{2.5}$$

*where* $0 \leq \alpha < p$.

*Proof.* Let us consider a function

$$g(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} = z^p + \sum_{k=p+1}^{\infty} \frac{\prod_{j=0}^{k-p-1}(2p-2\alpha-j)}{(k-p)!} z^k, \quad (0 \leq \alpha < p).$$

Since $f(z) \in T_{p,1}$ implies that

$$A_{p,1}^{-\lambda} f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)} a_k z^k,$$

if $f(z)$ satisfies (2.2), then

$$A_{p,1}^{-\lambda} f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}}, \quad (z \in \mathbb{U}).$$

Further, since $f(z) \in T_{p,1}$ satisfies

$$A_{p,1}^{\lambda} f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)} a_k z^k,$$

if $f(z)$ satisfies (2.4), we say that

$$A_{p,1}^{\lambda} f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}}, \quad (z \in \mathbb{U}).$$

□

**Remark 1.** *We consider the function*

$$g(z)=\frac{z^p}{(1-z)^{2(p-\alpha)}},\quad (0\leq \alpha<p).$$

*This function $g(z)$ satisfies*

$$Re\frac{zg'(z)}{g(z)}=Re\left(p+2(p-\alpha)\frac{z}{1-z}\right)>\alpha,\quad (z\in\mathbb{U}).$$

*Therefore, we say that the function $g(z)$ is $p$-valently starlike of order $\alpha$ in $\mathbb{U}$, and write that $g(z)\in\mathcal{S}^*(\alpha)$.*

Taking $\alpha=0$ in Theorem 2.1, we have the following corollary.

**Corollary 2.1.** *If $f(z)\in T_{p,1}$ satisfies*

$$|a_k|\leq\frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)\prod_{j=0}^{k-p-1}(2p-j)}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)(k-p)!},$$

*for $k=p+1,p+2,...,$ then*

$$A_{p,1}^{-\lambda}f(z)\ll\frac{z^p}{(1-z)^{2p}},\quad (z\in\mathbb{U}).$$

*If $f(z)\in T_{p,1}$ satisfies*

$$|a_k|\leq\frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)\prod_{j=0}^{k-p-1}(2p-j)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)(k-p)!},$$

*for $k=p+1,p+2,...,$ then*

$$A_{p,1}^{\lambda}f(z)\ll\frac{z^p}{(1-z)^{2p}},\quad (z\in\mathbb{U}).\tag{2.6}$$

Taking $\lambda=1$ in Theorem 2.1, we have the following corollary.

**Corollary 2.2.** *If $f(z)\in T_{p,1}$ satisfies*

$$|a_k|\leq\frac{(2k+1-p)\prod_{j=0}^{k-p-1}(2p-2\alpha-j)}{(k-p)!(p+1)},$$

*for $k=p+1,p+2,...,$ then*

$$A_{p,1}^{-1}f(z)\ll\frac{z^p}{(1-z)^{2(p-\alpha)}},\quad (z\in\mathbb{U})$$

*where $0\leq\alpha<p$.*

*If* $f(z) \in T_{p,1}$ *satisfies*

$$|a_k| \leq \frac{(p+1)\prod_{j=0}^{k-p-1}(2p-2\alpha-j)}{(k-p)!(2k+1-p)},$$

*for* $k = p+1, p+2, ...,$ *then*

$$A^1_{p,1}f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}}, \quad (z \in \mathbb{U})$$

*where* $0 \leq \alpha < p$.

Next, we derive the following theorem.

**Theorem 2.2.** *If* $f(z) \in T_{p,1}$ *satisfies*

$$|a_k| \leq \frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}, \tag{2.7}$$

*for* $k = p+1, p+2, ...,$ *then*

$$A^{-\lambda}_{p,1}f(z) \ll \frac{z^p}{1-z}, \quad (z \in \mathbb{U}). \tag{2.8}$$

*If* $f(z) \in T_{p,1}$ *satisfies*

$$|a_k| \leq \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}, \tag{2.9}$$

*for* $k = p+1, p+2, ...,$ *then*

$$A^{\lambda}_{p,1}f(z) \ll \frac{z^p}{1-z}, \quad (z \in \mathbb{U}). \tag{2.10}$$

*Proof.* Note that

$$g(z) = \frac{z^p}{1-z} = z^p + \sum_{k=p+1}^{\infty} z^k.$$

By means of the expansion for $A^{-\lambda}_{p,1}f(z)$ in (1.6), we know the dominant (2.8). Also, using (1.7), we have the dominant (2.10). □

**Remark 2.** *Let us consider the function*

$$g(z) = \frac{z^p}{1-z} = z^p + \sum_{k=p+1}^{\infty} z^k. \tag{2.11}$$

*This function* $g(z)$ *satisfies*

$$\frac{zg'(z)}{g(z)} = p + \frac{z}{1-z} \tag{2.12}$$

*and*

$$1 + \frac{zg''(z)}{g'(z)} = p + \frac{2z}{1-z} - \frac{(p-1)z}{p-(p-1)z}. \tag{2.13}$$

*It follows from (2.12) that*

$$Re\frac{zg'(z)}{g(z)} > p - \frac{1}{2}, \quad (z \in \mathbb{U}) \tag{2.14}$$

*and from (2.13)*

$$Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0, \quad (z \in \mathbb{U}). \tag{2.15}$$

*The inequality (2.14) means that $g(z)$ is p-valently starlike of order $p - \frac{1}{2}$, and the inequality (2.15) shows that $g(z)$ is p-valently convex in $\mathbb{U}$.*

Making $\lambda = 1$ in Theorem 2.2, we see the following corollary.

**Corollary 2.3.** *If $f(z) \in T_{p,1}$ satisfies*

$$|a_k| \leq \frac{2k+1-p}{p+1}, \quad (k = p+1, p+2, ...)$$

*then*

$$A_{p,1}^{-1} f(z) \ll \frac{z^p}{1-z}, \quad (z \in \mathbb{U})$$

*and if $f \in T_{p,1}$ satisfies*

$$|a_k| \leq \frac{p+1}{2k+1-p}, \quad (k = p+1, p+2, ...),$$

*then*

$$A_{p,1}^{1} f(z) \ll \frac{z^p}{1-z}, \quad (z \in \mathbb{U}).$$

Further we have the following theorem.

**Theorem 2.3.** *Let a function $f(z) \in T_{p,n}$ be given by*

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (n = 2p, 4p, 6p, ...). \tag{2.16}$$

*If $f(z)$ satisfies*

$$|a_k| \leq \frac{p\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}{k\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}, \tag{2.17}$$

*for* $k = 3p, 5p, 7p, ...,$ *then*

$$A_{p,1}^{-\lambda} f(z) \ll \frac{1}{2} log\left(\frac{1+z^p}{1-z^p}\right), \quad (z \in \mathbb{U}). \tag{2.18}$$

*If* $f(z)$ *satisfies*

$$|a_k| \leq \frac{p\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{k\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}, \tag{2.19}$$

*for* $k = 3p, 5p, 7p, ...,$ *then*

$$A_{p,1}^{\lambda} f(z) \ll \frac{1}{2} log\left(\frac{1+z^p}{1-z^p}\right), \quad (z \in \mathbb{U}). \tag{2.20}$$

*Proof.* We note that

$$\frac{1}{2} log\left(\frac{1+z^p}{1-z^p}\right) = z^p + \frac{1}{3}z^{3p} + \frac{1}{5}z^{5p} + ... = z^p + \sum_{k=p+n}^{\infty} \frac{p}{k} z^k. \tag{2.21}$$

Considering (1.6) and (1.7), we know that if $|a_k|$ satisfies (2.17) and (2.19), then we have (2.18) and (2.20). □

**Remark 3.** *We write that*

$$g(z) = \frac{1}{2} log\left(\frac{1+z^p}{1-z^p}\right).$$

*Then* $g(z)$ *satisfies*

$$Re\left(1 + \frac{zg''(z)}{g'(z)}\right) = Re\left(\frac{2p}{1-z^{2p}} - p\right) > 0, \quad (z \in \mathbb{U}).$$

*Therefore, the function* $g(z)$ *is p-valently convex in* $\mathbb{U}$. *Also,* $g(z)$ *satisfies* $-\frac{\pi}{4} < Img(z) < \frac{\pi}{4}$, $(z \in \mathbb{U})$.

Taking $\lambda = 1$ in Theorem 2.3, we know the following corollary.

**Corollary 2.4.** *Let a function* $f(z)$ *be given by (2.16). If* $f(z)$ *satisfies*

$$|a_k| \leq \frac{p(2k+1-p)}{k(p+1)}, \quad (k = 3p, 5p, 7p, ...),$$

*then*

$$A_{p,1}^{-1} f(z) \ll \frac{1}{2} log\left(\frac{1+z^p}{1-z^p}\right), \quad (z \in \mathbb{U}).$$

*If* $f(z)$ *satisfies*

$$|a_k| \leq \frac{p(p+1)}{k(2k+1-p)}, \quad (k = 3p, 5p, 7p, ...)$$

*for* $k = 3p, 5p, 7p, ...,$ *then*

$$A^1_{p,1}f(z) \ll \frac{1}{2}log\left(\frac{1+z^p}{1-z^p}\right), \quad (z \in \mathbb{U}).$$

## 3. Inclusion properties for the operators $A^{\lambda}_{p,1}$ and $A^{-\lambda}_{p,1}$

We would like to consider some interesting properties for operators $A^{-\lambda}_{p,n}f(z)$ and $A^{\lambda}_{p,n}f(z)$.

For $m$ different boundary points $z_s$ $(s = 1, 2, 3, ..., m)$ with $|z_s| = 1$, we define

$$d_m = \frac{1}{m}\sum_{s=1}^{m}\frac{A^{-\lambda}_{p,n}f(z_s)}{z_s^p}, \tag{3.1}$$

where $d_m \in e^{i\beta}A^{-\lambda}_{p,n}f(\mathbb{U}), d_m \neq 1$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$. For such $d_m$, if $f(z) \in T_{p,n}$ satisfies

$$\left|\frac{e^{i\beta}\frac{A^{-\lambda}_{p,n}f(z)}{z^p} - d_m}{e^{i\beta} - d_m} - 1\right| < \rho, \quad z \in \mathbb{U} \tag{3.2}$$

for some real $\rho > 0$, then we say that $f(z)$ belongs to $B^{-\lambda}_{p,n}(d_m, \beta, \rho) \subset T_{p,n}$. We note that our condition (3.2) is equivalent to

$$\left|\frac{A^{-\lambda}_{p,n}f(z)}{z^p} - 1\right| < \rho\left|e^{i\beta} - d_m\right|, \quad z \in \mathbb{U}. \tag{3.3}$$

Also, we define $e_m$ for $A^{\lambda}_{p,n}f(z)$ changing $\lambda$ by $-\lambda$ in $d_m$. If $f(z) \in T_{p,n}$ satisfies

$$\left|\frac{e^{i\beta}\frac{A^{\lambda}_{p,n}f(z)}{z^p} - e_m}{e^{i\beta} - e_m} - 1\right| < \rho, \quad z \in \mathbb{U} \tag{3.4}$$

for some real $\rho > 0$, then we say that $f(z)$ belongs to $B^{\lambda}_{p,n}(e_m, \beta, \rho) \subset T_{p,n}$. The above inequality (3.4) is equivalent to

$$\left|\frac{A^{\lambda}_{p,n}f(z)}{z^p} - 1\right| < \rho\left|e^{i\beta} - e_m\right|, \quad z \in \mathbb{U}.$$

Let us define $f(z) \in T_{p,n}$ by

$$f(z) = z^p + a_{p+n}z^{p+n}.$$

Then, if

$$|a_{p+n}| \leq \frac{\Gamma\left(\frac{p+2n-2+\lambda}{2}\right)\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma\left(\frac{p+2n-2-\lambda}{2}\right)\Gamma\left(\frac{p+2+\lambda}{2}\right)}\rho\left|e^{i\beta} - d_m\right|,$$

then $f \in B^{-\lambda}_{p,n}(d_m, \beta, \rho)$, and if

$$|a_{p+n}| \leq \frac{\Gamma\left(\frac{p+2n-2-\lambda}{2}\right)\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma\left(\frac{p+2n-2+\lambda}{2}\right)\Gamma\left(\frac{p+2-\lambda}{2}\right)}\rho\left|e^{i\beta} - e_m\right|,$$

then $f \in B^{\lambda}_{p,n}(e_m, \beta, \rho)$. Discussing our problems for $B^{-\lambda}_{p,n}(d_m, \beta, \rho)$ and $B^{\lambda}_{p,n}(e_m, \beta, \rho)$, we need the following lemma due to Miller and Mocanu [3, 4](also, by Jack [2]).

**Lemma 3.1.** *Let the function* $w(z)$ *given by*

$$w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad , \quad n \in \mathbb{N}$$

*be analytic in* $\mathbb{U}$ *with* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ *at a point* $z_0, (0 < |z_0| < 1)$ *then there exists a real number* $k \geq n$ *such that*

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

*and*

$$Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k.$$

Applying the above lemma, we derive the following theorem.

**Theorem 3.1.** *If* $f(z) \in T_{p,n}$ *satisfies*

$$\left|\frac{z(A^{-\lambda}_{p,n}f(z))'}{A^{-\lambda}_{p,n}f(z)} - p\right| < \frac{\left|e^{i\beta} - d_m\right|\rho}{1 + |e^{i\beta} - d_m|\rho}, \quad z \in \mathbb{U} \tag{3.5}$$

*for some* $d_m$ *defined by (3.1) with* $d_m \neq 1$ *such that* $z_s \in \partial\mathbb{U}$ $(s = 1, 2, 3, ..., m)$*, and for some real* $\rho > 1$*, then*

$$\left|\frac{A^{-\lambda}_{p,n}f(z)}{z^p} - 1\right| < \rho\left|e^{i\beta} - d_m\right|, \quad z \in \mathbb{U} \tag{3.6}$$

*that is,* $f \in B^{-\lambda}_{p,n}(d_m, \beta, \rho)$.

*Proof.* We define the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{e^{i\beta}\frac{A^{-\lambda}_{p,n}f(z)}{z^p} - d_m}{e^{i\beta} - d_m} - 1 \\ &= \frac{e^{i\beta}}{e^{i\beta} - d_m}\left(\sum_{k=p+n}^{\infty}\frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}a_k z^{k-p}\right). \end{aligned}$$

Then, we know that $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and

$$\frac{A_{p,n}^{-\lambda} f(z)}{z^p} = 1 + (1 - e^{-i\beta} d_m) w(z).$$

This gives us that

$$\frac{z(A_{p,n}^{-\lambda} f(z))'}{A_{p,n}^{-\lambda} f(z)} - p = \frac{(1 - e^{-i\beta} d_m) z w'(z)}{1 + (1 - e^{-i\beta} d_m) w(z)}.$$

that is, that

$$\left| \frac{z(A_{p,n}^{-\lambda} f(z))'}{A_{p,n}^{-\lambda} f(z)} - p \right| = \left| \frac{(1 - e^{-i\beta} d_m) z w'(z)}{1 + (1 - e^{-i\beta} d_m) w(z)} \right| < \frac{|e^{i\beta} - d_m|\rho}{1 + |e^{i\beta} - d_m|\rho}, \quad z \in \mathbb{U}$$

with (3.5). Assume that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1.$$

Then, using Lemma 3.1, we say that $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = k w(z_0), (k \geq n)$. For such $z_0 \in \mathbb{U}$, we have

$$\begin{aligned} \left| \frac{z_0(A_{p,n}^{-\lambda} f(z_0))'}{A_{p,n}^{-\lambda} f(z_0)} - p \right| &= \left| \frac{(1 - e^{-i\beta} d_m) z_0 w'(z_0)}{1 + (1 - e^{-i\beta} d_m) w(z_0)} \right| \\ &= \left| \frac{(1 - e^{-i\beta} d_m)\rho}{1 + (1 - e^{-i\beta} d_m)\rho e^{i\theta}} \right| \\ &\geq \frac{|1 - e^{-i\beta} d_m|\rho}{1 + |1 - e^{-i\beta} d_m|\rho} \\ &= \frac{|e^{i\beta} - d_m|\rho}{1 + |e^{i\beta} - d_m|\rho}. \end{aligned} \tag{3.7}$$

This contradicts our condition (3.5). Thus we say that there exists no $z_0, (0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. This gives that $|w(z)| < \rho$ for all $z \in \mathbb{U}$, and that

$$|w(z)| = \left| \frac{e^{i\beta}\left(\frac{A_{p,n}^{-\lambda} f(z)}{z} - 1\right)}{e^{i\beta} - d_m} \right| < \rho, \quad z \in \mathbb{U}.$$

This is equivalent to (3.6)and to $f \in B_{p,n}^{-\lambda}(d_m, \beta, \rho)$. □

**Example 3.1.** *We consider the function $f(z) \in T_{p,n}$ given by*

$$f(z) = z^p + a_{p+n} z^{p+n}.$$

*Then,* $f(z)$ *satisfies*

$$\left|\frac{z(A_{p,n}^{-\lambda}f(z))'}{A_{p,n}^{-\lambda}f(z)}-p\right|=\left|\frac{nV(p,n,\lambda)a_{p+n}z^n}{1+V(p,n,\lambda)a_{p+n}z^n}\right|<\frac{nV(p,n,\lambda)|a_{p+n}|}{1-V(p,n,\lambda)|a_{p+n}|},\quad z\in\mathbb{U}$$

*where*

$$V(p,n,\lambda)=\frac{\Gamma\left(\frac{p+2n+2-\lambda}{2}\right)\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma\left(\frac{p+2n+2+\lambda}{2}\right)\Gamma\left(\frac{p+2-\lambda}{2}\right)}$$

*and*

$$0<|a_{p+n}|<\frac{1}{(n+1)V(p,n,\lambda)}.$$

*Now we consider five boundary points*

$$z_1=e^{-i\frac{arg(a_{p+n})}{n}},$$
$$z_2=e^{i\frac{\pi-6arg(a_{p+n})}{6n}},$$
$$z_3=e^{i\frac{\pi-4arg(a_{p+n})}{4n}},$$
$$z_4=e^{i\frac{\pi-3arg(a_{p+n})}{3n}}$$

*and*

$$z_5=e^{i\frac{\pi-2arg(a_{p+n})}{2n}}.$$

*For such points* $z_s$ $(s=1,2,3,4,5)$*, we have*

$$\frac{A_{p,n}^{-\lambda}f(z_1)}{z_1^p}=1+V(p,n,\lambda)|a_{p+n}|,$$
$$\frac{A_{p,n}^{-\lambda}f(z_2)}{z_2^p}=1+V(p,n,\lambda)|a_{p+n}|\frac{\sqrt{3}+i}{2},$$
$$\frac{A_{p,n}^{-\lambda}f(z_3)}{z_3^p}=1+V(p,n,\lambda)|a_{p+n}|\frac{\sqrt{2}(1+i)}{2},$$
$$\frac{A_{p,n}^{-\lambda}f(z_4)}{z_4^p}=1+V(p,n,\lambda)|a_{p+n}|\frac{1+\sqrt{3}i}{2}$$

*and*

$$\frac{A_{p,n}^{-\lambda}f(z_5)}{z_5^p}=1+V(p,n,\lambda)|a_{p+n}|i.$$

*Thus, we obtain that*

$$d_5=\frac{1}{5}\sum_{s=1}^{5}\frac{A_{p,n}^{-\lambda}f(z_s)}{z_s^p}=1+\frac{1}{10}(3+\sqrt{2}+\sqrt{3})(1+i)V(p,n,\lambda)|a_{p+n}|.$$

*It follows from the above that*

$$\left|1-e^{-i\beta}d_5\right|=\frac{\sqrt{2}}{10}(3+\sqrt{2}+\sqrt{3})V(p,n,\lambda)|a_{p+n}|,$$

*for $\beta = 0$. For such $d_5$ and $\beta = 0$, we consider $\rho > 1$ such that*

$$\frac{nV(p,n,\lambda)|a_{p+n}|}{1 - V(p,n,\lambda)|a_{p+n}|} = \frac{|e^{i\beta} - d_5|\rho}{1 + |e^{i\beta} - d_5|\rho}.$$

*Then $\rho$ satisfies*

$$\rho = \frac{10\sqrt{2}n}{(3 + \sqrt{2} + \sqrt{3})(1 - (n+1)V(p,n,\lambda)|a_{p+n}|)} \geq \frac{10\sqrt{2}n}{3 + \sqrt{2} + \sqrt{3}} > 1.$$

*For such $d_5$ and $\rho > 1$, we see that*

$$\left|\frac{A_{p,n}^{-\lambda} f(z)}{z^p} - 1\right| < V(p,n,\lambda)|a_{p+n}| \leq \rho|e^{i\beta} - d_5| \quad , \quad z \in \mathbb{U}.$$

Letting $\lambda = 1$ in Theorem 3.1, we have the following corollary.

**Corollary 3.1.** *Let $f(z) \in T_{p,n}$ satisfies*

$$\left|\frac{z(A_{p,n}^{-\lambda} f(z))'}{A_{p,n}^{-\lambda} f(z)} - p\right| < \frac{\left|e^{i\beta} - d_m\right| \rho}{1 + |e^{i\beta} - d_m| \rho}, \quad z \in \mathbb{U}$$

*for some $d_m$ given by*

$$d_m = \frac{1}{m} \sum_{s=1}^{m} \frac{A_{p,n}^{-\lambda} f(z_s)}{z_s^p}, \qquad d_m \neq 1$$

*where $d_m \in e^{i\beta} A_{p,n}^{-\lambda} f(\mathbb{U})$ and $z_s \in \partial\mathbb{U}(s = 1, 2, 3, ..., m)$. Then, for some real $\rho > 1$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$*

$$\left|\frac{A_{p,n}^{-\lambda} f(z)}{z^p} - 1\right| < \rho \left|e^{i\beta} - d_m\right|, \quad z \in \mathbb{U}$$

*that is, $f \in B_{p,n}^{-1}(d_m, \beta, \rho)$.*

Finally, we derive the following theorem.

**Theorem 3.2.** *Let $f(z) \in T_{p,n}$ and*

$$e_m = \frac{1}{m} \sum_{s=1}^{m} \frac{A_{p,n}^{\lambda} f(z_s)}{z_s^p}, \qquad e_m \neq 1$$

*where $e_m \in e^{i\beta} A_{p,n}^{\lambda} f(\mathbb{U})$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ where $z_s \in \partial\mathbb{U}(s = 1, 2, 3, ..., m)$. If $f$ satisfies*

$$\left|\frac{z(A_{p,n}^{\lambda} f(z))'}{A_{p,n}^{\lambda} f(z)} - p\right| < \frac{\left|e^{i\beta} - e_m\right| \rho}{1 + |e^{i\beta} - e_m| \rho}, \quad z \in \mathbb{U}$$

*for some real* $\rho > 1$, *then*

$$\left|\frac{A_{p,n}^{\lambda}f(z)}{z^p} - p\right| < \rho\left|e^{i\beta} - e_m\right|, \quad z \in \mathbb{U}$$

*that is,* $f \in B_{p,n}^{\lambda}(d_m, \beta, \rho)$.

The proof of the Theorem 3.2 is same as the proof of Theorem 3.1.

**Example 3.2.** *We define the function* $f(z) \in T_{p,n}$ *given by*

$$f(z) = z^p + a_{p+n}z^{p+n}.$$

*Then,* $f(z)$ *satisfies*

$$\left|\frac{z(A_{p,n}^{\lambda}f(z))'}{A_{p,n}^{\lambda}f(z)} - p\right| = \left|\frac{nW(p,n,\lambda)a_{p+n}z^n}{1 + W(p,n,\lambda)a_{p+n}z^n}\right| < \frac{nW(p,n,\lambda)|a_{p+n}|}{1 - W(p,n,\lambda)|a_{p+n}|}, \quad z \in \mathbb{U}$$

*where*

$$W(p,n,\lambda) = \frac{\Gamma\left(\frac{p+2n+2+\lambda}{2}\right)\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma\left(\frac{p+2n+2-\lambda}{2}\right)\Gamma\left(\frac{p+2+\lambda}{2}\right)}$$

*and*

$$0 < |a_{p+n}| < \frac{1}{(n+1)W(p,n,\lambda)}.$$

*Considering five boundary points* $z_s(s = 1, 2, 3, 4, 5)$ *same as the Example 3.1, we have*

$$\left|1 - e^{-i\beta}e_5\right| = \frac{\sqrt{2}}{10}(3 + \sqrt{2} + \sqrt{3})W(p,n,\lambda)|a_{p+n}|,$$

*for* $\beta = 0$. *Taking* $\rho > 1$ *such that*

$$\frac{nW(p,n,\lambda)|a_{p+n}|}{1 - W(p,n,\lambda)|a_{p+n}|} = \frac{|e^{i\beta} - e_5|\rho}{1 + |e^{i\beta} - e_5|\rho},$$

*we have*

$$\rho = \frac{10\sqrt{2}n}{(3 + \sqrt{2} + \sqrt{3})(1 - (n+1)W(p,n,\lambda)|a_{p+n}|)} \geq \frac{10\sqrt{2}n}{3 + \sqrt{2} + \sqrt{3}} > 1.$$

*For such* $e_5$ *and* $\rho > 1$, $f$ *satisfies*

$$\left|\frac{A_{p,n}^{\lambda}f(z)}{z^p} - 1\right| < W(p,n,\lambda)|a_{p+n}| \leq \rho|e^{i\beta} - e_5| \quad , \quad z \in \mathbb{U}.$$

Making $\lambda = 1$ in Theorem 3.2, we see the following corollary.

**Corollary 3.2.** *Let* $f(z) \in T_{p,n}$ *satisfies*

$$\left|\frac{z(A^1_{p,n}f(z))'}{A^1_{p,n}f(z)} - p\right| < \frac{\left|e^{i\beta} - e_m\right|\rho}{1 + \left|e^{i\beta} - e_m\right|\rho}, \quad z \in \mathbb{U}$$

*for some* $e_m$ *given by*

$$e_m = \frac{1}{m}\sum_{s=1}^{m}\frac{A^1_{p,n}f(z_s)}{z_s^p}, \qquad d_m \neq 1$$

*where* $e_m \in e^{i\beta}A^1_{p,n}f(\mathbb{U})$ *and* $z_s \in \partial\mathbb{U}(s = 1, 2, 3, ..., m)$. *Then, for some real* $\rho > 1$ *and* $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$

$$\left|\frac{A^1_{p,n}f(z)}{z^p} - 1\right| < \rho\left|e^{i\beta} - e_m\right|, \quad z \in \mathbb{U}$$

*that is,* $f \in B^1_{p,n}(e_m, \beta, \rho)$.

## 4. Subordination Properties

Let the functions $F(z)$ and $G(z)$ be analytic in $\mathbb{U}$. Then $F(z)$ is said to be subordinate to $G(z)$ if there exists a Schwarz function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathbb{U})$ such that

(4.1) $$F(z) = G(w(z)), \quad z \in \mathbb{U}.$$

We denote this subordination by

(4.2) $$F(z) \prec G(z), \quad z \in \mathbb{U}.$$

Further, if $G(z)$ is univalent in $\mathbb{U}$, then the subordination (4.2) is equivalent to $F(0) = G(0)$ and $F(\mathbb{U}) \subset G(\mathbb{U})$.

For $f(z) \in T_{p,n}$, we consider the following subordination

(4.3) $$\frac{A^{-\lambda}_{p,n}f(z)}{z^p} \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}$$

for some real $\alpha > 1$. Many interesting properties for subordinations were considered by Miller and Mocanu [5, 6]. If we write that

$$G(z) = \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}$$

for $\alpha > 1$, then

$$Re(G(z)) = \frac{\alpha(\alpha + r^2 - r(\alpha + 1)cos\theta)}{\alpha^2 + r^2 - 2\alpha r cos\theta},$$

for $z = re^{i\theta}(0 \leq r < 1, 0 \leq \theta < 2\pi)$. This shows that

$$\frac{\alpha(1-r)}{\alpha-r} \leq Re(G(z)) \leq \frac{\alpha(1+r)}{\alpha+r} < \frac{2\alpha}{\alpha+1}. \tag{4.4}$$

Therefore, $f(z)$ defined by (4.3) satisfies

$$0 < Re\left(\frac{A_{p,n}^{-\lambda} f(z)}{z^p}\right) < \frac{2\alpha}{\alpha+1}, \quad z \in \mathbb{U}. \tag{4.5}$$

Now, we derive the following theorem.

**Theorem 4.1.** *If $f(z) \in T_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}|a_k| \leq \frac{\alpha-1}{\alpha+1}, \tag{4.6}$$

*for $\alpha > 1$, then $f(z)$ satisfies (4.5). The equality is attained for*

$$f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)(\alpha-1)\varepsilon}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)(\alpha+1)(k-p-n+1)(k-p-n+2)} z^k, \tag{4.7}$$

*where $|\varepsilon| = 1$.*

*Proof.* We consider a function $f(z) \in T_{p,n}$ which satisfies

$$\left|(\alpha^2-6\alpha+1)+4(\alpha+1)\frac{A_{p,n}^{-\lambda} f(z)}{z^p}\right| < \left|(\alpha^2+10\alpha+1)-4(\alpha+1)\frac{A_{p,n}^{-\lambda} f(z)}{z^p}\right|, \tag{4.8}$$

for $\alpha > 1$. From the inequality (4.8), we know that

$$\frac{A_{p,n}^{-\lambda} f(z)}{z^p} + \overline{\left(\frac{A_{p,n}^{-\lambda} f(z)}{z^p}\right)} < \frac{4\alpha}{\alpha+1}, \quad z \in \mathbb{U}.$$

It follows from the above that

$$0 < Re\left(\frac{A_{p,n}^{-\lambda} f(z)}{z^p}\right) < \frac{2\alpha}{\alpha+1}, \quad z \in \mathbb{U}.$$

If $f(z) \in T_{p,n}$ satisfies (4.8), then we know taht

$$\left|(\alpha^2-2\alpha+5)+4(\alpha+1)\sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)} a_k z^{k-p}\right|$$

$$< \left|(\alpha^2+6\alpha-3)-4(\alpha+1)\sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)} a_k z^{k-p}\right|,$$

where $|\varepsilon| = 1$. Therefore, to satisfy (4.8), $f(z)$ has to satisfy

$$(\alpha+1)\sum_{k=p+n}^{\infty}\frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}|a_k| \leq \alpha - 1.$$

Thus, if $f(z) \in T_{p,n}$ satisfies (4.6), then $f(z)$ satisfies (4.5). Further, we consider a function $f(z)$ given by (4.7). Then, we have

$$\begin{aligned}
&\sum_{k=p+n}^{\infty}\frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}|a_k| \\
&= \sum_{k=p+n}^{\infty}\frac{(\alpha-1)|\varepsilon|}{(\alpha+1)(k-p-n+1)(k-p-n+2)} \\
&= \left(\frac{\alpha-1}{\alpha+1}\right)\sum_{k=p+n}^{\infty}\left(\frac{1}{k-p-n+1}-\frac{1}{k-p-n+2}\right) \\
&= \frac{\alpha-1}{\alpha+1}.
\end{aligned}$$

Therefore, a function $f(z)$ given by (4.7) satisfies the equality in (4.6). □

Letting $\lambda = 1$ in Theorem 4.1, we have the following corollary.

**Corollary 4.1.** *If $f(z) \in T_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty}\frac{p+1}{2k+1-p}|a_k| \leq \frac{\alpha-1}{\alpha+1}, \quad z \in \mathbb{U}$$

*for $\alpha > 1$, then*

$$0 < Re\left(\frac{A_{p,n}^{-1}f(z)}{z^p}\right) < \frac{2\alpha}{\alpha+1}, \quad z \in \mathbb{U}.$$

*The equality holds true for*

$$f(z) = z^p + \sum_{k=p+n}^{\infty}\frac{(2k+1-p)(\alpha-1)\varepsilon}{(p+1)(\alpha+1)(k-p-n+1)(k-p-n+2)}z^k,$$

*where $|\varepsilon| = 1$.*

Next, we consider

$$0 < Re\left(\frac{A_{p,n}^{\lambda}f(z)}{z^p}\right) < \frac{2\alpha}{\alpha+1}, \quad z \in \mathbb{U}. \tag{4.9}$$

for $\alpha > 1$. Then we derive the following theorem.

**Theorem 4.2.** *If $f(z) \in T_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)}{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)}|a_k| \leq \frac{\alpha-1}{\alpha+1},$$

*for $\alpha > 1$, then $f(z)$ satisfies (4.9). The equality is attained for*

$$f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1-\frac{p+\lambda}{2})\Gamma\left(\frac{p+2+\lambda}{2}\right)(\alpha-1)\varepsilon}{\Gamma(k+1-\frac{p-\lambda}{2})\Gamma\left(\frac{p+2-\lambda}{2}\right)(\alpha+1)(k-p-n+1)(k-p-n+2)} z^k,$$

*where $|\varepsilon| = 1$.*

The proof of the theorem is the same as Theorem 4.1. Thus, we omited the proof of the theorem.

Taking $\lambda = 1$ in Theorem 4.2, we have the following corollary.

**Corollary 4.2.** *If $f(z) \in T_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} \frac{2k+1-p}{p+1}|a_k| \leq \frac{\alpha-1}{\alpha+1}, \quad z \in \mathbb{U}$$

*for $\alpha > 1$, then*

$$0 < Re\left(\frac{A_{p,n}^{1} f(z)}{z^p}\right) < \frac{2\alpha}{\alpha+1}, \quad z \in \mathbb{U}.$$

*The equality holds true for*

$$f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p+1)(\alpha-1)|\varepsilon|}{(2k+1-p)(\alpha+1)(k-p-n+1)(k-p-n+2)} z^k,$$

*where $|\varepsilon| = 1$.*

Next, we consider the following theorem.

**Theorem 4.3.** *If $f(z) \in T_{p,n}$ satisfies*

$$Re\left(\frac{z\left(A_{p,n}^{-\lambda} f(z)\right)'}{A_{p,n}^{-\lambda} f(z)}\right) < p + \frac{\alpha-1}{2(\alpha+1)}, \quad z \in \mathbb{U} \tag{4.10}$$

*for some real $\alpha > 1$, then*

$$\frac{A_{p,n}^{-\lambda} f(z)}{z^p} \prec \frac{\alpha(1-z)}{\alpha-z}, \quad z \in \mathbb{U}. \tag{4.11}$$

*Proof.* Consider an analytic function $w(z)$ in $\mathbb{U}$ given by

$$\frac{A_{p,n}^{-\lambda}f(z)}{z^p} = \frac{\alpha(1-w(z))}{\alpha - w(z)},$$

with $w(0)=0$. It follows that

$$\frac{z\left(A_{p,n}^{-\lambda}f(z)\right)'}{A_{p,n}^{-\lambda}f(z)} = p + \frac{zw'(z)}{w(z)}\left(\frac{w(z)}{\alpha - w(z)} - \frac{w(z)}{1-w(z)}\right)$$

and

$$w(z) = \frac{\alpha\left(\frac{A_{p,n}^{-\lambda}f(z)}{z^p} - 1\right)}{\frac{A_{p,n}^{-\lambda}f(z)}{z^p} - \alpha} = b_n z^n + \dots.$$

Suppose that there exists a point $z_0 (0 < |z_0| < 1)$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, using Lemma 3.1, we say that

$$z_0 w'(z_0) = kw(z_0), (k \geq n).$$

Letting $w(z_0) = e^{i\theta}(0 \leq \theta \leq 2\pi)$, we have that

$$\begin{aligned}\frac{z_0\left(A_{p,n}^{-\lambda}f(z_0)\right)'}{A_{p,n}^{-\lambda}f(z_0)} &= p + k\left(\frac{e^{i\theta}}{\alpha - e^{i\theta}} - \frac{e^{i\theta}}{1-e^{i\theta}}\right)\\ &= p + k\left(\frac{\alpha cos\theta - 1}{\alpha^2 + 1 - 2\alpha cos\theta} + \frac{1}{2}\right)\\ &\geq p + \frac{\alpha - 1}{2(\alpha+1)}.\end{aligned}$$

This contradicts our condition (4.10). Thus, we say that there is no $z_0, (0 < |z_0| < 1)$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$, that is that

(4.12) $$\frac{A_{p,n}^{-\lambda}f(z)}{z^p} \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}.$$

□

Taking $\lambda = 1$ in Theorem 4.3, we have the following corollary.

**Corollary 4.3.** *If $f(z) \in T_{p,n}$ satisfies*

$$Re\left(\frac{z\left(A_{p,n}^{-1}f(z)\right)'}{A_{p,n}^{-1}f(z)}\right) < p + \frac{\alpha - 1}{2(\alpha+1)}, \quad z \in \mathbb{U}$$

*for some real* $\alpha > 1$, *then*

$$\frac{p+1}{2} z^{-\frac{p+1}{2}} \int_0^z \frac{f(t)}{t^{\frac{p+1}{2}}} dt \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}.$$

Using the same method of Theorem 4.3, we derive the following theorem.

**Theorem 4.4.** *If* $f(z) \in T_{p,n}$ *satisfies*

$$Re\left(\frac{z\left(A^{\lambda}_{p,n} f(z)\right)'}{A^{\lambda}_{p,n} f(z)}\right) < p + \frac{\alpha - 1}{2(\alpha+1)}, \quad z \in \mathbb{U}$$

*for some real* $\alpha > 1$, *then*

$$\frac{A^{\lambda}_{p,n} f(z)}{z^p} \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}.$$

Taking $\lambda = 1$ in Theorem 4.4, we have the following corollary.

**Corollary 4.4.** *If* $f(z) \in T_{p,n}$ *satisfies*

$$Re\left(\frac{z\left(A^{1}_{p,n} f(z)\right)'}{A^{1}_{p,n} f(z)}\right) < p + \frac{\alpha - 1}{2(\alpha+1)}, \quad z \in \mathbb{U}$$

*for some real* $\alpha > 1$, *then*

$$\frac{2}{(p+1)z^p}\left(z f'(z) - \frac{p-1}{2} f(z)\right) \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}.$$

Finally, we prove the following theorem.

**Theorem 4.5.** *Let a function* $f(z) \in T_{p,n}$ *satisfies*

$$\frac{\beta(1-z)}{\beta - z} \prec \frac{f'(z)}{p z^{p-1}} \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U} \tag{4.13}$$

*for* $1 < \beta < \alpha$. *Then* $f(z)$ *satisfies*

$$\frac{\beta}{z}\left(z + log\left(\frac{\beta - z}{\beta}\right)^{\beta-1}\right) \prec \frac{1}{pz}\int_0^z \frac{f'(t)}{t^{p-1}} dt \prec \frac{\alpha}{z}\left(z + log\left(\frac{\alpha - z}{\alpha}\right)^{\alpha-1}\right), \quad z \in \mathbb{U}. \tag{4.14}$$

To prove the above theorem, we have to use the following lemma by Miller and Mocanu [7].

**Lemma 4.1.** *Let* $h_1(z)$ *and* $h_2(z)$ *be convex in* $\mathbb{U}$ *and let* $f(z)$ *be univalent in* $\mathbb{U}$ *with* $h_1(0) = h_2(0) = f(0)$. *Let* $\gamma \neq 0$ *with* $Re\gamma \geq 0$. *If*

$$h_1(z) \prec f(z) \prec h_2(z), \quad z \in \mathbb{U} \tag{4.15}$$

*then*

$$\frac{\gamma}{z^\gamma}\int_0^z h_1(t)t^{\gamma-1}dt \prec \frac{\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \prec \frac{\gamma}{z^\gamma}\int_0^z h_2(t)t^{\gamma-1}dt, \quad z \in \mathbb{U} \tag{4.16}$$

*when the middle integral is univalent.*

**Proof of the Theorem 4.5:** We write that

$$F(z) = \frac{1}{p}\int_0^z \frac{f'(t)}{t^{p-1}}dt, \quad z \in \mathbb{U}.$$

Then, we know that

$$F'(z) = \frac{f'(z)}{pz^{p-1}}, \quad z \in \mathbb{U}.$$

Since

$$\frac{\beta(1-z)}{\beta - z} \prec F'(z) \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}.$$

with (4.13), we say that

$$0 < Re(F'(z)) < \frac{2\alpha}{\alpha+1}, \quad z \in \mathbb{U},$$

that is, that $F(z)$ is close-to-convex (univalent) in $\mathbb{U}$. Defining $G(z)$ by (4), we obtain that

$$1 + \frac{zG''(z)}{G'(z)} = 1 + \frac{2z}{\alpha - z}, \quad z \in \mathbb{U}.$$

Letting $z = e^{i\theta}(0 \le \theta < 2\pi)$, we have

$$Re\left(1 + \frac{zG''(z)}{G'(z)}\right) = Re\left(1 + \frac{2e^{i\theta}}{\alpha - e^{i\theta}}\right)$$

$$= 1 + \frac{2(\alpha cos\theta - 1)}{\alpha^2 + 1 - 2\alpha cos\theta} > \frac{\alpha - 1}{\alpha + 1}.$$

This means that $G(z)$ is convex in $\mathbb{U}$ with $1 < \beta < \alpha$. Therefore, $\frac{\beta(1-z)}{\beta-z}$ and $\frac{\alpha(1-z)}{\alpha-z}$ are convex in $\mathbb{U}$ with $1 < \beta < \alpha$. Therefore, applying Lemma 4.1 for $\gamma = 1$, we have that

$$\frac{1}{z}\int_0^z \frac{\beta(1-t)}{\beta - t}dt \prec \frac{1}{z}\int_0^z F'(t)dt \prec \frac{1}{z}\int_0^z \frac{\alpha(1-t)}{\alpha - t}dt. \tag{4.17}$$

It is easy to see that the subordination (4.17) is same as the subordination (4.15).

Letting $p = 1$ in Theorem 4.5, we have the following corollary.

**Corollary 4.5.** *If* $f(z) \in T_{1,n}$ *satisfies*

$$\frac{\beta(1-z)}{\beta - z} \prec f'(z) \prec \frac{\alpha(1-z)}{\alpha - z}, \quad z \in \mathbb{U}$$

*for* $1 < \beta < \alpha$, *then*

$$\frac{\beta}{z}\left(z + log\left(\frac{\beta - z}{\beta}\right)^{\beta-1}\right) \prec \frac{f(z)}{z} \prec \frac{\alpha}{z}\left(z + log\left(\frac{\alpha - z}{\alpha}\right)^{\alpha-1}\right), \quad z \in \mathbb{U}.$$

## 5. Conclusions

Applying the fractional calculus (fractional integral and fractional derivative), two generalizations of the Alexander integral operator are introduced. Some interesting properties of these operators are discussed. Also, some simple examples of results for these operators are shown.

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