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ANALYTIC PROPERTIES OF THE APOSTOL-VU MULTIPLE LUCAS L-FUNCTIONS

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Abstract. In the article we study the analytic continuations of the Apostol-Vu multiple shifted Lucas zeta functions and Apostol-Vu multiple Lucas L-functions associated to Dirichlet characters. We also compute a complete list of exact singularities and residues of these functions at poles.

1. Introduction

For real numbers P and Q, the Lucas sequence $U_m(P,Q)$ is recursively defined by

$$U_0 = 0,\ U_1 = 1,\ U_{m+1} = PU_m - QU_{m-1},\ (m \geq 1).$$

The number U_m can be expressed explicitly as $U_m = U_m(P,Q) = \dfrac{a^m - b^m}{a-b}$, where $a = \frac{P+\sqrt{D}}{2}$ and $b = \frac{P-\sqrt{D}}{2}$ are the solutions of $x^2 - Px + Q = 0$ with $D = P^2 - 4Q$. It is certain that $a + b = P$, $a - b = \sqrt{D}$ and $ab = Q$. Notice that $U_m(1,-1)$ and $U_m(6,1)$ are the m-th Fibonacci and m-th balancing numbers. All through the paper, we consider

$$P > 0,\ Q \neq 0,\ \text{and } P > Q + 1 \text{ if } 0 < P \leq 2 \text{ and } P \geq Q + 1 \text{ if } P > 2.$$

From this supposition, we acquire

$$D > 0,\ a > 1,\ a > |b| > 0 \text{ and } U_m > 0,\ \text{for all } m \geq 1.$$

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The Euler-Zagier multiple zeta function $\zeta_{EZ,k}$ is defined by

$$\zeta_{EZ,k}(s_1, s_2, \ldots, s_k) = \sum_{0<m_1<m_2\cdots<m_k<\infty} \frac{1}{{m_1}^{s_1}{m_2}^{s_2}\cdots{m_k}^{s_k}}, \tag{1.1}$$

where $s_1, s_2, \ldots, s_k$ are complex variables [17]. This series is absolutely convergent in the region

$$\{(s_1, s_2, \ldots, s_k) \in \mathbb{C}^k | \mathrm{Re}(s_{k-r+1} + s_{k-r+2} + \cdots + s_k) > r,\ r = 1, 2, \ldots, k\}.$$

The analytic continuation of (1.1) has been studied by several researchers [1, 12, 18]. The series

$$\sum_{m=1}^{\infty}\sum_{n<m} \frac{1}{m^{s_1}n^{s_2}(m+n)}, \tag{1.2}$$

was first introduced by Apostol and Vu [3] and they obtained the partial results on its analytic continuation. The meromorphic continuation of (1.2) and more general series

$$\sum_{m=1}^{\infty}\sum_{n<m} \frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}, \tag{1.3}$$

were first proved in [8]. In [9], Matsumoto generalized (1.3) and introduced Apostol-Vu r-ple zeta function

(1.4)

$$\zeta_{AV,r}(s_1, \ldots, s_r; s_{r+1}) = \sum_{1\le m_1<m_2<\cdots<m_r<\infty} \frac{1}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}(m_1+m_2+\cdots+m_r)^{s_{r+1}}}.$$

He proved that the series (1.4) is convergent absolutely when $\mathrm{Re}(s_i) > 1$ $(1 \le i \le r)$, $\mathrm{Re}(s_{r+1}) > 0$ and can be continued meromorphically to the whole space $\mathbb{C}^{r+1}$.

Various generalizations of the multiple zeta functions were introduced and their analytic properties have been studied. One of the most valuable generalizations is the multiple series associated to Dirichlet characters which are called multiple Dirichlet L-function. The multiple Dirichlet L-function is defined as:

$$\mathcal{L}(s_1, \ldots, s_k \mid \chi_1, \ldots, \chi_k) = \sum_{0<m_1<m_2\cdots<m_k<\infty} \frac{\chi_1(m_1)}{{m_1}^{s_1}} \cdots \frac{\chi_k(m_k)}{{m_k}^{s_k}},$$

where $\chi_1, \ldots, \chi_k$ are Dirichlet characters of same modulo $q \in \mathbb{N}_{\ge 2}$. The analytic continuation of this multiple L-function has been studied by Akiyama and Ishikawa [2]. In [16], Waghamore and Vijaylaxmi investigated the uniqueness problems of meromorphic functions and L-functions in the extended Selberg class by introducing the concept of multiplicity along with weighted sharing of small functions or fixed points.

Zeta functions relating number sequences have been a source of attraction to many researchers. In [13], Navas presented Fibonacci Dirichlet series $\zeta_F(s) = \sum_{m=1}^{\infty} F_m^{-s}$, $Re(s) > 0$ for $s \in \mathbb{C}$, where F_m denotes the m-th Fibonacci number and obtained that $\zeta_F(s)$ is analytically continued to a meromorphic function on the complex plane $\mathbb{C}$. Rout and Panda [15] considered balancing zeta function $\zeta_B(s) = \sum_{m=1}^{\infty} B_m^{-s}$, $Re(s) > 0$ for $s \in \mathbb{C}$, where B_m denotes the m-th balancing number and proved that $\zeta_B(s)$ is meromorphically continued all over $\mathbb{C}$. In a consequent paper, Behera et al. [4] proved the analytical continuation of Lucas-balancing zeta function $\zeta_C(s) = \sum_{m=1}^{\infty} C_m^{-s}$, $Re(s) > 0$ for $s \in \mathbb{C}$, where C_m denotes the m-th Lucas-balancing number. In [7], Kamano considered the Lucas zeta function $\Phi^{(P,\,Q)}(s) = \sum_{m=1}^{\infty} U_m^{-s}$, $Re(s) > 0$, $s \in \mathbb{C}$, and determined its analytic continuation. Rout and Meher [14] defined the multiple Fibonacci zeta function of depth k as:

$$\zeta_F(s_1, s_2, \ldots, s_k) = \sum_{1 \le m_1 < m_2 < \cdots < m_k} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \cdots F_{m_k}^{s_k}}.$$

They derived the analytic continuation of $\zeta_F(s_1, s_2, \ldots, s_k)$ of depth 2 and found a complete list of poles and their corresponding residues. Later, Meher and Rout [10] proved the meromorphic continuation of multiple Lucas zeta functions of depth d :

$$\zeta_U(s_1, s_2, \ldots, s_d) = \sum_{0 < m_1 < m_2 \cdots < m_d} \frac{1}{U_{m_1}^{s_1} U_{m_2}^{s_2} \cdots U_{m_d}^{s_d}},$$

where U_m is the m-th Lucas number of first kind. Furthermore, Dutta and Ray [5] defined Euler-Zagier multiple Lucas-balancing zeta functions of depth k :

$$\zeta_{EZC}(s_1, \ldots, s_k) \;=\; \sum_{1 \le m_1 < m_2 < \cdots < m_k < \infty} \frac{1}{C_{m_1}^{s_1} \cdots C_{m_k}^{s_k}},$$

and studied its meromorphic continuation. Meher and Rout [11] studied the analytic continuation of multiple Lucas L-functions defined as:

$$\mathcal{L}_U^d(s_1, \ldots, s_k \mid \chi_1, \ldots, \chi_k) = \sum_{0 < m_1 < m_2 \cdots < m_k} \frac{\chi_1(m_1)}{{U_{m_1}}^{s_1}} \cdots \frac{\chi_k(m_k)}{{U_{m_k}}^{s_k}}, \tag{1.5}$$

where $\chi_1, \ldots, \chi_k$ are Dirichlet characters of same modulo $q \in \mathbb{N}_{\geq 2}$. Recently, Dutta and Ray [6] defined the Apostol-Vu multiple Fibonacci zeta functions $\zeta_{AVF,k}$ as:

$$\zeta_{AVF,k}(s_1, \ldots, s_k; s_{k+1}) = \sum_{1 \leq m_1 < \cdots < m_k < \infty} \frac{1}{F_{m_1}^{s_1} \cdots F_{m_k}^{s_k} F_{m_1+m_2+\cdots+m_k}^{s_{k+1}}},$$

and examined its analytic continuation.

In this paper we investigate the analytic continuations of the Apostol-Vu multiple shifted Lucas zeta functions and Apostol-Vu multiple Lucas L-functions associated to Dirichlet characters particularly in Section 2. We also estimate the residues of these functions corresponding to their respective poles in Section 3.

2. Analytic continuation of the Apostol-Vu multiple Lucas L-functions

In this section we introduce the Apostol-Vu multiple shifted Lucas zeta functions and Apostol-Vu multiple Lucas L-functions associated to Dirichlet characters and study their analytic continuations.

Let $\chi_1, \ldots, \chi_k$ be the Dirichlet characters of same modulus $t \in \mathbb{N}_{\geq 2}$ and χ_0 be the principal character. The Apostol-Vu multiple shifted Lucas zeta functions and Apostol-Vu multiple Lucas L-functions are respectively defined as:

$$\begin{aligned} &\zeta_{AVU,k}(s_1, \ldots, s_k; s_{k+1} \mid h_1, \ldots, h_k) \\ &= \sum_{0 \leq m_1 < \cdots < m_k < \infty} \frac{1}{U_{tm_1+h_1}^{s_1} \cdots U_{tm_k+h_k}^{s_k} U_{t(m_1+\cdots+m_k)+h_1+\cdots+h_k}^{s_{k+1}}}, \end{aligned} \tag{2.1}$$

and

$$\mathcal{L}_{AVU,k}(s_1, \ldots, s_k; s_{k+1} \mid \chi_1, \ldots, \chi_k) = \sum_{1 \leq m_1 < \cdots < m_k < \infty} \frac{\chi_1(m_1) \ldots \chi_k(m_k)}{U_{m_1}^{s_1} \cdots U_{m_k}^{s_k} U_{m_1+\cdots+m_k}^{s_{k+1}}}, \tag{2.2}$$

where $s_1, \ldots, s_{k+1} \in \mathbb{C}$ and $m_i, h_i \in \mathbb{N}$ for $1 \leq i \leq k$. The sum $s_1 + \cdots + s_{k+1}$ is called the weight of both $\zeta_{AVU,k}(s_1, \ldots, s_k; s_{k+1} \mid h_1, \ldots, h_k)$ and $\mathcal{L}_{AVU,k}(s_1, \ldots, s_k; s_{k+1} \mid \chi_1, \ldots, \chi_k)$ and k is called their depth. One can observe that $\mathcal{L}_{AVU,k}(s_1, \ldots, s_k; 0 \mid \chi_1, \ldots, \chi_k)$ is the multiple multiple Lucas L-functions (1.5) [11]. For the sake of convenience, we denote $\zeta_{AVU,k}(s|\mathbf{h})$ and $\mathcal{L}_{AVU,k}(s|\chi)$ for $\zeta_{AVU,k}(s_1, \ldots, s_k; s_{k+1} \mid h_1, \ldots, h_k)$ and $\mathcal{L}_{AVU,k}(s_1, \ldots, s_k; s_{k+1} \mid \chi_1, \ldots, \chi_k)$ respectively.

For any integer $k \geq 1$, we consider the open subset of D_{k+1} of $\mathbb{C}^{k+1}$, i.e.

$$D_{k+1} = \left\{ (s_1, \ldots, s_k; s_{k+1}) \in \mathbb{C}^{k+1} \Big| \sum_{j=d}^{k} Re(s_j) + (k+1-d) Re(s_{k+1}) > 0, 1 \leq d \leq k \right\}.$$

Proposition 2.1. *The series* $\displaystyle\sum_{1 \leq m_1 < \cdots < m_k} \frac{1}{U_{m_1}^{s_1} \cdots U_{m_k}^{s_k} U_{m_1+m_2+\cdots+m_k}^{s_{k+1}}}$ *is absolutely convergent in the domain* D_{k+1}.

Proof. Let $s_j = \sigma_j + it_j \in \mathbb{C}$ and $\sigma_j = Re(s_j) > 0$, $j = 1, 2, \ldots, k+1$. Now,

$$\sum_{1 \leq m_1 < \cdots < m_k} \frac{1}{U_{m_1}^{s_1} U_{m_2}^{s_2} \cdots U_{m_k}^{s_k} U_{m_1+m_2+\cdots+m_k}^{s_{k+1}}} = \sum_{m_1=1}^{\infty} \frac{1}{U_{m_1}^{s_1}} \sum_{m_2=1}^{\infty} \frac{1}{U_{m_1+m_2}^{s_1}}$$

$$\cdots \sum_{m_{k-1}=1}^{\infty} \frac{1}{U_{m_1+m_2+\cdots+m_{k-1}}^{s_{k-1}}} \sum_{m_k=1}^{\infty} \frac{1}{U_{m_1+m_2+\cdots+m_k}^{s_k} U_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}. \tag{2.3}$$

Now,

$$\begin{aligned} \sum_{m_1=1}^{\infty} \left| \frac{1}{U_{m_1}^{s_1}} \right| &= (a-b)^{\sigma_1} \textstyle\sum_{m_1=1}^{\infty} \left| \frac{1}{\left(a^{m_1} - b^{m_1}\right)^{s_1}} \right| \\ &\leq (a-b)^{\sigma_1} \textstyle\sum_{m_1=1}^{\infty} \frac{1}{a^{m_1\sigma_1} \left(1 - (|\frac{b}{a}|)^{m_1}\right)^{\sigma_1}} \\ &\leq \left(\frac{a-b}{1-|b/a|}\right)^{\sigma_1} \textstyle\sum_{m_1=1}^{\infty} \frac{1}{a^{\sigma_1 m_1}} = \Lambda_{\sigma_1} D^{\frac{\sigma_1}{2}} \frac{1}{a^{\sigma_1}-1}. \end{aligned} \tag{2.4}$$

Similarly,

$$\left| \frac{1}{U_{m_1+\cdots+m_d}^{s_d}} \right| \leq \Lambda_{\sigma_d} D^{\frac{\sigma_d}{2}} \frac{1}{a^{\sigma_d\left(m_1+\cdots+m_d\right)}}, \; 2 \leq d \leq k, \tag{2.5}$$

and

$$\left| \frac{1}{U_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}} \right| \leq \Lambda_{\sigma_{k+1}} D^{\frac{\sigma_{k+1}}{2}} \frac{1}{a^{\sigma_{k+1}\left(km_1+(k-1)m_2+\cdots+m_k\right)}}, \tag{2.6}$$

where $\Lambda_{\sigma_j} = \frac{1}{(1-|b/a|)^{\sigma_j}}$. By virtue of (2.3), (2.4), (2.5) and (2.6),

$$\begin{aligned} &\sum_{1 \leq m_1 < \cdots < m_k} \left| \frac{1}{U_{m_1}^{s_1} \cdots U_{m_k}^{s_k} U_{m_1+\cdots+m_k}^{s_{k+1}}} \right| \\ &\leq \sum_{m_1=1}^{\infty} \left| \frac{1}{U_{m_1}^{s_1}} \right| \sum_{m_2=1}^{\infty} \left| \frac{1}{U_{m_1+m_2}^{s_1}} \right| \cdots \sum_{m_{k-1}=1}^{\infty} \left| \frac{1}{U_{m_1+\cdots+m_{k-1}}^{s_{k-1}}} \right| \end{aligned}$$

$$\begin{aligned}
&\sum_{m_k=1}^{\infty}\left|\frac{1}{U^{s_k}_{m_1+\cdots+m_k}U^{s_{k+1}}_{km_1+(k-1)m_2+\cdots+m_k}}\right| \\
\leq\ & \Lambda_{\sigma_1}\Lambda_{\sigma_2}\cdots\Lambda_{\sigma_k}\Lambda_{\sigma_{k+1}}D^{\frac{\sigma_1}{2}}\cdots D^{\frac{\sigma_{k+1}}{2}}\sum_{m_1=1}^{\infty}\frac{1}{a^{\sigma_1 m_1}}\sum_{m_2=1}^{\infty}\frac{1}{a^{\sigma_2(m_1+m_2)}} \\
&\cdots\sum_{m_{k-1}=1}^{\infty}\frac{1}{a^{\sigma_{k-1}(m_1+m_2+\cdots+m_{k-1})}}\sum_{m_k=1}^{\infty}\frac{1}{a^{\sigma_k(m_1+\cdots+m_k)}}\frac{1}{a^{\sigma_{k+1}(km_1+(k-1)m_2+\cdots+m_k)}} \\
=\ & \Lambda_\sigma D^{\frac{\sum_{j=1}^{k+1}\sigma_j}{2}}\sum_{m_1=1}^{\infty}\frac{1}{a^{(\sigma_1+\cdots+\sigma_k+k\sigma_{k+1})m_1}}\sum_{m_2=1}^{\infty}\frac{1}{a^{(\sigma_2+\cdots+\sigma_k+(k-1)\sigma_{k+1})m_2}} \\
&\times\cdots\times\sum_{m_k=1}^{\infty}\frac{1}{a^{(\sigma_k+\sigma_{k+1})m_k}}
\end{aligned}$$

$$\begin{aligned}
=\ & \Lambda_\sigma D^{\frac{\sum_{j=1}^{k+1}\sigma_j}{2}}\frac{1}{\left(a^{\sigma_1+\cdots+\sigma_k+k\sigma_{k+1}}-1\right)}\frac{1}{\left(a^{\sigma_2+\cdots+\sigma_k+(k-1)\sigma_{k+1}}-1\right)}\cdots\frac{1}{\left(a^{\sigma_k+\sigma_{k+1}}-1\right)} \\
<\ & \infty,
\end{aligned}$$

as $a > 1$, where $\Lambda_\sigma = \Lambda_{\sigma_1}\ldots\Lambda_{\sigma_k}\Lambda_{\sigma_{k+1}}$ and $\Lambda_{\sigma_1},\ldots,\Lambda_{\sigma_k}$ and $\Lambda_{\sigma_{k+1}}$ are the positive constants depending on $\sigma_1,\ldots,\sigma_k$ and σ_{k+1} respectively. Therefore, the series (2.3) converges absolutely in the domain D_{k+1}. This completes the proof. □

Proposition 2.2. *The Apostol-Vu multiple shifted Lucas zeta functions $\zeta_{AVU,k}(s|\mathbf{h})$ is absolutely convergent in the domain D_{k+1}.*

Proof. Note that,

$$\begin{aligned}
&\sum_{0\le m_1<\cdots<m_k<\infty}\left|\frac{1}{U^{s_1}_{tm_1+h_1}\cdots U^{s_k}_{tm_k+h_k}U^{s_{k+1}}_{t(m_1+\cdots+m_k)+h_1+\cdots+h_k}}\right| \\
=\ & \sum_{m_1=0,\ldots,m_k=0}^{\infty}\left|\frac{1}{U^{s_1}_{tm_1+h_1}}\cdots\frac{1}{U^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}}\right. \\
&\left.\frac{1}{U^{s_{k+1}}_{t(km_1+(k-1)m_2+\cdots+m_k)+kh_1+(k-1)h_2+\cdots+h_k}}\right| \\
\leq\ & \sum_{m_1=0,\ldots,m_k=0}^{\infty}\left|\frac{1}{U^{s_1}_{tm_1+h_1}}\right|\cdots\left|\frac{1}{U^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}}\right| \\
&\times\left|\frac{1}{U^{s_{k+1}}_{t(km_1+(k-1)m_2+\cdots+m_k)+kh_1+(k-1)h_2+\cdots+h_k}}\right|
\end{aligned}$$

$$\leq \sum_{m_1=1,\dots,m_k=1}^{\infty} \left|\frac{1}{U_{m_1}^{s_1}}\right| \cdots \left|\frac{1}{U_{m_1+\cdots+m_k}^{s_k}}\right| \left|\frac{1}{U_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}\right|.$$

Thus, the result follows from Proposition 2.1 and (2.7). □

Proposition 2.3. *For any positive integer $k \geq 1$, let $\chi_1, \dots, \chi_k$ be the Dirichlet characters of same modulus $t \in \mathbb{N}_{\geq 2}$. The Apostol-Vu multiple Lucas L-functions $\mathcal{L}_{AVU,k}(s|\chi)$ are absolutely convergent in the domain D_{k+1}.*

Proof. From (2.2), it is observed that

$$\begin{aligned}
&\mathcal{L}_{AVU,k}(s|\chi)\\
&= \sum_{1\leq m_1<\cdots<m_k<\infty} \frac{\chi_1(m_1)\dots\chi_k(m_k)}{U_{m_1}^{s_1}\cdots U_{m_k}^{s_k} U_{m_1+\cdots+m_k}^{s_{k+1}}}\\
&= \sum_{m_1=1,\dots,m_k=1}^{\infty} \frac{\chi_1(m_1)}{U_{m_1}^{s_1}} \frac{\chi_2(m_1+m_2)}{U_{m_1+m_2}^{s_2}} \cdots \frac{\chi_k(m_1+\cdots+m_k)}{U_{m_1+\cdots+m_k}^{s_k}} \frac{1}{U_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}.
\end{aligned}$$

For any $m \in \mathbb{N}$, $|\chi(m)| \leq 1$. Therefore, by Proposition 2.1, the function $\mathcal{L}_{AVU,k}(s|\chi)$ converges absolutely in D_{k+1}. □

Theorem 2.1. *The Apostol-Vu multiple shifted Lucas zeta function $\zeta_{AVU,k}(s|\mathbf{h})$ can be analytically continued to a meromorphic function on all of $\mathbb{C}^{k+1}$ with exact list of poles on the hyperplanes*

$$\begin{aligned}
&s_d + \cdots + s_k + (k+1-d)s_{k+1} = -2(r_d + \cdots + r_k + (k+1-d)r_{k+1})\\
&+\frac{(r_d + \cdots + r_k + (k+1-d)r_{k+1})\log|Q|}{\log a} + \left(\frac{2c}{t} + W(Q, r_d, r_{d+1}, \dots, r_{k+1})\right)\frac{\pi i}{\log a},
\end{aligned}$$

$1 \leq d \leq k$, with $r_1, \dots, r_{k+1} \in \mathbb{Z}_{\geq o}$ and $c \in \mathbb{Z}$, where

$$W(Q, r_d, r_{d+1}, \dots, r_{k+1}) = \begin{cases} r_d + \cdots + r_k + (k+1-d)r_{k+1}, & \text{if } Q < 0\\ 0, & \text{if } Q > 0.\end{cases}$$

Proof. Using $a - b = \sqrt{D}$ and $ab = Q$, for any $s \in \mathbb{C}$, we have

$$\begin{aligned}
U_m^s = \left(\frac{a^m - b^m}{a-b}\right)^s &= (a-b)^{-s} a^{ms}\left(1 + (-1)\left(\frac{b}{a}\right)^m\right)^s\\
&= D^{-\frac{s}{2}} a^{ms} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^k \left(\frac{b}{a}\right)^{mk}\\
&= D^{-\frac{s}{2}} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^k Q^{mk} a^{m\left(s-2k\right)}.
\end{aligned}$$

Now,

$$U^s_{tm+h} \quad = \quad D^{-\frac{s}{2}} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^k Q^{(tm+h)k} a^{(tm+h)\left(s-2k\right)}.$$

Using the above identity in (2.1), we get

$$\begin{aligned}
\zeta_{AVU,k}(s|\mathbf{h}) = & \sum_{m_1=0,\ldots,m_k=0}^{\infty} \frac{1}{U^{s_1}_{tm_1+h_1}} \cdots \frac{1}{U^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}} \\
& \frac{1}{U^{s_{k+1}}_{t(km_1+(k-1)m_2+\cdots+m_k)+kh_1+(k-1)h_2+\cdots+h_k}} \\
& = \sum_{m_1,\ldots,m_k=0}^{\infty} D^{\frac{s_1}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} Q^{(tm_1+h_1)r_1} a^{-(tm_1+h_1)(s_1+2r_1)} \\
& \times \cdots \times \ D^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{r_k} Q^{\left(t\sum_{n=1}^{k} m_n + \sum_{n=1}^{k} h_n\right) r_k} \\
& \times a^{-\left(t\sum_{n=1}^{k} m_n + \sum_{n=1}^{k} h_n\right)\left(s_k+2r_k\right)} \\
& \times \ D^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} Q^{\left(t\sum_{n=1}^{k}(k+1-n)m_n + \sum_{n=1}^{k}(k+1-n)h_n\right) r_{k+1}} \\
& \times a^{-\left(t\sum_{n=1}^{k}(k+1-n)m_n + \sum_{n=1}^{k}(k+1-n)h_n\right)\left(s_{k+1}+2r_{k+1}\right)}.
\end{aligned} \tag{2.7}$$

Since the series $\zeta_{AVU,k}(s|\mathbf{h})$ is absolutely convergent, interchanging the order of summation in (2.7) gives

$$\begin{aligned}
& \zeta_{AVU,k}(s|\mathbf{h}) \\
= \ & D^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \cdots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} Q^{(r_1+\cdots+r_k+kr_{k+1})h_1} \\
& \times a^{-\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1}\right)h_1} \times \cdots \times Q^{(r_k+r_{k+1})h_k} a^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)h_k} \\
& \times \sum_{m_1,\ldots,m_k=0}^{\infty} \left(Q^{t(r_1+\cdots+r_k+kr_{k+1})} a^{-t\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})\right)} \right)^{m_1} \\
& \times \cdots \times \left(Q^{t(r_k+r_{k+1})} a^{-t\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)} \right)^{m_k}
\end{aligned}$$

$$= D^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \cdots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}}$$

$$\times \frac{Q^{(r_1+\cdots+r_k+kr_{k+1})h_1} a^{-\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1}\right)h_1}}{\left(1-Q^{t(r_1+\cdots+r_k+kr_{k+1})} a^{-t\left(s_1+\cdots+s_k+ks_{k+1}+2(r_1+\cdots+r_k+kr_{k+1})\right)}\right)}$$

$$\times \cdots \times \frac{Q^{(r_k+r_{k+1})h_k} a^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)h_k}}{\left(1-Q^{t(r_k+r_{k+1})} a^{-t\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}\right)}.$$

The infinite series (2.8) is holomorphic function on $\mathbb{C}^{k+1}$ except for the poles derived from the functions

$$F_{r_d,\ldots,r_{k+1}}(s_d,\ldots,s_{k+1})$$
$$= \frac{Q^{\left(r_1+\cdots+r_k+(k+1-d)r_{k+1}\right)h_d} a^{-h_d\left(s_d+\cdots+s_k+(k+1-d)s_{k+1}+2(r_d+\cdots+r_k+(k+1-d)r_{k+1})\right)}}{1-Q^{t\left(r_1+\cdots+r_k+(k+1-d)r_{k+1}\right)} a^{-t\left(s_d+\cdots+s_k+(k+1-d)s_{k+1}+2(r_d+\cdots+r_k+(k+1-d)r_{k+1})\right)}} \tag{2.8}$$

for $d = 1, \ldots, k$. Therefore, $\zeta_{AVU,k}(s|\mathbf{h})$ is meromorphically continued on $\mathbb{C}^{k+1}$ with poles on the hyperplanes

$$s_d + \cdots + s_k + (k+1-d)s_{k+1} = -2(r_d + \cdots + r_k + (k+1-d)r_{k+1})$$
$$+\frac{(r_d + \cdots + r_k + (k+1-d)r_{k+1}) \log |Q|}{\log a} + \left(\frac{2c}{t} + W(Q, r_d, r_{d+1}, \ldots, r_{k+1})\right) \frac{\pi i}{\log a}, \tag{2.9}$$

$1 \leq d \leq k$, with $r_1, \ldots, r_{k+1} \in \mathbb{Z}_{\geq o}$ and $c \in \mathbb{Z}$, where

$$W\big(Q, r_d, r_{d+1}, \ldots, r_{k+1}\big) = \begin{cases} r_d + \cdots + r_k + (k+1-d)r_{k+1}, & \text{if } Q < 0 \\ 0, & \text{if } Q > 0. \end{cases}$$

This finishes the proof. □

Now,

$$\mathcal{L}_{AVU,k}(s \mid \chi)$$
$$= \sum_{1 \leq m_1 < \cdots < m_k < \infty} \frac{\chi_1(m_1) \ldots \chi_k(m_k)}{U_{m_1}^{s_1} \cdots U_{m_k}^{s_k} U_{m_1+\cdots+m_k}^{s_{k+1}}}$$
$$= \sum_{m_1=1,\ldots,m_k=1}^{\infty} \frac{\chi_1(m_1)}{U_{m_1}^{s_1}} \frac{\chi_2(m_1+m_2)}{U_{m_1+m_2}^{s_2}} \cdots \frac{\chi_k(m_1+\cdots+m_k)}{U_{m_1+\cdots+m_k}^{s_k}} \frac{1}{U_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}}$$

$$= \sum_{h_1=1}^{t} \cdots \sum_{h_k=1}^{t} \sum_{m_1=1}^{\infty} \frac{\chi_1(h_1)}{U_{tm_1+h_1}^{s_1}}$$
$$\times \cdots \times \sum_{m_k=1}^{\infty} \frac{\chi_k(h_1+\cdots+h_k)}{U_{t(m_1+\cdots+m_k)+h_1+\cdots+h_k}^{s_k}} \frac{1}{U_{t(km_1+(k-1)m_2+\cdots+m_k)+kh_1+(k-1)h_2+\cdots+h_k}^{s_{k+1}}}$$
$$= \sum_{h_1=1}^{t} \cdots \sum_{h_k=1}^{t} \chi_1(h_1) \cdots \chi_k(h_1+\cdots+h_k) \zeta_{AVU,k}(s|\mathbf{h}).$$

From the above expression, it is found that the function $\mathcal{L}_{AVU,k}(s \mid \chi)$ is a linear combination of $\zeta_{AVU,k}(s|\mathbf{h})$. Thus, we get the following result.

Theorem 2.2. *For any positive integer $k \geq 1$, let $\chi_1, \ldots, \chi_k$ be the Dirichlet characters of same modulus $t \in \mathbb{N}_{\geq 2}$, then the function $\mathcal{L}_{AVU,k}(s \mid \chi)$ can be meromorphically continued on all of $\mathbb{C}^{k+1}$ with possible poles on the hyperplanes given by (2.9).*

3. Residues of the Apostol-Vu multiple Lucas L-functions

In this section we calculate the residues of $\mathcal{L}_{AVU,k}(s \mid \chi)$ at the poles lying on the hyperplanes given by (2.9). For $1 \leq d \leq k$, let us denote

$$s_{k+1}(d) = s_d + \cdots + s_k + (k+1-d)s_{k+1}, r_{k+1}(d) = r_d + \cdots + r_k + (k+1-d)r_{k+1},$$

$r'_{k+1}(d) = r'_d + \cdots + r'_k + (k+1-d)r'_{k+1}$, $h_k(d) = h_d + \cdots + h_k$ and $\zeta_t = e^{\frac{2\pi i}{t}}$ with the assumption that

$$s_{k+1}(d) = 0, r_{k+1}(d) = 0, r'_{k+1}(d) = 0 \text{ for } d \geq k+1.$$

We define the residue of $\mathcal{L}_{AVU,k}(s \mid \chi)$ along the hyperplanes (2.9) to be the restriction of the meromorphic function

$$\left(s_{k+1}(d) + 2r_{k+1}(d) - \frac{r_{k+1}(d) \log |Q|}{\log a} - \left(\frac{2c}{t} + W(Q, r_d, \ldots, r_{k+1}) \right) \frac{\pi i}{\log a} \right) \mathcal{L}_{AVU,k}(s \mid \chi)$$

to the hyperplanes (2.9).

Theorem 3.1. *Let $r'_d, \dots, r'_k, r'_{k+1}$ be non-negative integers for $1 \le d \le k$ and $e_{k+1}(d) = -2r'_{k+1}(d) + r'_{k+1}(d)\frac{\log|Q|}{\log a} + \left(\frac{2c}{t} + W(Q, r'_d, \dots, r'_{k+1})\right)\frac{\pi i}{\log a}$. Then*

$$\begin{aligned}
&\underset{s_{k+1}(d)=e_{k+1}(d)}{Res} \mathcal{L}_{AVU,k}(s \mid \chi) \\
= \ & D^{\frac{e_{k+1}(d)}{2}}(-1)^{r'_{k+1}(d)} \sum_{\substack{r_1,\dots,r_{k+1}\ge 0 \\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i)+(d-k)r_{k+1}} \\
& \times D^{\frac{(\sum_{i=1}^{d-1} s_i)+(d-k)s_{k+1}}{2}} \sum_{h_1=1}^{t} \chi_1(h_1) \sum_{h_2=1}^{t} \chi_2(h_2(1)) \cdots \sum_{h_k=1}^{t} \chi_k(h_k(1)) \frac{\zeta_t^{-ch_d}}{t \log a} \\
& \times \prod_{\substack{l=1 \\ l\ne d}}^{k} \frac{Q^{r_{k+1}(l)h_l} a^{-\left(s_{k+1}(l)+2r_{k+1}(l)\right)h_l}}{\left(1 - Q^{tr_{k+1}(l)} a^{-t\left(s_{k+1}(l)+2r_{k+1}(l)\right)}\right)}.
\end{aligned}$$

Proof. Let $s_{k+1}(d) = e_{k+1}(d) = -2r'_{k+1}(d) + r'_{k+1}(d)\frac{\log|Q|}{\log a} + \left(\frac{2c}{t} + W(Q, r'_d, \dots, r'_{k+1})\right)\frac{\pi i}{\log a}$. Now, $s_{k+1}(d) + 2r'_{k+1}(d) = r'_{k+1}(d)\frac{\log|Q|}{\log a} + \left(\frac{2c}{t} + W(Q, r'_k, r'_{k+1})\right)\frac{\pi i}{\log a}$ which implies that $a^{s_{k+1}(d)+2r'_{k+1}(d)} = Q^{r'_{k+1}(d)}\zeta_t^c$. Thus, for $1 \le d \le k$, we obtain

$$a^{-h_d\left(s_{k+1}(d)+2r'_{k+1}(d)\right)} = Q^{-h_d r'_{k+1}(d)} \zeta_t^{-h_d c}$$

and $a^{-t\left(s_{k+1}(d)+2r'_{k+1}(d)\right)} = Q^{-tr'_{k+1}(d)}$. Therefore, $a^{t\left(s_{k+1}(d)+2r'_{k+1}(d)\right)} - Q^{tr'_{k+1}(d)}$ is an analytic function with simple zeros at $e_{k+1}(d)$ for $1 \le d \le k$. Now,

$$\begin{aligned}
&\underset{s_{k+1}(d)=e_{k+1}(d)}{\text{Res}} \frac{1}{\left(1 - Q^{tr_{k+1}(d)} a^{-t\left(s_{k+1}(d)+2r_{k+1}(d)\right)}\right)} \\
= \ & \lim_{s_{k+1}(d)\to e_{k+1}(d)} \frac{s_{k+1}(d) - e_{k+1}(d)}{\left(1 - Q^{tr_{k+1}(d)} a^{-t\left(s_{k+1}(d)+2r_{k+1}(d)\right)}\right)} = \frac{1}{t \log a}.
\end{aligned}$$

Note that,

$$\begin{aligned}
&\lim_{s_{k+1}(d)\to e_{k+1}(d)} \left(s_{k+1}(d) - e_{k+1}(d)\right) \sum_{r_d,\dots,r_{k+1}=0}^{\infty} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_{k+1}}{r_{k+1}}(-1)^{r_{k+1}} \\
\times \ & \frac{Q^{r_{k+1}(d)h_d} a^{-\left(s_{k+1}(d)+2r_{k+1}(d)\right)h_d}}{\left(1 - Q^{tr_{k+1}(d)} a^{-t\left(s_{k+1}(d)+2r_{k+1}(d)\right)}\right)} \times \cdots \times \frac{Q^{r_{k+1}(k)h_k} a^{-\left(s_{k+1}(k)+2r_{k+1}(k)\right)h_k}}{\left(1 - Q^{tr_{k+1}(k)} a^{-t\left(s_{k+1}(k)+2r_{k+1}(k)\right)}\right)}.
\end{aligned}$$

In the above calculation after applying the limit, only those terms containing $r_d, \dots, r_{k+1}$ will survive when $r_{k+1}(d) = r'_{k+1}(d)$ and rest of the terms will vanish. Therefore, the above limit reduces to

$$\sum_{\substack{r_d\geq 0,\ldots,r_{k+1}\geq 0\\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_d}{r_d}\cdots\binom{-s_{k+1}}{r_{k+1}} \frac{Q^{r_{k+1}(d+1)h_{d+1}} a^{-\left(s_{k+1}(d+1)+2r_{k+1}(d+1)\right)h_{d+1}}}{\left(1-Q^{tr_{k+1}(d+1)}a^{-t\left(s_{k+1}(d+1)+2r_{k+1}(d+1)\right)}\right)}$$

$$\times\cdots\times \frac{Q^{r_{k+1}(k)h_k} a^{-\left(s_{k+1}(k)+2r_{k+1}(k)\right)h_k}}{\left(1-Q^{tr_{k+1}(k)}a^{-t\left(s_{k+1}(k)+2r_{k+1}(k)\right)}\right)} \times \frac{\zeta_t^{-ch_d}}{t\log a}.$$

Therefore, the residue of $\zeta_{AVU,k}(s|\mathbf{h})$ along the hyper plane $s_{k+1}(d)=e_{k+1}(d)$ is given by

$$\begin{aligned}
&\operatorname*{Res}_{s_{k+1}(d)=e_{k+1}(d)} \zeta_{AVU,k}(s|\mathbf{h})\\
=\ & \lim_{s_{k+1}(d)\to e_{k+1}(d)} \left(s_{k+1}(d)-e_{k+1}(d)\right) D^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\cdots\sum_{r_{k+1}=0}^{\infty}\binom{-s_{k+1}}{r_{k+1}}\\
& (-1)^{r_{k+1}} \frac{Q^{r_{k+1}(1)h_1} a^{-\left(s_{k+1}(1)+2r_{k+1}(1)\right)h_1}}{\left(1-Q^{tr_{k+1}(1)}a^{-t\left(s_{k+1}(1)+2r_{k+1}(1)\right)}\right)} \cdots \frac{Q^{r_{k+1}(k)h_k} a^{-\left(s_{k+1}(k)+2r_{k+1}(k)\right)h_k}}{\left(1-Q^{tr_{k+1}(k)}a^{-t\left(s_{k+1}(k)+2r_{k+1}(k)\right)}\right)}\\
=\ & \sum_{r_1,\ldots,r_{k+1}=0}^{\infty}\binom{-s_1}{r_1}\cdots\binom{-s_{k+1}}{r_{k+1}} \lim_{s_{k+1}(d)\to e_{k+1}(d)} (s_{k+1}(d)-e_{k+1}(d)) D^{\frac{(\sum_{n=1}^{k+1} s_n)}{2}}\\
& (-1)^{(\sum_{i=1}^{k+1} r_i)} \prod_{d=1}^{k} \frac{Q^{r_{k+1}(d)h_d} a^{-\left(s_{k+1}(d)+2r_{k+1}(d)\right)h_d}}{\left(1-Q^{tr_{k+1}(d)}a^{-t\left(s_{k+1}(d)+2r_{k+1}(d)\right)}\right)}\\
=\ & D^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}\zeta_t^{-ch_d}}{t\log a} \sum_{\substack{r_1,\ldots,r_{k+1}\geq 0\\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_1}{r_1}\cdots\binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i)+(d-k)r_{k+1}}
\end{aligned}$$

$$D^{\frac{(\sum_{i=1}^{d-1} s_i)+(d-k)s_{k+1}}{2}} \prod_{\substack{l=1\\ l\neq d}}^{k} \frac{Q^{r_{k+1}(l)h_l} a^{-\left(s_{k+1}(l)+2r_{k+1}(l)\right)h_l}}{\left(1-Q^{tr_{k+1}(l)}a^{-t\left(s_{k+1}(l)+2r_{k+1}(l)\right)}\right)}. \tag{3.1}$$

By virtue of (2.10) and (3.1), we have

$$\begin{aligned}
&\operatorname*{Res}_{s_{k+1}(d)=e_{k+1}(d)} \mathcal{L}_{AVU,k}(s\mid\chi)\\
=\ & \sum_{h_1=1}^{t}\chi_1(h_1)\sum_{h_2=1}^{t}\chi_2(h_2(1))\cdots\sum_{h_k=1}^{t}\chi_k(h_k(1)) \operatorname*{Res}_{s_{k+1}(d)=e_{k+1}(d)} \zeta_{AVU,k}(s|\mathbf{h})\\
=\ & \sum_{h_1=1}^{t}\chi_1(h_1)\sum_{h_2=1}^{t}\chi_2(h_2(1))\cdots\sum_{h_k=1}^{t}\chi_k(h_k(1)) D^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}\zeta_t^{-ch_d}}{t\log a}
\end{aligned}$$

$$\times \sum_{\substack{r_1,\ldots,r_{k+1}\geq 0\\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_1}{r_1}\cdots\binom{-s_{k+1}}{r_{k+1}}(-1)^{(\sum_{i=1}^{d-1} r_i)+(d-k)r_{k+1}}$$

$$\times D^{\frac{(\sum_{i=1}^{d-1} s_i)+(d-k)s_{k+1}}{2}} \prod_{\substack{l=1\\ l\neq d}}^{k} \frac{Q^{r_{k+1}(l)h_l} a^{-\left(s_{k+1}(l)+2r_{k+1}(l)\right)h_l}}{\left(1-Q^{tr_{k+1}(l)}a^{-t\left(s_{k+1}(l)+2r_{k+1}(l)\right)}\right)}$$

$$= D^{\frac{e_{k+1}(d)}{2}}(-1)^{r'_{k+1}(d)} \sum_{\substack{r_1,\ldots,r_{k+1}\geq 0\\ r_{k+1}(d)=r'_{k+1}(d)}} \binom{-s_1}{r_1}\cdots\binom{-s_{k+1}}{r_{k+1}}(-1)^{(\sum_{i=1}^{d-1} r_i)+(d-k)r_{k+1}}$$

$$\times D^{\frac{(\sum_{i=1}^{d-1} s_i)+(d-k)s_{k+1}}{2}} \sum_{h_1=1}^{t}\chi_1(h_1)\sum_{h_2=1}^{t}\chi_2(h_2(1))\cdots\sum_{h_k=1}^{t}\chi_k(h_k(1))\frac{\zeta_t^{-ch_d}}{t\log a}$$

$$\times \prod_{\substack{l=1\\ l\neq d}}^{k} \frac{Q^{r_{k+1}(l)h_l} a^{-\left(s_{k+1}(l)+2r_{k+1}(l)\right)h_l}}{\left(1-Q^{tr_{k+1}(l)}a^{-t\left(s_{k+1}(l)+2r_{k+1}(l)\right)}\right)}.$$

This ends the proof. □

Conclusion and scope of future work

In this article we introduced the Apostol-Vu multiple shifted Lucas zeta functions and Apostol-Vu multiple Lucas L-functions associated to Dirichlet characters which are generalizations of some specific functions defined in [11]. For instance, the Apostol-Vu multiple Lucas L-functions generate various multiple L-functions relating to number sequences such as Apostol-Vu multiple Fibonacci L-functions, Apostol-Vu multiple balancing L-functions etc. The analytic continuation of these multiple functions along with the poles and their corresponding residues have been studied in this work. One can investigate the values of these multiple zeta functions and L-functions at integer arguments. There are many other interesting analytic properties that we may consider. Interested researchers can explore some more analytic properties of these type multiple zeta functions, L-functions and can also address similar types of problems implementing different tools.

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