Jordan Journal of Mathematics and Statistics (JJMS), 15(4A), 2022, pp 925 - 939
DOI: https://doi.org/10.47013/15.4.9

VAGUE MODULES ON THE BASE OF $([0,1], \leq, \wedge)$

SEVDA SEZER[(1)] AND MURAT YÜKSEL[(2)]

Abstract. In this paper, the concepts of vague module, vague submodule, vague module homomorphism and vague module isomorphism based on the Demirci's vague groups are defined. Then various elementary properties of these concepts are obtained, and the validity of some relevant classical results in these settings are investigated.

1. Introduction

The concept of fuzzy set was defined in [12] by Zadeh after reconsideration of the concept of classical mathematics began. Thereafter, the concept of fuzzy subgroup was introduced in [9] by Rosenfeld as a natural generalization of the concept of subgroup and have been widely studied.

Following this, a new object related to groups called vague groups was introduced and studied by Demirci in [2] by forcing the operations of the group to be compatible with a given fuzzy equality. After, the theory of some vague algebraic notions was established in [3–8, 10, 11]. This work introduces some elementary properties of vague module, vague submodule, vague module homomorphism and vague module isomorphism and establishes some new results.

After this introductory Section, Section 2 is devoted to some definitions and properties related to vague groups, generalized vague subgroups, vague rings, vague subrings, vague ideals and vague homomorphisms that will be needed later. In Section 3, the definitions of vague module, vague submodule, vague module homomorphism

2010 *Mathematics Subject Classification.* 03E7, 08A72, 16D99, 20N25.
Key words and phrases. Vague group, generalized vague subgroup, vague ring, vague homomorphism, vague module, vague submodule, vague module homomorphism, vague module isomorphism.

Received: Jun. 14, 2021 Accepted: April 14, 2022 .

and vague module isomorphism are given on the basis of $([0,1], \leq, \wedge)$ and some basic properties of these concepts are investigated.

2. Preliminaries

The notions of fuzzy equality, strong fuzzy function, vague group, generalized vague subgroup, vague rings, vague subrings, vague ideals, vague homomorphisms, vague isomorphisms and their fundamental properties are introduced in [1–3, 10, 11]. Our aim in this section is to recall these notions and some of their elementary properties, which will be needed in this paper.

The symbols "$\wedge$" and "$\vee$" will always stand for the minimum and maximum operations between finitely many real numbers, respectively; and X, Y, G will always stand for crisp and nonempty sets in this paper.

Definition 2.1. [1] A mapping $E_X : X \times X \to [0,1]$ is called a fuzzy equality on X if the following conditions are satisfied:

(E.1) $E_X(x,y) = 1 \Longleftrightarrow x = y$, $\forall x, y \in X$,

(E.2) $E_X(x,y) = E_X(y,x)$, $\forall x, y \in X$,

(E.3) $E_X(x,y) \wedge E_X(y,z) \leq E_X(x,z)$, $\forall x, y, z \in X$.

For $x, y \in X$, the real number $E_X(x,y)$ shows the degree of the equality of x and y. One can always define a fuzzy equality on X with respect to (abbreviated to "w.r.t.") the classical equality of the elements of X. Indeed, the mapping $E^c_X : X \times X \to [0,1]$, defined by

$$E^c_X(x,y) = \begin{cases} 1 & , \ if\ x = y \\ 0 & , \ otherwise \end{cases}$$

is obviously a fuzzy equality on X.

Furthermore, for $k \in (0,1]$ the mapping $E_X : X \times X \to [0,1]$, defined by

$$E_X(x,y) = \begin{cases} 1 & , \ if\ x = y \\ k & , \ otherwise \end{cases}$$

is a fuzzy equality on X.

Definition 2.2. [3] Let E_X and E_Y be two fuzzy equalities on X and Y, respectively. Then a fuzzy relation $\tilde{\circ}$ from X to Y (i.e., a fuzzy subset $\tilde{\circ}$ of $X \times Y$) is called a strong fuzzy function from X to Y w.r.t. the fuzzy equalities E_X and E_Y, denoted by $\tilde{\circ} : X \rightsquigarrow Y$, if the characteristic function $\mu_{\tilde{\circ}} : X \times Y \to [0,1]$ of $\tilde{\circ}$ satisfies the following two conditions:

(F.1) For each $x \in X$, there exists $y \in Y$ such that $\mu_{\tilde{\circ}}(x,y) = 1$,

(F.2) For each $x_1, x_2 \in X$, $y_1, y_2 \in Y$,

$$\mu_{\tilde{\circ}}(x_1, y_1) \wedge \mu_{\tilde{\circ}}(x_2, y_2) \wedge E_X(x_1, x_2) \leq E_Y(y_1, y_2).$$

The concepts of vague binary operation on X and transitivity of a vague binary operation are defined by Demirci as follows.

Definition 2.3. [2, 3]

(i) A strong fuzzy function $\tilde{\circ} : X \times X \rightsquigarrow X$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X is called a vague binary operation on X w.r.t. $E_{X\times X}$ and E_X. (For all $(x_1, x_2) \in X \times X$, $x_3 \in X$, $\mu_{\tilde{\circ}}((x_1, x_2), x_3)$ will be denoted by $\mu_{\tilde{\circ}}(x_1, x_2, x_3)$ for the sake of simplicity.)

(ii) A vague binary operation $\tilde{\circ}$ on X w.r.t. $E_{X\times X}$ and E_X is said to be transitive of the first order if $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(c,d) \leq \mu_{\tilde{\circ}}(a,b,d)$ for all $a,b,c,d \in X$.

(iii) A vague binary operation $\tilde{\circ}$ on X w.r.t. $E_{X\times X}$ and E_X is said to be transitive of the second order if $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(b,d) \leq \mu_{\tilde{\circ}}(a,d,c)$ for all $a,b,c,d \in X$.

(iv) A vague binary operation $\tilde{\circ}$ on X w.r.t. $E_{X\times X}$ and E_X is said to be transitive of the third order if $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(a,d) \leq \mu_{\tilde{\circ}}(d,b,c)$ for all $a,b,c,d \in X$.

Definition 2.4. [2] Let $\tilde{\circ}$ be a vague binary operation on G w.r.t. a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality E_G on G. Then

(i) G together with $\tilde{\circ}$, denoted by $< G, \tilde{\circ}, E_{G\times G}, E_G >$ or simply $< G, \tilde{\circ} >$, is called a vague semigroup if the characteristic function $\mu_{\tilde{\circ}} : G \times G \times G \to [0,1]$ of $\tilde{\circ}$ fulfills the condition: For all $a,b,c,d,m,q,w \in G$,

$$\mu_{\tilde{\circ}}(b,c,d) \wedge \mu_{\tilde{\circ}}(a,d,m) \wedge \mu_{\tilde{\circ}}(a,b,q) \wedge \mu_{\tilde{\circ}}(q,c,w) \leq E_G(m,w).$$

(ii) A vague semigroup $< G, \tilde{\circ} >$ is called a vague monoid if there exists a two-sided identity element $e_{\tilde{\circ}} \in G$, that is an element $e_{\tilde{\circ}}$ satisfying $\mu_{\tilde{\circ}}(e_{\tilde{\circ}}, a, a) \wedge \mu_{\tilde{\circ}}(a, e_{\tilde{\circ}}, a) = 1$ for each $a \in G$.

(iii) A vague monoid $< G, \tilde{\circ} >$ is called a vague group if for each $a \in G$, there exists a two-sided inverse element $a^{-1} \in G$, that is an element a^{-1} satisfying $\mu_{\tilde{\circ}}(a^{-1}, a, e_{\tilde{\circ}}) \wedge \mu_{\tilde{\circ}}(a, a^{-1}, e_{\tilde{\circ}}) = 1$.

(iv) A vague semigroup $< G, \tilde{\circ} >$ is said to be commutative (Abelian) if $\mu_{\tilde{\circ}}(a, b, m) \wedge \mu_{\tilde{\circ}}(b, a, w) \leq E_G(m, w)$ for each $a, b, m, w \in G$.

In the rest of this paper, the notation $< G, \tilde{\circ} >$ always stands for the vague group $< G, \tilde{\circ} >$ w.r.t. a fuzzy equality $E_{G \times G}$ on $G \times G$ and a fuzzy equality E_G on G.

Proposition 2.1. *[2] For a given vague group $< G, \tilde{\circ} >$, there exists a unique binary operation in the classical sense, denoted by $\circ$, on G such that $< G, \circ >$ is a group in the classical sense.*

The binary operation "$\circ$" in Proposition 2.1 is explicitly given by the equivalence

$$a \circ b = c \iff \mu_{\tilde{\circ}}(a, b, c) = 1, \quad \forall a, b, c \in G. \tag{2.1}$$

The binary operation "$\circ$", defined by the equivalence (2.1), is called the ordinary description of $\tilde{\circ}$, and is denoted by $\circ = ord(\tilde{\circ})$ in [3, 5].

If $\tilde{\circ}$ is a vague binary operation on G w.r.t. a fuzzy equality $E_{G \times G}$ on $G \times G$ and a fuzzy equality E_G on G, in the rest of this paper the ordinary description of $\tilde{\circ}$ will be denoted by $\circ$. In this case, from [3, 5] we have the following property

$$\mu_{\tilde{\circ}}(a, b, a \circ b) = 1 \quad \text{and} \quad \mu_{\tilde{\circ}}(a, b, c) \leq E_G(a \circ b, c)\ ,\ \forall a, b, c \in G. \tag{2.2}$$

For a given fuzzy equality E_G on G and for a crisp subset A of G, the restriction of the mapping E_G to $A \times A$, denoted by E_A, is obviously a fuzzy equality on A.

Definition 2.5. [10] Let $< G, \tilde{\circ} >$ be a vague group and A be a nonempty, crisp subset of G. Let $\tilde{\odot}$ be a vague binary operation on A such that

$$\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c), \ \forall a, b, c \in A.$$

If $< A, \tilde{\odot} >$ is itself a vague group w.r.t. the fuzzy equalities $E_{A \times A}$ on $A \times A$ and E_A on A, then $< A, \tilde{\odot} >$ is said to be a generalized vague subgroup of $< G, \tilde{\circ} >$, denoted by $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.

For a given vague group $< G, \tilde{\circ} >$, because of the uniqueness of the identity and the inverse of an element of $< G, \tilde{\circ} >$, it can be easily seen that if $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$, then the identity of $< A, \tilde{\odot} >$ and the inverse of $x \in A$ w.r.t. $< A, \tilde{\odot} >$ are the identity of $< G, \tilde{\circ} >$ and the inverse of $x \in A$ w.r.t. $< G, \tilde{\circ} >$, i.e., $e_A = e_G$ and $x_A^{-1} = x_G^{-1}$, respectively.

Definition 2.6. [2] Let $< G, \tilde{\circ} >$ and $< H, \tilde{\star} >$ be two vague semigroups. A function (in the classical sense) $f : G \to H$ is called a vague homomorphism if $\mu_{\tilde{\circ}}(a, b, c) \leq \mu_{\tilde{\star}}(f(a), f(b), f(c))$, $\forall a, b, c \in G$. In this case, the crisp set $\{g \in G : f(g) = e_H\}$ is called the vague kernel of f, and is denoted by $VKerf$.

In a similar fashion to classical algebra, the concepts of vague ring, vague subring and vague ideal are defined in [11] as follows:

Definition 2.7. [11] Let $E_{\mathcal{R}\times\mathcal{R}}$ and $E_{\mathcal{R}}$ be fuzzy equalities on $\mathcal{R} \times \mathcal{R}$ and $\mathcal{R}$, respectively. Let $\tilde{\oplus}$, $\tilde{\odot}$ be two vague binary operations on $\mathcal{R}$. Then, the 3-tuple $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ is called a vague ring w.r.t. $E_{\mathcal{R}\times\mathcal{R}}$ and $E_{\mathcal{R}}$ if the following three conditions are satisfied:

(VR.1) $< \mathcal{R}, \tilde{\oplus} >$ is a commutative vague group,

(VR.2) $< \mathcal{R}, \tilde{\odot} >$ is a vague semigroup,

(VR.3) $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ satisfies distributive laws, i.e., $\forall a, b, c, d, t, x, y, z \in \mathcal{R}$,

$$\mu_{\tilde{\odot}}(x, y, a) \wedge \mu_{\tilde{\odot}}(x, z, b) \wedge \mu_{\tilde{\oplus}}(a, b, c) \wedge \mu_{\tilde{\oplus}}(y, z, d) \wedge \mu_{\tilde{\odot}}(x, d, t) \leq E_{\mathcal{R}}(t, c),$$

$$\mu_{\tilde{\odot}}(x, z, a) \wedge \mu_{\tilde{\odot}}(y, z, b) \wedge \mu_{\tilde{\oplus}}(a, b, c) \wedge \mu_{\tilde{\oplus}}(x, y, d) \wedge \mu_{\tilde{\odot}}(d, z, t) \leq E_{\mathcal{R}}(t, c).$$

(VR.4) A vague ring $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ is said to be a vague ring with identity if there exists $e_{\tilde{\odot}} \in \mathcal{R}$ such that $\mu_{\tilde{\odot}}(x, e_{\tilde{\odot}}, x) \wedge \mu_{\tilde{\odot}}(e_{\tilde{\odot}}, x, x) = 1$ for each $x \in \mathcal{R}$.

(VR.5) A vague ring $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ is said to be a commutative (Abelian) if

$$\mu_{\tilde{\odot}}(x, y, s) \wedge \mu_{\tilde{\odot}}(y, x, t) \leq E_{\mathcal{R}}(s, t) \ , \ \forall x, y, s, t \in \mathcal{R}.$$

In the rest of this paper, the notation $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ always stands for the vague ring $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ w.r.t. $E_{\mathcal{R}\times\mathcal{R}}$ and $E_{\mathcal{R}}$. If $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ is a vague ring, then we denote the inverse of a by $-a$ w.r.t. the vague group $< \mathcal{R}, \tilde{\oplus} >$; additionally if $< \mathcal{R}, \tilde{\odot} >$ is a vague group, then we denote the inverse of a by a^{-1} w.r.t. the vague group $< \mathcal{R}, \tilde{\odot} >$.

Definition 2.8. [11] Let $<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$ be a vague ring and A be a nonempty, crisp subset of $\mathcal{R}$. Let $\tilde{+}$ and $\tilde{\cdot}$ be two vague binary operations on A such that

$$\mu_{\tilde{+}}(a,b,c)\leq\mu_{\tilde{\oplus}}(a,b,c)\ ,\quad\mu_{\tilde{\cdot}}(a,b,c)\leq\mu_{\tilde{\odot}}(a,b,c),\quad\forall a,b,c\in A.$$

If $<A,\tilde{+},\tilde{\cdot}>$ is itself a vague ring w.r.t. $E_{A\times A}$ and E_A, then $<A,\tilde{+},\tilde{\cdot}>$ is said to be a vague subring of $<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$, denoted by $<A,\tilde{+},\tilde{\cdot}>\overset{v.r}{\leq}<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$.

Definition 2.9. [11] Let $<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$ be a vague ring and $<A,\tilde{+},\tilde{\cdot}>\overset{v.r}{\leq}<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$. If for all $a\in A$ and for all $h,t,s\in\mathcal{R}$

$$\mu_{\tilde{\odot}}(a,h,t)=1\Longrightarrow t\in A\quad\text{and}\quad\mu_{\tilde{\odot}}(h,a,s)=1\Longrightarrow s\in A,$$

then $<A,\tilde{+},\tilde{\cdot}>$ is said to be a vague ideal of $<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$, it is denoted by $<A,\tilde{+},\tilde{\cdot}>\overset{v.i}{\leq}<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$.

It is clear from Definition 2.9 that if $E_{\mathcal{R}}=E^c_{\mathcal{R}}$, $E_{\mathcal{R}\times\mathcal{R}}=E^c_{\mathcal{R}\times\mathcal{R}}$, $\mu_{\tilde{\oplus}}(\mathcal{R}\times\mathcal{R}\times\mathcal{R})\in\{0,1\}$ and $<A,\tilde{+},\tilde{\cdot}>\overset{v.i}{\leq}<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$, then $<A,+,.>$ is an ideal of $<\mathcal{R},\oplus,\odot>$. Therefore, in this case, a vague ideal $<A,\tilde{+},\tilde{\cdot}>$ of $<\mathcal{R},\tilde{\oplus},\tilde{\odot}>$ is nothing but an ideal of the classical ring $<\mathcal{R},\oplus,\odot>$ in the classical sense.

3. Vague Module

In this section, we will define the concepts of vague module, vague submodule, vague homomorphism, vague isomorphism, which are some of the basic concepts of this work, and we will obtain some fundamental properties of these concepts.

Definition 3.1. Let $<\mathcal{R},\tilde{\oplus},\tilde{\odot},E_{\mathcal{R}\times\mathcal{R}},E_{\mathcal{R}}>$ be a vague ring and $<A,\tilde{+}>$ be an Abelian vague group. For each $a,a_1,a_2,a'_1,a'_2,u,v,w\in A$ and $r,r',r_1,r_2\in\mathcal{R}$,

(1) If $f:\mathcal{R}\times A\rightsquigarrow A$ is a fuzzy function such that

(VM 1) $\mu_{\tilde{+}}(a_1,a_2,u)\wedge\mu_f(r,u,v)\wedge\mu_f(r,a_1,a'_1)\wedge\mu_f(r,a_2,a'_2)\wedge\mu_{\tilde{+}}(a'_1,a'_2,w)\leq E_A(v,w)$,

(VM 2) $\mu_{\tilde{\oplus}}(r_1,r_2,r)\wedge\mu_f(r,a,v)\wedge\mu_f(r_1,a,a_1)\wedge\mu_f(r_2,a,a_2)\wedge\mu_{\tilde{+}}(a_1,a_2,w)\leq E_A(v,w)$,

(VM 3) $\mu_{\tilde{\odot}}(r_1,r_2,r')\wedge\mu_f(r',a,v)\wedge\mu_f(r_2,a,u)\wedge\mu_f(r_1,u,w)\leq E_A(v,w)$

then $<A,\tilde{+}>$ is said to be a vague left $\mathcal{R}$-module.

(2) If $g:A\times\mathcal{R}\rightsquigarrow A$ is a fuzzy function such that

(VM 1 ′) $\mu_{\tilde{+}}(a_1, a_2, u) \wedge \mu_g(u, r, v) \wedge \mu_g(a_1, r, a_1') \wedge \mu_g(a_2, r, a_2') \wedge \mu_{\tilde{\oplus}}(a_1', a_2', w) \leq E_A(v, w)$,

(VM 2 ′) $\mu_{\tilde{\oplus}}(r_1, r_2, r) \wedge \mu_g(a, r, v) \wedge \mu_g(a, r_1, a_1) \wedge \mu_g(a, r_2, a_2) \wedge \mu_{\tilde{+}}(a_1, a_2, w) \leq E_A(v, w)$,

(VM 3 ′) $\mu_{\tilde{\odot}}(r_1, r_2, r') \wedge \mu_g(a, r', v) \wedge \mu_g(a, r_1, u) \wedge \mu_g(u, r_2, w) \leq E_A(v, w)$

then $< A, \tilde{+} >$ is said to be a vague right $\mathcal{R}$-module.

(3) If $< A, \tilde{+} >$ is both a vague left $\mathcal{R}$-module and a vague right $\mathcal{R}$-module then $< A, \tilde{+} >$ (in shortly, A) is said to be a vague $\mathcal{R}$-module.

(4) Let $\mathcal{R}$ be a vague ring with identity.

a) If $< A, \tilde{+} >$ is a vague left $\mathcal{R}$-module and

$$\mu_f(1_{\mathcal{R}}, a, s) \leq E_A(s, a), \ \forall a, s \in A$$

then $< A, \tilde{+} >$ is said to be a unitary vague left $\mathcal{R}$-module.

b) If $< A, \tilde{+} >$ is a vague right $\mathcal{R}$-module and

$$\mu_g(a, 1_{\mathcal{R}}, t) \leq E_A(a, t), \ \forall a, t \in A$$

then $< A, \tilde{+} >$ is said to be a unitary vague right $\mathcal{R}$-module.

c) If $< A, \tilde{+} >$ is both a unitary vague left $\mathcal{R}$-module and a unitary vague right $\mathcal{R}$-module then $< A, \tilde{+} >$ is said to be a unitary vague $\mathcal{R}$-module.

It is clear from Definition 3.1 that, if $< A, \tilde{+}, E_{A\times A}, E_A >$ is a vague $\mathcal{R}$-module such that $E_{\mathcal{R}\times\mathcal{R}} = E^c_{\mathcal{R}\times\mathcal{R}}$, $E_{\mathcal{R}} = E^c_{\mathcal{R}}$ and fuzzy functions $\mu_{\tilde{\oplus}}, \mu_f$ are classical functions then a vague $\mathcal{R}$-module $< A, \tilde{+} >$ is a classical $\mathcal{R}$-module. Therefore, in this case, a vague $\mathcal{R}$-module is nothing but a module in the classical case.

Notation . *If the binary operation $\tilde{+}$ on A, fuzzy equalities $E_{A\times A}$ and E_A are known, then the sentence "A is a vague left (right) $\mathcal{R}$-module" will be written instead of "$< A, \tilde{+} >$ is a vague left (right) $\mathcal{R}$-module".*

Example 3.1. *Let* $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ *be a vague ring,* $< A, \tilde{+} >$ *be an Abelian vague group and* $f : \mathcal{R} \times A \rightsquigarrow A$ *be a fuzzy function such that*

$$\mu_f(r, a, a') = \begin{cases} 1 & , \quad a' = e_A \\ 0 & , \quad a' \neq e_A \end{cases}$$

then we can obtain A *is a vague* $\mathcal{R}$*-module. Indeed,* μ_f *is a (strong) fuzzy function because for* $a' \neq e_A$ *or* $a'' \neq e_A$

$$\mu_f(r, a, a') \wedge \mu_f(r, a, a'') \wedge E_{\mathcal{R}\times A}((r, a), (r_1, a_1)) \leq E_A(a', a'') \tag{3.1}$$

i.e. the condition (F.2) is satisfied. On the other hand, if $a' = a'' = e_A$*, then inequality (3.1) is satisfied from* $E_A(a', a'') = 1$ *. Also, for all* $(r, a) \in \mathcal{R} \times A$*,* $\mu_f(r, a, e_A) = 1$*, then the condition (F.3) is obtained, i.e.,* $f : \mathcal{R} \times A \rightsquigarrow A$ *is a strong vague function. Now, let us show that* A *is a vague* $\mathcal{R}$*-module:*

For $v = a_1' = a_2' = e_A$ *at the condition (VM 1), we obtain*

$$\begin{aligned} & \mu_{\tilde{+}}(a_1, a_2, u) \wedge \mu_f(r, u, v) \wedge \mu_f(r, a_1, a_1') \wedge \mu_f(r, a_2, a_2') \wedge \mu_{\tilde{+}}(a_1', a_2', w) \\ = \; & \mu_{\tilde{+}}(a_1, a_2, u) \wedge \mu_f(r, u, e_A) \wedge \mu_f(r, a_1, e_A) \wedge \mu_f(r, a_2, e_A) \wedge \mu_{\tilde{+}}(e_A, e_A, w) \\ = \; & \mu_{\tilde{+}}(a_1, a_2, u) \wedge \mu_{\tilde{+}}(e_A, e_A, w). \end{aligned}$$

From $\mu_{\tilde{+}}(e_A, e_A, w) \leq E_A(e_A, w)$ *we can write that*

$$\mu_{\tilde{+}}(a_1, a_2, u) \wedge E_A(e_A, w) \leq E_A(e_A, w).$$

For $v = a_1 = a_2 = e_A$ *at the condition (VM 2), we have*

$$\begin{aligned} & \mu_{\tilde{\oplus}}(r_1, r_2, r) \wedge \mu_f(r, a, v) \wedge \mu_f(r_1, a, a_1) \wedge \mu_f(r_2, a, a_2) \wedge \mu_{\tilde{+}}(a_1, a_2, w) \\ = \; & \mu_{\tilde{\oplus}}(r_1, r_2, r) \wedge \mu_f(r, a, e_A) \wedge \mu_f(r_1, a, e_A) \wedge \mu_f(r_2, a, e_A) \wedge \mu_{\tilde{+}}(e_A, e_A, w) \\ = \; & \mu_{\tilde{\oplus}}(r_1, r_2, r) \wedge \mu_{\tilde{+}}(e_A, e_A, w) \\ \leq \; & E_A(e_A, w). \end{aligned}$$

Also, for $v = u = w = e_A$ *at the condition (VM 3) we can obtain that*

$$\begin{aligned} & \mu_{\tilde{\odot}}(r_1, r_2, r') \wedge \mu_f(r', a, v) \wedge \mu_f(r_2, a, u) \wedge \mu_f(r_1, u, w) \\ = \; & \mu_{\tilde{\odot}}(r_1, r_2, r') \wedge \mu_f(r', a, e_A) \wedge \mu_f(r_2, a, e_A) \wedge \mu_f(r_1, u, e_A) \\ \leq \; & E_A(e_A, e_A) = 1, \end{aligned}$$

i.e., A is a vague $\mathcal{R}$-module.

In here, A is not a unitary vague $\mathcal{R}$-module, because the inequality $\mu_f(1_{\mathcal{R}}, a, s) \leq E_A(a, s)$ is not satisfied for $s = e_A \neq a$.

Proposition 3.1. *Let $< S, \tilde{+}, \tilde{\cdot} > \overset{v.r.}{\leq} < \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ and $f : S \times \mathcal{R} \rightsquigarrow \mathcal{R}$ be a fuzzy function such that*

$$\mu_f(s, r, r') = \mu_{\tilde{\odot}}(s, r, r'), \ \forall s \in S, \forall r, r' \in \mathcal{R}. \tag{3.2}$$

Then, $\mathcal{R}$ is a vague S-module.

Proof. Since $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ is a vague ring then we have that $< \mathcal{R}, \tilde{\oplus} >$ is an Abelian vague group and $\forall r_1, r_2, r, r'_0, r'_1, r'_2, r' \in \mathcal{R}$; $\forall s \in S$

$$\begin{aligned} & \mu_{\tilde{\oplus}}(r_1, r_2, r) \wedge \mu_f(s, r, r') \wedge \mu_f(s, r_1, r'_1) \wedge \mu_f(s, r_2, r'_2) \wedge \mu_{\tilde{\oplus}}(r'_1, r'_2, r'_0) \\ = \ & \mu_{\tilde{\oplus}}(r_1, r_2, r) \wedge \mu_{\tilde{\odot}}(s, r, r') \wedge \mu_{\tilde{\odot}}(s, r_1, r'_1) \wedge \mu_{\tilde{\odot}}(s, r_2, r'_2) \wedge \mu_{\tilde{\oplus}}(r'_1, r'_2, r'_0) \\ \leq \ & E_{\mathcal{R}}(r', r'_0). \end{aligned}$$

So, the condition (VM 1) of Definition 3.1 is satisfied. From Definition 2.7, we have $\mu_{\tilde{+}}(s_1, s_2, s) \leq \mu_{\tilde{\oplus}}(s_1, s_2, s)$ and

$$\begin{aligned} & \mu_{\tilde{+}}(s_1, s_2, s) \wedge \mu_f(s, r, r') \wedge \mu_f(s_1, r, r_1) \wedge \mu_f(s_2, r, r_2) \wedge \mu_{\tilde{\oplus}}(r_1, r_2, r_0) \\ = \ & \mu_{\tilde{+}}(s_1, s_2, s) \wedge \mu_{\tilde{\odot}}(s, r, r') \wedge \mu_{\tilde{\odot}}(s_1, r, r_1) \wedge \mu_{\tilde{\odot}}(s_2, r, r_2) \wedge \mu_{\tilde{\oplus}}(r_1, r_2, r_0) \\ \leq \ & \mu_{\tilde{\oplus}}(s_1, s_2, s) \wedge \mu_{\tilde{\odot}}(s, r, r') \wedge \mu_{\tilde{\odot}}(s_1, r, r_1) \wedge \mu_{\tilde{\odot}}(s_2, r, r_2) \wedge \mu_{\tilde{\oplus}}(r_1, r_2, r_0) \\ \leq \ & E_{\mathcal{R}}(r', r_0). \end{aligned}$$

So, the condition (VM 2) is satisfied. Using Definition 2.8, we have $\mu_{\tilde{\cdot}}(s_1, s_2, s') \leq \mu_{\tilde{\odot}}(s_1, s_2, s')$ and from $< \mathcal{R}, \tilde{\odot} >$ is a vague semigroup;

$$\begin{aligned} & \mu_{\tilde{\cdot}}(s_1, s_2, s') \wedge \mu_f(s', r, r') \wedge \mu_f(s_1, r, r_0) \wedge \mu_f(r_0, s_2, r'_0) \\ = \ & \mu_{\tilde{\cdot}}(s_1, s_2, s') \wedge \mu_{\tilde{\odot}}(s', r, r') \wedge \mu_{\tilde{\odot}}(s_1, r, r_0) \wedge \mu_{\tilde{\odot}}(r_0, s_2, r'_0) \\ \leq \ & \mu_{\tilde{\odot}}(s_1, s_2, s') \wedge \mu_{\tilde{\odot}}(s', r, r') \wedge \mu_{\tilde{\odot}}(s_1, r, r_0) \wedge \mu_{\tilde{\odot}}(r_0, s_2, r'_0) \\ \leq \ & E_S(r', r'_0) \end{aligned}$$

i.e., the condition (VM 3) is satisfied. Hence, we have $\mathcal{R}$ is a vague left S-module. In a similar way, we can obtain that $\mathcal{R}$ is a vague right S-module. Therefore, $\mathcal{R}$ is a vague S-module. ☐

Corollary 3.1. *If* $< I, \tilde{+}, \tilde{\cdot} > \overset{v.i.}{\leq} < \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$, *then* $\mathcal{R}$ *is a vague I-module.*

Proof. If we define $f : I \times \mathcal{R} \rightsquigarrow \mathcal{R}$ such that $\mu_f(a, r, r') = \mu_{\tilde{\odot}}(a, r, r')$, then we obtained $\mathcal{R}$ is a vague I-module from Proposition 3.1. ☐

In the rest of this paper, the sentence "A is a vague $\mathcal{R}$-module" will be understood that "A is a vague left $\mathcal{R}$-module".

Proposition 3.2. *Let* $< I, \tilde{+}, \tilde{\cdot} > \overset{v.i.}{\leq} < \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$. *Let* $f : \mathcal{R} \times I \rightsquigarrow I$ *such that* $\mu_f(r, a, a') \leq \mu_{\tilde{\odot}}(r, a, a')$, *then* I *is a vague* $\mathcal{R}$*-module.*

Proof. Since $< I, \tilde{+}, \tilde{\cdot} >$ is a vague ideal, $< I, \tilde{+} >$ is an Abelian vague group. From the distributive laws of Definition 2.7, the conditions (VM 1) and (VM 2) are satisfied. Finally, we have that

$$\begin{aligned} \mu_{\tilde{\odot}}(r_1, r_2, r') \;\;\wedge\;\; & \mu_f(r', a, v) \wedge \mu_f(r_2, a, u) \wedge \mu_f(r, u, w) \\ = \;\; & \mu_{\tilde{\odot}}(r_1, r_2, r') \wedge \mu_{\tilde{\odot}}(r', a, v) \wedge \mu_{\tilde{\odot}}(r_2, a, u) \wedge \mu_{\tilde{\odot}}(r, u, w) \\ \leq \;\; & E_{\mathcal{R}}(v, w) \end{aligned}$$

from $< \mathcal{R}, \tilde{\odot} >$ is a vague semigroup, Hence, the condition (VM 3) is satisfied. ☐

Definition 3.2. Let $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot}, E_{\mathcal{R}\times\mathcal{R}}, E_{\mathcal{R}} >$ be a vague ring, $< A, \tilde{+} >$ be a vague $\mathcal{R}$-module, $< B, \tilde{+}' > \overset{v.g.}{\leq} < A, \tilde{+} >$ and $f : \mathcal{R} \times A \rightsquigarrow A$ be a fuzzy function. If $\forall r \in \mathcal{R}$, $\forall b \in B$, $\mu_f(r, b, b') = 1$ implies $b' \in B$, then $< B, \tilde{+}' >$ is said to be a vague $\mathcal{R}$-submodule of $< A, \tilde{+} >$ and it is denoted by $B \underset{\mathcal{R}}{\overset{v.m.}{\leq}} A$.

Example 3.2. *Let* $A = \mathbb{Z}$, $B = 2\mathbb{Z}$, $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $0 \leq \gamma \leq \beta \leq \alpha < 1$. *We define*

$$E_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to [0, 1], \quad E_{\mathbb{Z}}(u, v) = \begin{cases} 1 & , \quad u = v \\ \alpha & , \quad otherwise \end{cases}$$

$E_{\mathbb{Z}\times\mathbb{Z}} = E^c_{\mathbb{Z}\times\mathbb{Z}}$, $E_{2\mathbb{Z}} : 2\mathbb{Z}\times 2\mathbb{Z} \to [0,1]$, $E_{2\mathbb{Z}}(u,v) = E_{\mathbb{Z}}(u,v)$, $E_{2\mathbb{Z}\times 2\mathbb{Z}} = E^c_{2\mathbb{Z}\times 2\mathbb{Z}}$, $\tilde{+} : \mathbb{Z}\times\mathbb{Z} \rightsquigarrow \mathbb{Z}$,

$$\mu_{\tilde{+}}(x,y,z) = \begin{cases} 1 & , \quad x+y=z \\ \beta & , \quad otherwise \end{cases}$$

and $\tilde{+}' : 2\mathbb{Z}\times 2\mathbb{Z} \rightsquigarrow 2\mathbb{Z}$,

$$\mu_{\tilde{+}'}(a,b,c) = \begin{cases} 1 & , \quad a+b=c \\ \gamma & , \quad otherwise. \end{cases}$$

From Example 22 in [10], $< A, \tilde{+} >$ *is an Abelian vague group and* $< B, \tilde{+}' > \overset{v.g.}{\leq} < A, \tilde{+} >$*. Let* $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot} >$ *be a vague ring and* $f : \mathcal{R}\times A \rightsquigarrow A$ *be a fuzzy function such that*

$$\mu_f(r,a,a') = \begin{cases} 1 & , \quad a' = e_A \\ 0 & , \quad otherwise. \end{cases}$$

From Example 3.1, A *is a vague* $\mathcal{R}$*-module and for each* $r \in \mathcal{R}$, $b \in B$, $\mu_f(r,b,b') = 1$ *implies* $b' = e_A \in B$*. Thus,* $< B, \tilde{+}' >$ *is a vague* $\mathcal{R}$*-submodule of* $< A, \tilde{+} >$*, i.e.,* $2\mathbb{Z} \underset{\mathcal{R}}{\overset{v.m.}{\leq}} \mathbb{Z}$.

Proposition 3.3. *Let* $< \mathcal{R}, \tilde{\oplus}, \tilde{\odot}, E_{\mathcal{R}\times\mathcal{R}}, E_{\mathcal{R}} >$ *be a vague ring,* $< A, \tilde{+} >$ *be a vague* $\mathcal{R}$*-module,* $f : \mathcal{R}\times A \rightsquigarrow A$ *be a vague function,* $B \subseteq A$ *and* $\tilde{+}'$ *be a fuzzy binary operation on* B*. In this case, the following two statements are equivalent:*

(i) $< B, \tilde{+}' > \underset{\mathcal{R}}{\overset{v.m.}{\leq}} < A, \tilde{+} >$

(ii) *(a)* $e_A \in B$,

(b) $\forall b_1, b_2 \in B \ \exists b' \in B$ *such that* $\mu_{\tilde{+}'}(b_1, b_2^{-1}, b') \leq E_B(b_1 +' b_2^{-1}, b')$,

(c) $\forall r \in \mathcal{R}$, $\forall b \in B$ *and* $\mu_f(r,b,b') = 1 \Rightarrow b' \in B$.

Proof. $((i) \Rightarrow (ii))$: It is obvious from Definition 3.2.
$((ii) \Rightarrow (i))$ By making use of (a) and (b), we can write $< B, \tilde{+}' > \overset{v.g.}{\leq} < A, \tilde{+} >$. And we have $< B, \tilde{+}' > \underset{\mathcal{R}}{\overset{v.m.}{\leq}} < A, \tilde{+} >$ from Definition 3.2 under the assumption (c). □

Definition 3.3. Let $< A, \tilde{+} >$ and $< B, \tilde{+}' >$ be two vague $\mathcal{R}$-modules and $h : A \to B$ be a function. If $f_A : \mathcal{R}\times A \rightsquigarrow A$, $f_B : \mathcal{R}\times B \rightsquigarrow B$ are fuzzy functions such that $\forall r \in \mathcal{R}$ and $\forall a, a', b, c \in A$

(i) $\mu_{\tilde{+}}(a,b,c) \ \leq \ \mu_{\tilde{+}'}(h(a), h(b), h(c))$,

(ii) $\mu_{f_A}(r,a,a') \leq \mu_{f_B}(r,h(a),h(a'))$

then the function h is said to be a vague $\mathcal{R}$-module homomorphism from A to B.

(iii) If $h : A \to B$ is a vague $\mathcal{R}$-module homomorphism and $E_B(h(a),h(b)) \leq E_A(a,b)$ for all $a,b \in A$ then h is said to be a vague $\mathcal{R}$-module monomorphism.

(iv) If $h : A \to B$ is a onto vague $\mathcal{R}$-module monomorphism and $h^{-1} : B \to A$ is a vague homomorphism then h is said to be a vague $\mathcal{R}$-module isomorphism.

Proposition 3.4. *Let* $< A, \tilde{+} >$ *and* $< B, \tilde{+}' >$ *be two vague* $\mathcal{R}$*-modules, the vague binary operation* $\tilde{+}'$ *be a transitive of the first order,* $f_B : \mathcal{R} \times B \rightsquigarrow B$ *be a fuzzy function and* $h : A \to B$ *be a vague* $\mathcal{R}$*-module homomorphism. Then, vague subgroup* $< VKerh, \tilde{\star} >$ *of* $< A, \tilde{+} >$ *is a vague* $\mathcal{R}$*-submodule of* $< A, \tilde{+} >$*.*

Proof. Indeed $< VKerh, \tilde{\star} > \overset{v.g.}{\leq} < A, \tilde{+} >$ from Proposition 34 in [10]. So, it is sufficient to show that $\forall r \in \mathcal{R}$, $\forall a \in VKerh$, $\mu_{f_A}(r,a,a') = 1$ implies $a' \in VKerh$ from Proposition 3.3.(ii).(c). Since h is a vague $\mathcal{R}$-module homomorphism $\forall r \in \mathcal{R}$, $\forall a \in VKerh$

$$1 = \mu_{f_A}(r,a,a') \leq \mu_{f_B}(r,h(a),h(a')) = \mu_{f_B}(r,e_B,h(a'))$$

i.e.,

$$\mu_{f_B}(r,e_B,h(a')) = 1. \tag{3.3}$$

Furthermore; $\exists s \in B$ such that $\mu_{\tilde{+}}(h(a'),h(a'),s) = 1$, so

$$\begin{aligned} 1 = \mu_{\tilde{+}'}(e_B,e_B,e_B) \wedge \mu_{f_B}(r,e_B,h(a')) \wedge \mu_{f_B}(r,e_B,h(a')) \\ \wedge \mu_{f_B}(r,e_B,h(a')) \wedge \mu_{\tilde{+}'}(h(a'),h(a'),s) \leq E_B(h(a'),s) \end{aligned}$$

i.e.,

$$E_B(h(a'),s) = 1. \tag{3.4}$$

Since $\tilde{+}'$ is transitive of the first order, by making use of (3.3) and (3.4), we can write

$$1 = \mu_{\tilde{+}'}(h(a'),h(a'),s) \wedge E_B(h(a'),s) \leq \mu_{\tilde{+}'}(h(a'),h(a'),h(a'))$$

i.e., $\mu_{\tilde{+}'}(h(a'), h(a'), h(a')) = 1$. On the other hand, from

$$1 = \mu_{\tilde{+}'}(h(a'), h(a'), h(a')) \wedge \mu_{\tilde{+}'}(h(a')^{-1}, h(a'), e_B) \wedge \\ \mu_{\tilde{+}'}(h(a')^{-1}, h(a'), e_B) \wedge \mu_{\tilde{+}'}(e_B, h(a'), h(a')) \leq E_B(e_B, h(a'))$$

we can write

$$E_B(e_B, h(a')) = 1 \tag{3.5}$$

So, $h(a') = e_B$, i.e., $a' \in Ker(h)$. □

If μ_{f_B} is transitive of the first order in Proposition 3.4, then by making use of (3.3) and (3.5), we obtain

$$1 = \mu_{f_B}(r, e_B, h(a')) \wedge E_B(e_B, h(a')) \leq \mu_{f_B}(r, e_B, e_B)$$

i.e., $\mu_{f_B}(r, e_B, e_B) = 1$.

Proposition 3.5. *Let* $< A, \tilde{+} >$ *and* $< B, \tilde{+}' >$ *be two vague* $\mathcal{R}$*-modules. Let* $f_A : \mathcal{R} \times A \rightsquigarrow A$ *and* $f_B : \mathcal{R} \times B \rightsquigarrow B$ *be two fuzzy functions. Let* $h : A \to B$ *be a function such that* $h(a) = e_B$*. Then,* h *is a vague* $\mathcal{R}$*-module homomorphism from* A *to* B*.*

Proof. It is clear that,

$$\mu_{\tilde{+}}(a, b, c) \leq \mu_{\tilde{+}'}(h(a), h(b), h(c)) = \mu_{\tilde{+}'}(e_B, e_B, e_B) = 1, \ \forall a, b, c \in A.$$

And, $\forall r \in \mathcal{R}$, $\exists k_r \in B$ such that $\mu_{f_B}(r, e_B, k_r) = 1$. Since B is a vague $\mathcal{R}$-module, we can write

$$\begin{aligned} 1 &= \mu_{f_B}(r, e_B, k) \wedge \mu_{f_B}(r, e_B, k) \wedge \mu_{f_B}(r, e_B, k) \wedge \mu_{\tilde{+}'}(k, k, k +' k) \\ &\leq E_B(k, k +' k). \end{aligned}$$

Thus, $k = k +' k$, i.e., $k = e_B$. So, $\mu_{f_B}(r, e_B, e_B) = 1$. In this case, we obtain

$$\mu_{f_A}(r, a, b) \leq \mu_{f_B}(r, h(a), h(b)) = \mu_{f_B}(r, e_B, e_B) = 1.$$

This completes the proof. □

Example 3.3. *Let* $\mathcal{R}, A, B, \alpha, \beta, \gamma, E_{\mathbb{Z}}, E_{\mathbb{Z}\times\mathbb{Z}}, E_{2\mathbb{Z}}, E_{2\mathbb{Z}\times 2\mathbb{Z}}, \tilde{+}$ *and* $\tilde{+}'$ *be as in Example 3.2. Let* $f_A = f$ *as in Example 3.2. We define* $f_B : \mathcal{R} \times B \rightsquigarrow B$ *be a fuzzy function such that*

$$\mu_{f_B}(r, b, b') = \begin{cases} 1 \quad , \quad b' = e_B \\ 0 \quad , \quad otherwise. \end{cases}$$

and $h : A \to B$ *be a function such that* $h(a) = e_B$. *Then,* h *is a vague* $\mathcal{R}$*-module homomorphism from* A *to* B *from Proposition 3.5.*

4. Conclusion

In this paper, the concepts of vague module, vague submodule, vague module homomorphism and vague module isomorphism are introduced, and then various elementary properties of these concepts are investigated.

Although the results in this paper are formulated on $([0, 1], \leq, \wedge)$, it seems that most of them can be restated for any t-norm instead of the minimum t-norm. This topic is left to the readers for future investigations.

Acknowledgement

We would like to thank the editor and the referees for their valuable comments.

References

[1] M. Demirci, Fuzzy Functions and Their Fundamental Properties, *Fuzzy Sets and Systems*, **106** (1999), 239–246.

[2] M. Demirci, Vague Groups, *Journal of Mathematical Analysis and Applications*, **230** (1999), 142–156 .

[3] M. Demirci, Fundamentals of M-vague Algebra and M-vague Arithmetic Operations, International Journal of Uncertainty, *Fuzziness and Knowledge-Based Systems*, **10** (2002), 25–75.

[4] M. Demirci, Foundations of Fuzzy Functions and Vague Algebra Based on Many-valued Equivalence Relations, Part I: Fuzzy Functions and Their Applications, *Int. J. Gen. Syst.*, **32** (2003), 123–155 .

[5] M. Demirci, Foundations of Fuzzy Functions and Vague Algebra Based on Many-valued Equivalence Relations, Part II: Vague Algebraic Notions, *International Journal of General Systems*, **32**(2) (2003), 157–175 .

[6] M. Demirci, Foundations of Fuzzy Functions and Vague Algebra Based on Many-valued Equivalence Relations. Part III: Constructions of Vague Algebraic Notions and Vague Arithmetic Operations, *International Journal of General Systems*, **32**(2) (2003), 177–201 .

[7] M. Demirci and D. Coker, Remarks on Vague Groups, *Journal of Fuzzy Mathematics*, **10**(2002), 657–668.

[8] M. Demirci and J. Recasens, Fuzzy Groups, Fuzzy Functions and Fuzzy equvalence Relation, *Fuzzy Sets and Systems*, **144** (2004), 441–458.

[9] A. Rosenfeld, Fuzzy Groups, *Journal of Mathematical Analysis and Applications*, **35** (1971) , 512–517 .

[10] S. Sezer, Vague Groups and Generalized Vague Subgroups on the Basis of $([0,1], \leq, \wedge)$, *Information Sciences*, **174** (2005), 123–142.

[11] S. Sezer, Vague Rings and Vague Ideals, *Iranian Journal of Fuzzy Systems*, **8** (2011) , 145–157.

[12] L. A. Zadeh, Fuzzy Sets, *Information and Control*, **8** (1965), 338–353.

(1) Faculty of Education, Akdeniz University, Antalya-TURKEY

Email address: sevdasezer@akdeniz.edu.tr

(2) Math Teacher

Email address: yukselmurat@outlook.com.tr