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K-PRODUCT CORDIAL LABELING OF POWERS OF PATHS

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Abstract. Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$, where k is an integer and $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges, respectively labeled with x ($x = 0, 1, ..., k-1$). In this paper, we add some new results on k-product cordial labeling and prove that the graph P_n^2 is 4-product cordial. Further, we study the k-product cordial behaviour of powers of paths P_n^3, P_n^4 and P_n^5 for $k = 3$ and 4.

1. Introduction and Terminology

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [4]. The concepts of labeling of graph has gained a lot of popularity in the field of graph theory during the last 60 years due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [16] published a pioneering paper on graph labeling problems. Thereafter, many types of graph labeling techniques have been studied by several authors. All these labelings are beautifully classified by Gallian [3] in his survey. Cordial labeling is a weaker version of graceful and harmonious labeling was defined by Cahit [1]: Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges

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labeled 0 and the number of edges labeled 1 differ at most by 1. Motivated by the concept of cordial labeling, Sundaram et al. [17] introduced the concept of product cordial labeling: Let f be a function from $V(G)$ to $\{0,1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges, respectively labeled with $i(i = 0, 1)$. Several results have been published on this topic (see [3]).

In 2012, Ponraj et al. [15] extended the concept of product cordial labeling and introduced k-product cordial labeling: Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$, where k is an integer and $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges, respectively labeled with x $(x = 0, 1, ..., k-1)$. They proved that k-product cordial labeling of stars, bistars and also 4-product cordial labeling behavior of paths, complete graphs and combs. Jeyanthi and Maheswari [13] gave the maximum number of edges in a 3-product cordial graph of order p is $\frac{p^2-3p+6}{3}$ if $p \equiv 0(mod\ 3)$, $\frac{p^2-2p+7}{3}$ if $p \equiv 1(mod\ 3)$ and $\frac{p^2-p+4}{3}$ if $p \equiv 2(mod\ 3)$. The same authors [14] proved that the graph P_n^2 is 3-product cordial. Inspired by the concept of k-product cordial labeling and also the results in [13, 14, 15], we made an attempt to study further on k-product cordial labeling. We have established that the following graphs admit k-product cordial labeling: union of graphs [5]; Napier bridge graphs [7]; fan and double fan graphs [11]; cone and double cone graphs [6]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil\frac{p-1}{4}\rceil\lfloor\frac{p-1}{4}\rfloor + 3$ [10]; path graphs [12] and product of graphs [8].

In this paper, we study the k-product cordial labeling of powers of paths. We use the following definition in the present study. Given a graph $G = (V, E)$ and a positive integer d, the d^{th} power of G is the graph $G^d = (V, E')$ in which two vertices are adjacent when they have distance at most d in G [2]. The path of order n is denoted P_n. The d^{th} power of a path is denoted P_n^d.

2. 3-PRODUCT CORDIAL LABELING OF POWERS OF PATHS

In this section, we establish the 3-product cordial labeling of powers of paths P_n^3, P_n^4 and P_n^5.

Theorem 2.1. For $n \geq 3$, the graph P_n^3 is 3-product cordial if and only if $n \equiv 2\ (mod\ 3)$ and $n \neq 5$.

Proof. Let the vertex set and the edge set of P_n^3 be $V(P_n^3) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n^3) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+2})\ ;\ 1 \leq i \leq n-2\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\}$, respectively. We have the following four cases.
Define $f : V(P_n^3) \to \{0, 1, 2\}$ as follows:

Case (i): If $n \equiv 2(mod\ 3)$ for $n \geq 8$, then

$$f(v_i) = \begin{cases} 0 & if\ 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ 1 & if\ \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 3 \\ 2 & if\ \lfloor \frac{n}{3} \rfloor + 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 6. \end{cases}$$

For $i = \lfloor \frac{n}{3} \rfloor + 6 + j\ ;\ 1 \leq j \leq 2\left(\lfloor \frac{n}{3} \rfloor - 2\right)$ and $n > 8$,

$$f(v_i) = \begin{cases} 1 & if\ j \equiv 2, 4, 5, 7(mod\ 8) \\ 2 & if\ j \equiv 1, 3, 6, 0(mod\ 8). \end{cases}$$

Thus we get,
$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1$,
$e_f(0) = e_f(1) = e_f(2) = 3\lfloor \frac{n}{3} \rfloor$.
Hence, P_n^3 is a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n > 5$.

Case (ii): If $n \equiv 0(mod\ 3)$ for $n \geq 3$, then $|V(P_n^3)| = 3t$ and $|E(P_n^3)| = 9t-6$. Thus, $v_f(i) = t\ (i = 0, 1, 2)$ and $e_f(i) = 3t-2\ (i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 3t-2$ for $t \geq 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^3 is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$ for $n \geq 3$.

Case (iii): If $n \equiv 1(mod\ 3)$ for $n \geq 4$, then $|V(P_n^3)| = 3t+1$ and $|E(P_n^3)| = 9t-3$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0, 1, 2)$ and $e_f(i) = 3t-1\ (i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 3t-1$ for $t \geq 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^3 is not a

3-product cordial graph if $n \equiv 1(mod\ 3)$ for $n \geq 4$.

Case (iv): If $n = 5$, then $|V(P_5^3)| = 5$ and $|E(P_5^3)| = 9$. But the maximum number of edges in a 3-product cordial graph of order 5 is 8. Hence, P_n^3 is not a 3-product cordial graph if $n = 5$. □

Example 2.1. An example of 3-product cordial labeling of P_8^3 is shown in Figure 1.

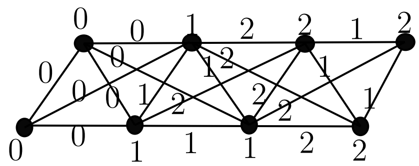

Figure 1

Theorem 2.2. For $n \geq 3$, the graph P_n^4 is 3-product cordial if and only if $n \equiv 2\ (mod\ 3)$ and $n \neq 5, 8$.

Proof. Let the vertex set and the edge set of P_n^4 be $V(P_n^4) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n^4) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+2})\ ;\ 1 \leq i \leq n-2\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\} \cup \{(v_i, v_{i+4})\ ;\ 1 \leq i \leq n-4\}$, respectively. We have the following five cases.

Define $f : V(P_n^4) \rightarrow \{0, 1, 2\}$ as follows:

Case (i): If $n \equiv 2(mod\ 3)$ for $n \geq 11$, then

$$f(v_i) = \begin{cases} 0 & if\ 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ 1 & if\ \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 2,\ \lfloor \frac{n}{3} \rfloor + 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 5 \\ 2 & if\ i = \lfloor \frac{n}{3} \rfloor + 3,\ \lfloor \frac{n}{3} \rfloor + 6 \leq i \leq \lfloor \frac{n}{3} \rfloor + 8. \end{cases}$$

For $i = \lfloor \frac{n}{3} \rfloor + 8 + j\ ;\ 1 \leq j \leq 2\left(\lfloor \frac{n}{3} \rfloor - 3\right)$ and $n > 11$,

$$f(v_i) = \begin{cases} 1 & if\ j \equiv 1, 0(mod\ 4) \\ 2 & if\ j \equiv 2, 3(mod\ 4). \end{cases}$$

From the above labeling we get,

$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1$,

$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = 4\lfloor \frac{n}{3} \rfloor$.

Hence, P_n^4 is a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n \geq 11$.

Case (ii): If $n \equiv 0(mod\ 3)$ for $n \geq 6$, then $|V(P_n^4)| = 3t$ and $|E(P_n^4)| = 12t - 10$. Thus, $v_f(i) = t$ $(i = 0, 1, 2)$ and $e_f(i) = 4t - 3$ *or* $4t - 4$ $(i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 4t - 3$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^4 is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$ for $n \geq 6$.

Case (iii): If $n \equiv 1(mod\ 3)$ for $n \geq 4$, then $|V(P_n^4)| = 3t + 1$ and $|E(P_n^4)| = 12t - 6$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2)$ and $e_f(i) = 4t - 2$ $(i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 4t - 2$ for $t \geq 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^4 is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$ for $n \geq 4$.

Case (iv): If $n = 3$, then $|V(P_3^4)| = 3$ and $|E(P_3^4)| = 3$. Thus, $v_f(i) = 1$ $(i = 0, 1, 2)$ and $e_f(i) = 1$ $(i = 0, 1, 2)$. If $v_f(0) = 1$, then $e_f(0) > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^4 is not a 3-product cordial graph if $n = 3$.

Case (v): If $n = 5$ *or* 8, then $|V(P_5^4)| = 5$, $|E(P_5^4)| = 10$, $|V(P_8^4)| = 8$ and $|E(P_8^4)| = 22$. But the maximum number of edges in a 3-product cordial graph of order 5 and 8 are 8 and 20, respectively. Hence, P_n^4 is not a 3-product cordial graph if $n = 5$ *or* 8. □

Example 2.2. An example of 3-product cordial labeling of P_{11}^4 is shown in Figure 2.

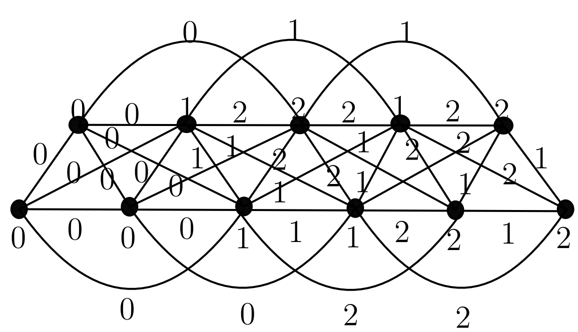

Figure 2

Theorem 2.3. The graph P_n^5 is not 3-product cordial for all $n \geq 3$.

Proof. Let the vertex set and the edge set of P_n^5 be $V(P_n^5) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n^5) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+2})\ ;\ 1 \leq i \leq n-2\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\} \cup \{(v_i, v_{i+4})\ ;\ 1 \leq i \leq n-4\} \cup \{(v_i, v_{i+5})\ ;\ 1 \leq i \leq n-5\}$, respectively. We have the following five cases.

Define $f : V(P_n^5) \to \{0, 1, 2\}$ as follows:

Case (i): If $n \equiv 0(mod\ 3)$ for $n \geq 6$, then $|V(P_n^5)| = 3t$ and $|E(P_n^5)| = 15t - 15$. Thus, $v_f(i) = t$ $(i = 0, 1, 2)$ and $e_f(i) = 5t - 5$ $(i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 5t - 5$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^5 is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$ for $n \geq 6$.

Case (ii): If $n \equiv 1(mod\ 3)$ for $n \geq 7$, then $|V(P_n^5)| = 3t+1$ and $|E(P_n^5)| = 15t - 10$. Thus, $v_f(i) = t\ or\ t + 1$ $(i = 0, 1, 2)$ and $e_f(i) = 5t - 3\ or\ 5t - 4$ $(i = 0, 1, 2)$. If $v_f(0) = t\ or\ t + 1$, then $e_f(0) > 5t - 3$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^5 is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$ for $n \geq 7$.

Case (iii): If $n \equiv 2(mod\ 3)$ for $n \geq 8$, then $|V(P_n^5)| = 3t+2$ and $|E(P_n^5)| = 15t - 5$. Thus, $v_f(i) = t\ or\ t + 1$ $(i = 0, 1, 2)$ and $e_f(i) = 5t - 2\ oe\ 5t - 1$ $(i = 0, 1, 2)$. If $v_f(0) = t\ or\ t + 1$, then $e_f(0) > 5t - 1$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^5 is not a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n \geq 8$.

Case (iv): If $n = 3\ or\ 4$. For $n = 3$, $|V(P_3^5)| = 3$ and $|E(P_3^5)| = 3$. Thus, $v_f(i) = 1$ $(i = 0, 1, 2)$ and $e_f(i) = 1$ $(i = 0, 1, 2)$. If $v_f(0) = 1$, then $e_f(0) > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. For $n = 4$, $|V(P_4^5)| = 4$ and $|E(P_4^5)| = 6$. Thus, $v_f(i) = 1\ or\ 2$ $(i = 0, 1, 2)$ and $e_f(i) = 2$ $(i = 0, 1, 2)$. If $v_f(0) = 1\ or\ 2$, then $e_f(0) > 2$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, P_n^5 is not a 3-product cordial graph if $n = 3\ or\ 4$.

Case (v): If $n = 5$, then $|V(P_5^5)| = 5$ and $|E(P_5^5)| = 10$. But the maximum number of edges in a 3-product cordial graph of order 5 is 8. Hence, P_n^5 is not a 3-product cordial graph if $n = 5$. □

3. 4-PRODUCT CORDIAL LABELING OF POWERS OF PATHS

In this section, we find the 4-product cordial labeling of powers of paths P_n^2, P_n^3, P_n^4 and P_n^5.

Theorem 3.1. For $n \geq 3$, the graph P_n^2 is 4-product cordial if and only if $n = 14$ or $5 \leq n \leq 11$ except 8.

Proof. Let the vertex set and the edge set of P_n^2 be $V(P_n^2) = \{v_i \; ; \; 1 \leq i \leq n\}$ and $E(P_n^2) = \{(v_i, v_{i+1}) \; ; \; 1 \leq i \leq n-1\} \; \cup \; \{(v_i, v_{i+2}) \; ; \; 1 \leq i \leq n-2\}$, respectively. We have the following five cases.
Define $f : V(P_n^2) \rightarrow \{0, 1, 2, 3\}$ as follows:

Case (i): If $n = 14$ or $5 \leq n \leq 11$ except 8, then the 4-product cordial labelings of P_n^2 are shown in Table 1.

Table 1.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
5	0	2	1	3	1									
6	0	2	1	1	3	3								
7	0	2	2	1	1	3	3							
9	0	0	2	1	1	1	3	3	2					
10	0	0	2	1	1	3	3	3	1	2				
11	0	0	2	2	1	1	3	1	3	3	2			
14	0	0	0	2	2	1	1	1	3	3	3	1	2	3

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2, 3$.
Hence, P_n^2 is a 4-product cordial graph if $n = 14$ or $5 \leq n \leq 11$ except 8.

Case (ii): If $n \equiv 0(mod\ 4)$ for $n \geq 4$, then $|V(P_n^2)| = 4t$ and $|E(P_n^2)| = 8t - 3$. Thus, $v_f(i) = t$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t$ *or* $2t - 1$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^2. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t$ and 2 must be assigned nonconsecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) \geq 2t$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, P_n^2 is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n \geq 4$.

Case (iii): If $n \equiv 1(mod\ 4)$ for $n \geq 13$, then $|V(P_n^2)| = 4t + 1$ and $|E(P_n^2)| = 8t - 1$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t$ *or* $2t - 1$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^2. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t$ and 2 must be assigned nonconsecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) \geq 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, P_n^2 is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 13$.

Case (iv): If $n \equiv 2(mod\ 4)$ for $n \geq 18$, then $|V(P_n^2)| = 4t+2$ and $|E(P_n^2)| = 8t+1$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t$ *or* $2t+1$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and we assign 0 to the vertices of the P_n^2 in such a way that $e_f(0) = 2t$ *or* $2t+1$. If $e_f(0) = 2t$. Now $v_f(2) = t$ *or* $t+1$. Let $v_f(2) = t$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t+1$. Then, $e_f(2) \geq 2t+1$ for $t \geq 4$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. Also, $e_f(0) = 2t+1$ can be dealt with the similar way. Hence, P_n^2 is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 18$.

Case (v): If $n \equiv 3(mod\ 4)$ for $n \geq 15$ and $n = 3$, then $|V(P_n^2)| = 4t+3$ and $|E(P_n^2)| = 8t+3$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t$ *or* $2t+1$ $(i = 0, 1, 2, 3)$. For $n = 3$, $v_f(0) = 0$. Otherwise $e_f(0) > 1$. Now $v_f(2) = 1$. Then we have $e_f(2) > 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. For $n \geq 15$, $v_f(0) = t$ and we assign 0 to the vertices of the P_n^2 in such a way that $e_f(0) = 2t$ *or* $2t+1$. Otherwise $e_f(0) > 2t+1$. If $e_f(0) = 2t$. Clearly, $v_f(2) = t+1$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t+1$. Then, $e_f(2) \geq 2t+1$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, P_n^2 is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 15$ and $n = 3$. □

Example 3.1. An example of 4-product cordial labeling of P_6^2 is shown in Figure 3.

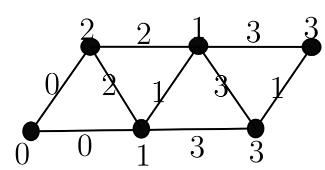

Figure 3

Theorem 3.2. For $n \geq 3$, the graph P_n^3 is 4-product cordial if and only if $n = 10$.

Proof. Let the vertex set and the edge set of P_n^3 be $V(P_n^3) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n^3) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+2})\ ;\ 1 \leq i \leq n-2\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\}$, respectively. We have the following five cases.

Define $f : V(P_n^3) \to \{0, 1, 2, 3\}$ as follows:

Case (i): If $n = 10$, then the 4-product cordial labeling of P_n^3 is shown in Table 2.

Table 2.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
10	0	0	2	1	1	3	1	3	3	2

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2, 3$.
Hence, P_n^3 is a 4-product cordial graph if $n = 10$.

Case (ii): If $n \equiv 0(mod\ 4)$ for $n \geq 4$, then $|V(P_n^3)| = 4t$ and $|E(P_n^3)| = 12t - 6$. Thus, $v_f(i) = t$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t - 1$ *or* $3t - 2$ $(i = 0, 1, 2, 3)$. If $v_f(0) = t$, then $e_f(0) > 3t - 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, P_n^3 is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n \geq 4$.

Case (iii): If $n \equiv 1(mod\ 4)$ for $n \geq 5$, then $|V(P_n^3)| = 4t + 1$ and $|E(P_n^3)| = 12t - 3$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t$ *or* $3t - 1$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^3. Otherwise $e_f(0) > 3t$. Thus, $e_f(0) = 3t$. Now $v_f(2) = t$ *or* $t + 1$. If $v_f(2) = t$, then 2 must be assigned nonconsecutively. Otherwise $e_f(0) > 3t$. Then, $e_f(2) > 3t - 1$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t + 1$. Hence, P_n^3 is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 5$.

Case (iv): If $n \equiv 2(mod\ 4)$ for $n \geq 14$ or $n = 6$, then $|V(P_n^3)| = 4t + 2$ and $|E(P_n^3)| = 12t$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^3. Otherwise $e_f(0) > 3t$. Thus, $e_f(0) = 3t$. Now $v_f(2) = t$ *or* $t + 1$. If $v_f(2) = t$, then 2 must be assigned nonconsecutively. Otherwise $e_f(0) > 3t$. Thus, $e_f(2) > 3t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t + 1$. For $n = 6$, $|E(P_6^3)| = 12$. But the maximum number of edges in a 4-product cordial graph of order 6 is 11. Hence, P_n^3 is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 14$ or $n = 6$.

Case (v): If $n \equiv 3(mod\ 4)$ for $n \geq 3$, then $|V(P_n^3)| = 4t+3$ and $|E(P_n^3)| = 12t+3$. Thus, $v_f(i) = t\ or\ t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 3t\ or\ 3t+1$ $(i = 0,1,2,3)$. Obviously, $v_f(0) = t$ and we assign 0 to the vertices of the P_n^3 in such a way that $e_f(0) = 3t\ or\ 3t+1$. If $e_f(0) = 3t$. Now $v_f(2) = t+1$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 3t+1$. Then, $e_f(2) > 3t+1$ for $t \geq 0$ Therefore $|e_f(i) - e_f(j)| > 1$ for all $i,j = 0,1,2,3$. Hence, P_n^3 is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 3$. □

Example 3.2. An example of 4-product cordial labeling of P_{10}^3 is shown in Figure 4.

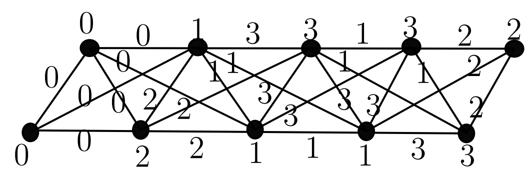

Figure 4

Theorem 3.3. The graph P_n^4 is not 4-product cordial for all $n \geq 3$.

Proof. Let the vertex set and the edge set of P_n^4 be $V(P_n^4) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n^4) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+2})\ ;\ 1 \leq i \leq n-2\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\} \cup \{(v_i, v_{i+4})\ ;\ 1 \leq i \leq n-4\}$, respectively. We have the following four cases.

Define $f : V(P_n^4) \to \{0,1,2,3\}$ as follows:

Case (i): If $n \equiv 0(mod\ 4)$ for $n \geq 4$, then $|V(P_n^4)| = 4t$ and $|E(P_n^4)| = 16t-10$. Thus, $v_f(i) = t$ $(i = 0,1,2,3)$ and $e_f(i) = 4t-2\ or\ 4t-3$ $(i = 0,1,2,3)$. If $v_f(0) = t$, then $e_f(0) > 4t-2$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i,j = 0,1,2,3$. Hence, P_n^4 is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n \geq 4$.

Case (ii): If $n \equiv 1(mod\ 4)$ for $n \geq 5$, then $|V(P_n^4)| = 4t+1$ and $|E(P_n^4)| = 16t-6$. Thus, $v_f(i) = t\ or\ t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 4t-1\ or\ 4t-2$ $(i = 0,1,2,3)$. If $v_f(0) = t\ or\ t+1$, then $e_f(0) > 4t-1$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i,j = 0,1,2,3$. Hence, P_n^4 is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 5$.

Case (iii): If $n \equiv 2(mod\ 4)$ for $n \geq 6$. For $n \geq 14$, $|V(P_n^4)| = 4t+2$ and $|E(P_n^4)| = 16t-2$. Thus, $v_f(i) = t\ or\ t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 4t\ or\ 4t-1$ $(i = 0,1,2,3)$.

Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^4. Otherwise $e_f(0) > 4t$. Thus, $e_f(0) = 4t$. Now $v_f(2) = t \ or \ t+1$. If $v_f(2) = t$, then 2 must be assigned nonconsecutively. Otherwise $e_f(0) > 4t$. Then, $e_f(2) > 4t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. For $n = 6$, $|E(P_6^4)| = 14$. But the maximum number of edges in a 4-product cordial graph of order 6 is 11. For $n = 10$, $|E(P_{10}^4)| = 30$. But the maximum number of edges in a 4-product cordial graph of order 10 is 27. Hence, P_n^4 is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 6$.

Case (iv): If $n \equiv 3(mod\ 4)$ for $n \geq 3$. For $n \geq 7$, $|V(P_n^4)| = 4t+3$ and $|E(P_n^4)| = 16t+2$. Thus, $v_f(i) = t \ or \ t+1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 4t \ or \ 4t+1$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^4. Otherwise $e_f(0) > 4t$. Thus, $e_f(0) = 4t$. Clearly, $v_f(2) = t+1$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 4t+1$. Then, $e_f(2) > 4t+1$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. For $n = 3$, $|V(P_3^4)| = 3$ and $|E(P_3^4)| = 3$. Thus, $v_f(i) = 0 \ or \ 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 0 \ or \ 1$ $(i = 0, 1, 2, 3)$. If $v_f(0) = 0$, then $e_f(0) = 0$. Otherwise $e_f(0) > 1$. If $v_f(2) = 1$, then $e_f(2) > 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, P_n^4 is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 3$. □

Theorem 3.4. The graph P_n^5 is not 4-product cordial for all $n \geq 3$.

Proof. Let the vertex set and the edge set of P_n^5 be $V(P_n^5) = \{v_i \ ; \ 1 \leq i \leq n\}$ and $E(P_n^5) = \{(v_i, v_{i+1}) \ ; \ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+2}) \ ; \ 1 \leq i \leq n-2\} \cup \{(v_i, v_{i+3}) \ ; \ 1 \leq i \leq n-3\} \cup \{(v_i, v_{i+4}) \ ; \ 1 \leq i \leq n-4\} \cup \{(v_i, v_{i+5}) \ ; \ 1 \leq i \leq n-5\}$, respectively. We have the following four cases.
Define $f : V(P_n^5) \rightarrow \{0, 1, 2, 3\}$ as follows:

Case (i): If $n \equiv 0(mod\ 4)$ for $n \geq 4$. For $n \geq 8$, $|V(P_n^5)| = 4t$ and $|E(P_n^5)| = 20t-15$. Thus, $v_f(i) = t$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 5t-3 \ or \ 5t-4$ $(i = 0, 1, 2, 3)$. If $v_f(0) = t$, then $e_f(0) > 5t-3$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. For $n = 4$, $|V(P_4^5)| = 4$ and $|E(P_4^5)| = 6$. Thus, $v_f(i) = 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 1 \ or \ 2$ $(i = 0, 1, 2, 3)$. If $v_f(0) = 1$, then $e_f(0) > 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $j = 0, 1, 2, 3$. Hence, P_n^5 is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n \geq 4$.

Case (ii): If $n \equiv 1(mod\ 4)$ for $n \geq 5$, then $|V(P_n^5)| = 4t+1$ and $|E(P_n^5)| = 20t-10$. Thus, $v_f(i) = t\ or\ t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 5t-2\ or\ 5t-3$ $(i = 0,1,2,3)$. If $v_f(0) = t\ or\ t+1$, then $e_f(0) > 5t-2$ for $t \geq 1$ Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0,1,2,3$. Hence, P_n^5 is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 5$.

Case (iii): If $n \equiv 2(mod\ 4)$ for $n \geq 6$, then $|V(P_n^5)| = 4t+2$ and $|E(P_n^5)| = 20t-5$. Thus, $v_f(i) = t\ or\ t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 5t-1\ or\ 5t-2$ $(i = 0,1,2,3)$. If $v_f(0) = t\ or\ t+1$, then $e_f(0) > 5t-1$ for $t \geq 1$ Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0,1,2,3$. Hence, P_n^5 is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 6$.

Case (iv): If $n \equiv 3(mod\ 4)$ for $n \geq 3$. For $n \geq 7$ then $|V(P_n^5)| = 4t+3$ and $|E(P_n^5)| = 20t$. Thus, $v_f(i) = t\ or\ t+1$ $(i = 0,1,2,3)$ and $e_f(i) = 5t$ $(i = 0,1,2,3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or at the end of P_n^5. Otherwise $e_f(0) > 5t$. Thus, $e_f(0) = 5t$. Clearly, $v_f(2) = t+1$ and 2 must be assigned nonconsecutively. Otherwise $e_f(0) > 5t$. Then, $e_f(2) > 5t$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0,1,2,3$. For $n = 3$, $|V(P_3^5)| = 3$ and $|E(P_3^5)| = 3$. Thus, $v_f(i) = 0\ or\ 1$ $(i = 0,1,2,3)$ and $e_f(i) = 0\ or\ 1$ $(i = 0,1,2,3)$. If $v_f(0) = 0$, then $e_f(0) = 0$. Otherwise $e_f(0) > 1$. If $v_f(2) = 1$, then $e_f(2) > 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0,1,2,3$. Hence, P_n^5 is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 3$. □

4. Conclusion

In this paper, we find some new results on k-product cordial labeling and establish the 3-product cordial behaviour of the powers of paths P_n^3, P_n^4 and P_n^5. Also, we study the 4-product cordial behaviour of powers of paths P_n^2, P_n^3, P_n^4 and P_n^5. We conclude this paper with the following open problem.

Open problem:

Find k-product cordial labeling of P_n^d for $k \geq 5$.

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