Jordan Journal of Mathematics and Statistics (JJMS), 15(4A), 2022, pp 941 - 953
DOI: https://doi.org/10.47013/15.4.10

PERIODIC OSCILLATION OF THE SOLUTIONS FOR A MODEL OF FOUR-DISK DYNAMO SYSTEM WITH DELAYS

CHUNHUA FENG

ABSTRACT. In this paper, the periodic oscillatory behavior of the solutions for a model of four-disk dynamo system with delays is investigated. By means of the mathematical analysis method, some sufficient conditions to guarantee the periodic oscillation of the solutions are obtained. Computer simulations are provided to demonstrate our results.

1. INTRODUCTION

It is well known that the disk dynamo plays an important role in studying the geographical phenomenon of the irregular reversal of the earth's magnetic pole. Many researchers have devoted to the understanding of dynamo dynamics [1-15]. For example, in 1958, Rikitake firstly proposed a two coupled disk dynamo system, and the coupling between electro-magnetism, motion and heat were discussed [1]. Then Cook and Roberts [2] promulgated that the Rikitake's model may have a periodic orbit. In 1972, Cook [3] modified the Rikitake's model by considering the viscous friction and time delay as the following system:

$$
(1.1)\qquad \begin{cases} x_1'(t) = -kx_1 + x_2(t-\tau)x_3, \\ x_2'(t) = -kx_2 + x_1(t-\tau)x_4, \\ x_3'(t) = 1 - x_1x_2(t-\tau), \\ x_4'(t) = 1 - x_1(t-\tau)x_2. \end{cases}
$$

The chaotic dynamics of the Rikitake two-disk dynamo system was studied for a wide range of parameters and results were compared with the sequence of geomagnetic

2010 *Mathematics Subject Classification.* 34K11.

Key words and phrases. a four-disk dynamo system, delay, instability, periodic oscillation.

Received: July 1, 2021 Accepted: April 14, 2022 .

polarity reversals. The chaos of the Rikitake system belongs to the Lorenz type, which was characterized by irregular traveling of an orbit between two unstable fixed points. Since then, extensive research works on modeling, dynamics analysis, and circuit experiment were conducted for a further understanding of such systems. For instance, Feng et al. considered the following system [4]:

$$\begin{cases} x_1'(t) = -\mu_1 x_1 + x_2 x_3, \\ x_2'(t) = -\mu_2 x_2 + x_1 x_4, \\ x_3'(t) = q_1 - \varepsilon_1 x_3 - x_1 x_2, \\ x_4'(t) = q_2 - \varepsilon_2 x_4 - x_1 x_2. \end{cases} \tag{1.2}$$

where q_1 and q_2 are the torques applied to the rotors, $\mu_1, \mu_2, \varepsilon_1$, and ε_2 are the positive constants representing the power consumption and mechanical damping dissipation of disk dynamo, respectively. The globally attractive and positive invariant set including ultimate boundary of model (1.2) were provided. A set of sufficient conditions was derived for all solutions of the stochastic disk dynamo system being global convergent to the equilibrium point. It is known that time delays are often very small in practical situations, they cannot be ignored and can cause a series of complex phenomena. Therefore, Wei et al. considered the case that three communication delays due to diffusion may be incorporated into the Rikitake model [5]:

$$\begin{cases} x_1'(t) = -kx_1 + x_2(t-\tau_2)x_3, \\ x_2'(t) = -kx_2 + x_1(t-\tau_1)x_4, \\ x_3'(t) = 1 - x_1 x_2(t-\tau_3) - \nu_1 x_3, \\ x_4'(t) = 1 - x_1(t-\tau_1)x_2 - \nu_2 x_4. \end{cases} \tag{1.3}$$

The authors considered the stability of equilibrium states for different delay values, and determined the location of relevant Hopf bifurcations by means of the normal form method and the center manifold theory. A classical period-doubling route towards deterministic chaos in system (1.3) has been revealed. Recently, Deng et al.

investigated a three-disk dynamo system as follows [6]:

$$\begin{cases} x_1'(t) = -kx_1 + y_1 x_3(t-\tau), \\ x_2'(t) = -kx_2 + y_2 x_1(t-\tau), \\ x_3'(t) = -kx_3 + y_3 x_2(t-\tau), \\ y_1'(t) = 1 - x_1 x_3(t-\tau) - cy_1, \\ y_2'(t) = 1 - x_2 x_1(t-\tau) - cy_2. \\ y_3'(t) = 1 - x_3 x_2(t-\tau) - cy_3. \end{cases} \tag{1.4}$$

where terms $-cy_i (i = 1, 2, 3)$ represent the viscous frictions in the disks and c denotes the coefficient of friction. τ designates the interaction delay caused by the electromagnetic diffusion. By carrying out a comparative analysis, the dynamic behaviors of the coupled dynamo system (1.4) were studied. Novel and complex nonlinear dynamic phenomena in the coupled delayed dynamo system have been explored. The authors found that the double Hopf bifurcations can be induced in the delayed dynamo system. Three different topological structures of the unfolding were obtained under different time delays. For a system of four disk dynamos with eight degrees of freedom, the chaos behavior was confirmed by Lyapunov exponents method, the stability at zero equilibrium point of four disk dynamos was proved [7]. However, the system of four disk dynamos in [7] did not incorporate time delays. It is known that time delay may induce the instability of the system. The time delay system is highly important, not only to model various physical phenomena, but also to affect the stability of the physical systems. Therefore, in this paper, we consider the following four-disk dynamo system with four delays:

$$\begin{cases} x_1'(t) = -k_1 x_1 + y_1 x_4(t-\tau_4), \\ x_2'(t) = -k_2 x_2 + y_2 x_1(t-\tau_1), \\ x_3'(t) = -k_3 x_3 + y_3 x_2(t-\tau_2), \\ x_4'(t) = -k_4 x_4 + y_4 x_3(t-\tau_3), \\ y_1'(t) = \mu_1 - x_1 x_4(t-\tau_4) - c_1 y_1, \\ y_2'(t) = \mu_2 - x_2 x_1(t-\tau_1) - c_2 y_2, \\ y_3'(t) = \mu_3 - x_3 x_2(t-\tau_2) - c_3 y_3, \\ y_4'(t) = \mu_4 - x_4 x_3(t-\tau_3) - c_4 y_4. \end{cases} \tag{1.5}$$

where k_i, μ_i, and $c_i(i = 1, \cdots, 4)$ are positive numbers. Our goal is to investigate the instability of the equilibrium point and the existence of periodic solutions. Noting that system (1.5) has four different delays, the bifurcating method is hard to deal with model (1.5) if delays will be used as bifurcating parameters. By means of the mathematical analysis method, the property of the solutions of system (1.5) has been discussed.

This paper is organized as follows: In section 2, the boundedness of the solutions and the uniqueness of the equilibrium point of system (1.5) are studied. In section 3, the instability of the solutions and periodic oscillation of the solutions of system (1.5) are investigated. In section 4, numerical simulations are carried out to illustrate our results. Some main conclusions are drawn in section 5.

2. Preliminaries

For convenience, if $(x_{1*}, x_{2*}, x_{3*}, x_{4*}, y_{1*}, y_{2*}, y_{3*}, y_{4*})^T$ is an equilibrium point. Let $u_i(t) = x_i(t) - x_{i*}, u_j(t) = y_j(t) - y_{j*}(i = 1, \cdots, 4, j = 5, \cdots, 8)$, then system (1.5) can be written as follows:

$$
\begin{cases}
u_1'(t) = -k_1u_1(t) + x_{4*}u_5(t) + y_{1*}u_4(t-\tau_4) + u_5(t)u_4(t-\tau_4), \\
u_2'(t) = -k_2u_2(t) + x_{1*}u_6(t) + y_{2*}u_1(t-\tau_1) + u_6(t)u_1(t-\tau_1), \\
u_3'(t) = -k_3u_3(t) + x_{2*}u_7(t) + y_{3*}u_2(t-\tau_2) + u_7(t)u_2(t-\tau_2), \\
u_4'(t) = -k_4u_4(t) + x_{3*}u_8(t) + y_{4*}u_3(t-\tau_3) + u_8(t)u_3(t-\tau_3), \\
u_5'(t) = -x_{4*}u_1(t) - x_{1*}u_4(t-\tau_4) - u_1(t)u_4(t-\tau_4) - c_1u_5(t), \\
u_6'(t) = -x_{1*}u_2(t) - x_{2*}u_1(t-\tau_1) - u_2(t)u_1(t-\tau_1) - c_2u_6(t), \\
u_7'(t) = -x_{2*}u_3(t) - x_{3*}u_2(t-\tau_2) - u_3(t)u_2(t-\tau_2) - c_3u_7(t), \\
u_8'(t) = -x_{3*}u_4(t) - x_{4*}u_3(t-\tau_3) - u_4(t)u_3(t-\tau_3) - c_4u_8(t).
\end{cases}
\tag{2.1}
$$

The linearized system of (2.1) around $(x_{1*}, x_{2*}, x_{3*}, x_{4*}, y_{1*}, y_{2*}, y_{3*}, y_{4*})^T$ is

$$
\left\{
\begin{aligned}
u_1'(t) &= -k_1 u_1(t) + x_{4*} u_5(t) + y_{1*} u_4(t-\tau_4),\\
u_2'(t) &= -k_2 u_2(t) + x_{1*} u_6(t) + y_{2*} u_1(t-\tau_1),\\
u_3'(t) &= -k_3 u_3(t) + x_{2*} u_7(t) + y_{3*} u_2(t-\tau_2),\\
u_4'(t) &= -k_4 u_4(t) + x_{3*} u_8(t) + y_{4*} u_3(t-\tau_3),\\
u_5'(t) &= -x_{4*} u_1(t) - x_{1*} u_4(t-\tau_4) - c_1 u_5(t),\\
u_6'(t) &= -x_{1*} u_2(t) - x_{2*} u_1(t-\tau_1) - c_2 u_6(t),\\
u_7'(t) &= -x_{2*} u_3(t) - x_{3*} u_2(t-\tau_2) - c_3 u_7(t),\\
u_8'(t) &= -x_{3*} u_4(t) - x_{4*} u_3(t-\tau_3) - c_4 u_8(t).
\end{aligned}
\right.
\tag{2.2}
$$

System (2.2) can be written as a matrix form:

$$
u'(t) = Au(t) + Bu(t-\tau)
\tag{2.3}
$$

where $u = (u_1, u_2, \cdots, u_8)^T$, $u(t-\tau) = (u_1(t-\tau_1), u_2(t-\tau_2), u_3(t-\tau_3), u_4(t-\tau_4), 0, 0, 0, 0)^T$, A and B both are 8×8 matrices:

$$
A = (a_{ij})_{8\times 8} = \begin{pmatrix}
-k_1 & 0 & 0 & 0 & x_{4*} & 0 & 0 & 0\\
0 & -k_2 & 0 & 0 & 0 & x_{1*} & 0 & 0\\
0 & 0 & -k_3 & 0 & 0 & 0 & x_{2*} & 0\\
0 & 0 & 0 & -k_4 & 0 & 0 & 0 & x_{3*}\\
-x_{4*} & 0 & 0 & 0 & -c_1 & 0 & 0 & 0\\
0 & -x_{1*} & 0 & 0 & 0 & -c_2 & 0 & 0\\
0 & 0 & -x_{2*} & 0 & 0 & 0 & -c_3 & 0\\
0 & 0 & 0 & -x_{3*} & 0 & 0 & 0 & -c_4
\end{pmatrix},
$$

$$
B = (b_{ij})_{8\times 8} = \begin{pmatrix}
0 & 0 & 0 & y_{1*} & 0 & 0 & 0 & 0\\
y_{2*} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & y_{3*} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & y_{4*} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -x_{1*} & 0 & 0 & 0 & 0\\
-x_{2*} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -x_{3*} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -x_{4*} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Lemma 2.1. If matrix R $(= A+B)$ is a nonsingular matrix, then there exists a unique positive equilibrium point for system (1.5).

Proof. Obviously, zero is an equilibrium point of system (2.2) (or (2.3)), and if system (2.2) has a unique zero equilibrium point, then corresponding system (1.5) has a unique positive equilibrium point $(x_{1*}, x_{2*}, x_{3*}, x_{4*}, y_{1*}, y_{2*}, y_{3*}, y_{4*})^T$. Assume that v_* is another equilibrium point of system (2.2), then we have the following algebraic equation

$$A(v_*) + B(v_*) - A\cdot\mathbf{0} - B\cdot\mathbf{0} = R(v_*) = \mathbf{0} \tag{2.4}$$

From (2.4) since R $(= A+B)$ is a nonsingular matrix, by basic algebraic knowledge we get

$$v_* = \mathbf{0} \tag{2.5}$$

This means that v_* must be zero. The proof is completed.

□

Lemma 2.2. All solutions of system (1.5) are uniformly bounded.

Proof. To prove the boundedness of the solutions in system (1.5), we construct a Lyapunov function $V(t) = \sum_{i=1}^{4} \frac{1}{2}[x_i^2(t) + y_i^2(t)]$. Calculating the derivative of $V(t)$ through system (1.5) we get

$$\begin{aligned} V'(t)|_{(1.5)} &= \sum_{i=1}^{4}[x_i'(t)x_i(t) + y_i'(t)y_i(t)] \\ &= -k_1x_1^2 + y_1x_4x_1 - k_2x_2^2 + y_2x_1x_2 - k_3x_3^2 + y_3x_2x_3 - k_4x_4^2 \\ &\quad +y_4x_3x_4 + \mu_1y_1 - y_1x_1x_4 - c_1y_1^2 + \mu_2y_2 - y_2x_2x_1 - c_2y_2^2 \\ &\quad +\mu_3y_3 - y_3x_3x_2 - c_3y_3^2 + \mu_4y_4 - y_4x_4x_3 - c_4y_4^2 \\ &= -\sum_{i=1}^{4} k_ix_i^2 - \sum_{i=1}^{4} c_iy_i^2 + \sum_{i=1}^{4} \mu_iy_i \end{aligned} \tag{2.6}$$

Obviously, when $x_i(t), y_i(t) (1 \le i \le 4)$ tend to infinity, $x_i^2(t), y_i^2(t)$, are higher order infinity. Noting that k_1, μ_i, and c_i are positive constants, therefore, there exists suitably large $L > 0$ such that $V'(t)|_{(1.5)} < 0$ as $|x_i| > L, |y_i| > L$. This means that all solutions of system (1.5) are uniformly bounded. □

3. Periodic oscillation of the solutions

Theorem 3.1. Assume that zero is the unique equilibrium point of system (2.2) (or(2.3)) for selecting parameter values. Let $\alpha_1, \alpha_2, \cdots, \alpha_8$ and $\beta_1, \beta_2, \cdots, \beta_8$ be characteristic values of matrix A and B, respectively. If all $Re(\alpha_i) < 0$, and $Re(\beta_i) \leq 0$, then the trivial solution of system (2.2) is stable, implying that the unique positive equilibrium point of system (2.1) is stable. If there exists some $Re(\alpha_k) > 0$, or $Re(\alpha_k) < 0$, but $|Re(\alpha_k)| < Re(\beta_k)$ or $|Re(\alpha_k)| < |Im(\beta_k)|$, then the unique equilibrium point of system (2.2) is unstable, implying that the unique equilibrium point of system (1.5) is unstable. System (1.5) generates a periodic oscillatory solution.

Proof. Obviously, based on the property of differential equation, since α_i and β_i $(i = 1, 2, \cdots, 8)$ are characteristic values of matrix A and B, respectively. Noting that there exist four characteristic values are zeros of matrix B, then the characteristic equation corresponding to system (2.2) is the following:

$$\Pi_{i=1}^{4}(\lambda - \alpha_i) = 0 \tag{3.1}$$

and

$$\Pi_{j=5}^{8}(\lambda - \alpha_j - \beta_j e^{-\lambda\tau_j}) = 0 \tag{3.2}$$

If for some $Re(\alpha_i) > 0, i \in \{1, 2, 3, 4\}$, we have $Re(\lambda) = Re(\alpha_i) > 0$. Thus, the trivial solution is unstable. If for some $|Re(\alpha_k)| < Re(\beta_k)$ or $|Re(\alpha_k)| < |Im(\beta_k)|, k \in \{5, 6, 7, 8\}$, we are led to an investigation of the nature of the roots for some k

$$\lambda - \alpha_k - \beta_k e^{-\lambda\tau_k} = 0 \tag{3.3}$$

Equation (3.3) is a transcendental equation which is hard to find all solutions for the equation. However, we show that there exists a positive real part eigenvalue of equation (3.3) under the assumption of Theorem 1. If $|Re(\alpha_k)| < Re(\beta_k)$ or $|Re(\alpha_k)| < |Im(\beta_k)|$, let $\lambda = \sigma + i\theta, \alpha_k = \alpha_{k1} + i\alpha_{k2}, \beta_k = \beta_{k1} + i\beta_{k2}$, where $Re(\alpha_k) =$

$\alpha_{k1}, Re(\beta_k) = \beta_{k1}$. Separating the real and imaginary parts from equation (3.3), we have the real part

$$\sigma = \alpha_{k1} + \beta_{k1} e^{-\sigma\tau_k} \cos(\theta\tau_k) - \beta_{k2} e^{-\sigma\tau_k} \sin(\theta\tau_k). \tag{3.4}$$

We show that equation (3.3) has a positive real part root. Let

$$h(\sigma) = \sigma - \alpha_{k1} - \beta_{k1} e^{-\sigma\tau_k} \cos(\theta\tau_k) + \beta_{k2} e^{-\sigma\tau_k} \sin(\theta\tau_k). \tag{3.5}$$

Obviously, $h(\sigma)$ is a continuous function of σ. When $\sigma = 0$ we have $h(0) = -\alpha_{k1} - \beta_{k1} \cos(\theta\tau_k) + \beta_{k2} \sin(\theta\tau_k) < 0$. Noting that $\lim_{\sigma \to +\infty} e^{-\sigma\tau_j} = 0$, so there exists a suitably large $\tilde{\sigma}(> 0)$ such that $h(\tilde{\sigma}) = \tilde{\sigma} - \alpha_{k1} - \beta_{k1} e^{-\tilde{\sigma}\tau_k} \cos(\theta\tau_k) + \beta_{k2} e^{-\tilde{\sigma}\tau_k} \sin(\theta\tau_k) > 0$. By means of the Intermediate Value Theorem, there exists a $\bar{\sigma} \in (0, \tilde{\sigma})$ such that $h(\bar{\sigma}) = \bar{\sigma} - \alpha_{k1} - \beta_{k1} e^{-\bar{\sigma}\tau_k} \cos(\theta\tau_k) + \beta_{k2} e^{-\bar{\sigma}\tau_k} \sin(\theta\tau_k) = 0$. This means that the characteristic value λ has a positive real part. Therefore, the trivial solution of system (2.2) is unstable for any time delays, implying that the unique positive equilibrium point of system (1.5) is unstable [16]. Since all solutions of system (1.5) are bounded, the instability of the positive equilibrium point and the boundedness of the solutions will force system (1.5) to generate a limit cycle, namely, a periodic oscillatory solution. The proof is completed.

□

Theorem 3.2. Assume that there exists a unique equilibrium point of system (1.5) for selecting parameter values. Let matrix $R = A + B$. If R has a positive real eigenvalue or there has at least one eigenvalue which has a positive real part. Then the unique equilibrium point of system (1.5) is unstable, implying that system (1.5) has a periodic oscillatory solution.

Proof. Consider a special case in system (2.2): $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$. Then system (2.2) can be written as a without time delay system:

$$u'(t) = Au(t) + Bu(t) = Ru(t) \tag{3.6}$$

The characteristic equation corresponding to system (3.6) is the following

$$det(\lambda I_{ij} - R) = 0 \tag{3.7}$$

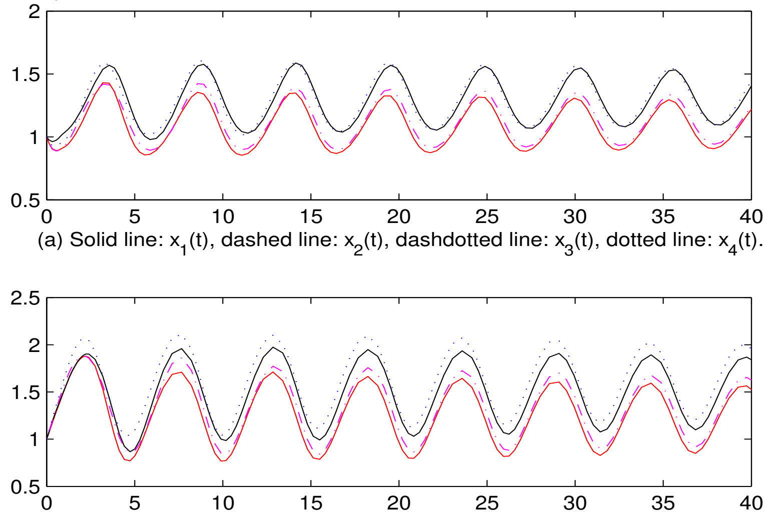
where I_{ij} is the 8×8 identity matrix. Since R has a positive real eigenvalue or there has at least one eigenvalue which has a positive real part, so the characteristic equation (3.7) has a positive real eigenvalue or there has at least one eigenvalue which has a positive real part. Therefore, the trivial solution $u_1(t), u_2(t) \cdots, u_8(t)$ of system (3.6) is unstable, implying that the trivial solution $u_1(t-\tau_1), \cdots, u_4(t-\tau_4), u_5(t), \cdots, u_8(t)$ of system (2.2) is unstable. On the other hand, when t is sufficiently large, we have $u_1(t) \sim u_1(t-\tau_1), u_2(t) \sim u_2(t-\tau_2), u_3(t) \sim u_3(t-\tau_3), u_4(t) \sim u_4(t-\tau_4)$. Hence the trivial solution of system (2.2) is unstable, implying that the unique equilibrium point of system (1.5) is unstable. It suggests that system (1.5) has a periodic oscillatory solution. This concludes the proof.

□

4. Simulation results

The simulation is based on the system (1.5), first the parameters are selected as follows: $k_1 = 1.45, k_2 = 1.24, k_3 = 1.48, k_4 = 1.38, \mu_1 = 1.75, \mu_2 = 1.85, \mu_3 = 1.80, \mu_4 = 1.88, c_1 = 0.26, c_2 = 0.28, c_3 = 0.25, c_4 = 0.24$. The time delays $\tau_1 = 0.75, \tau_2 = 0.65, \tau_3 = 0.55, \tau_4 = 0.72$. Then the unique positive equilibrium point is $x_{1*} = 1.0578, x_{2*} = 1.3170, x_{3*} = 1.1265, x_{4*} = 1.3245, y_{1*} = 1.3421, y_{2*} = 1.6317, y_{3*} = 1.2656, y_{4*} = 1.6165$. The characteristic values of $R = A + B$ are $1.0104, -0.5517 \pm 1.7882i, -1.2988 \pm 0.9111i, -0.7578 \pm 0.5152i, -2.3737$. Therefore, R is a nonsingular matrix. the characteristic values of A are $0.5912, -2.3212, -0.8550 \pm 1.1783i, -0.7600 \pm 0.9451i, -0.8100 \pm 0.9757i$, the characteristic values of B are $0, 0, 0, 0, 1.4531, -1.4531, \pm 1.4531i$. Since there is a characteristic value $\alpha_1 = 0.5912 > 0$, the condition of Theorem 3.1 are satisfied. There exists a periodic oscillatory solution for system (1.5) (see Fig.1). Figure 1 indicates that this periodic oscillation is stable around the equilibrium point. The oscillatory frequency of each component of the solution is the same. In order to see the effect of time delays, we change time delays as $\tau_1 = 0.45, \tau_2 = 0.48, \tau_3 = 0.42, \tau_4 = 0.46$. The other parameters are the same as Fig.1, we see that the oscillatory behavior is still maintained but the oscillatory frequency is changed (see Fig.2). Figure 2 suggests that oscillatory frequency increases slightly when time delays are slightly reduced. Then

Fig. 1 Periodic oscillatory solutions, delays: 0.75, 0.65, 0.55, 0.72, equilibrium: 1.0578, 1.3170, 1.1265, 1.3245, 1.3421, 1.6317, 1.2656, 1.6165.

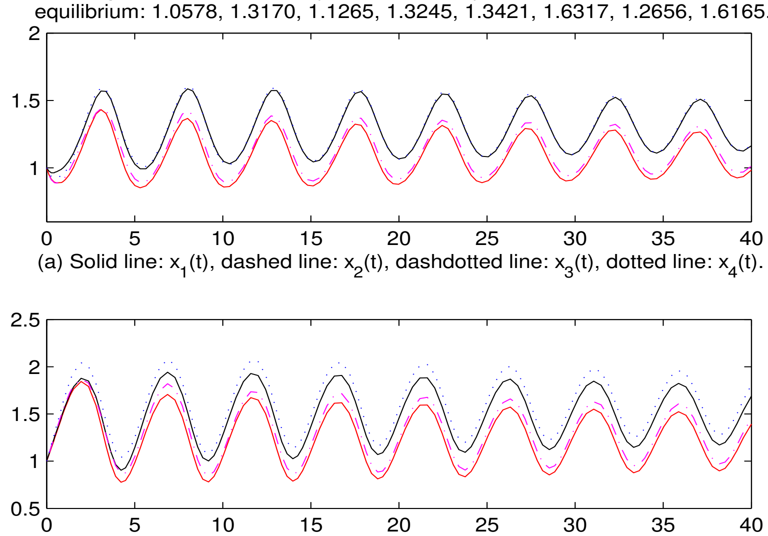


(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dashdotted line: $x_3(t)$, dotted line: $x_4(t)$.

(b) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dashdotted line: $y_3(t)$, dotted line: $y_4(t)$.

we change the parameters as $k_1 = 1.15, k_2 = 1.12, k_3 = 1.18, k_4 = 1.16, \mu_1 = 2.45, \mu_2 = 2.42, \mu_3 = 2.40, \mu_4 = 2.38, c_1 = 0.36, c_2 = 0.38, c_3 = 0.35, c_4 = 0.34$. The time delays $\tau_1 = 1.15, \tau_2 = 1.18, \tau_3 = 1.12, \tau_4 = 1.16$, and $\tau_1 = 1.05, \tau_2 = 1.25, \tau_3 = 1.08, \tau_4 = 1.12$, respectively, and the unique positive equilibrium point is $x_{1*} = 1.4441, x_{2*} = 1.3915, x_{3*} = 1.4215, x_{4*} = 1.4011, y_{1*} = 1.1852, y_{2*} = 1.0804, y_{3*} = 1.2057, y_{4*} = 1.1422$. The characteristic values of $R = A + B$ are $0.3954, -0.0334 \pm 1.0451i, -0.1671 \pm 1.9531i, -1.4766 \pm 2.2207i, -2.2909$. Noting that there exists a characteristic value 0.3954, therefore, R is a nonsingular matrix and the conditions of Theorem 3.2 are satisfied, system (1.5) generates an oscillatory solution (see Fig.3 and Fig.4). In Figure 3, time delays are selected as $1.15, 1.18, 1.12$, and 1.16. In Figure 4, we change time delays as $1.05, 1.25, 1.08$, and 1.12. We see that oscillatory amplitude and frequency both are slightly changed. This means that a change slightly of time delay does not affect the oscillatory behavior of the solutions. We point out that our criterion only is a sufficient condition.

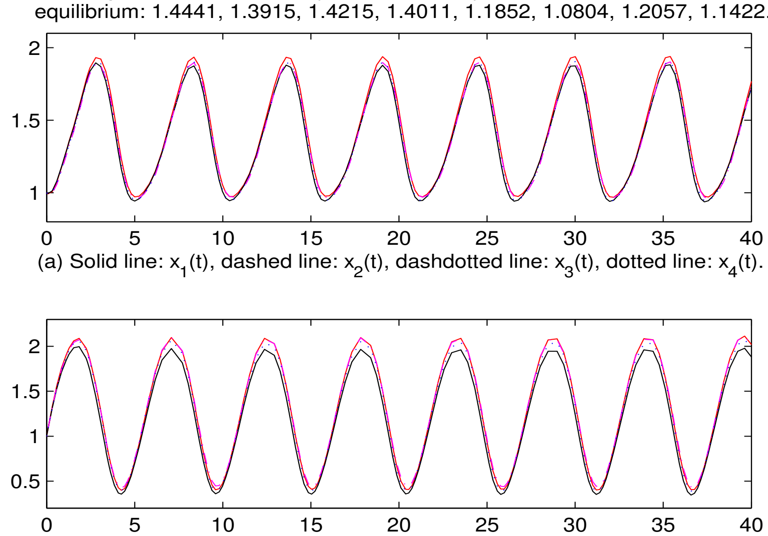
Fig. 2 Periodic oscillatory solutions, delays: 0.45, 0.48, 0.42, 0.46, equilibrium: 1.0578, 1.3170, 1.1265, 1.3245, 1.3421, 1.6317, 1.2656, 1.6165.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dashdotted line: $x_3(t)$, dotted line: $x_4(t)$.

(b) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dashdotted line: $y_3(t)$, dotted line: $y_4(t)$.

Fig. 3 Periodic oscillatory solutions, delays: 1.15, 1.18, 1.12, 1.16, equilibrium: 1.4441, 1.3915, 1.4215, 1.4011, 1.1852, 1.0804, 1.2057, 1.1422.



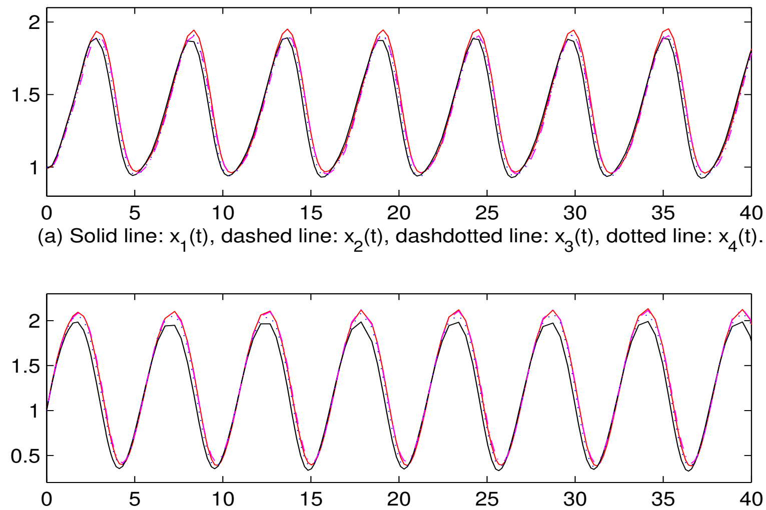
(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dashdotted line: $x_3(t)$, dotted line: $x_4(t)$.

(b) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dashdotted line: $y_3(t)$, dotted line: $y_4(t)$.

5. Conclusion

In this paper, we have discussed the periodic oscillatory behavior of the solutions for a model of four-disk dynamo system with delays. From mathematical analysis point of view, for a time delayed system, if all solutions of the system are uniformly bounded, and there exists a unique unstable equilibrium point, which will force the system to generate a limit cycle, namely, a periodic solution. A set of restrictive conditions for parameters of the system is provided to guarantee the boundedness of

Fig. 4 Periodic oscillatory solutions, delays: 1.05, 1.25, 1.08, 1.12, equilibrium: 1.4441, 1.3915, 1.4215, 1.4011, 1.1852, 1.0804, 1.2057, 1.1422.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dashdotted line: $x_3(t)$, dotted line: $x_4(t)$.

(b) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dashdotted line: $y_3(t)$, dotted line: $y_4(t)$.

the solutions and the uniqueness of the equilibrium point of the system. Two theorems are given to ensure the instability of the equilibrium point. It can be seen that the theoretical analysis is in good agreement with computer simulation, which indicates that the criterion is valid and the obtained result is correct. From the numerical simulation, time delays affect the oscillatory frequency. Chaos can be explored with big delays. The present criterion is different from the bifurcation method. We may extend this method to any n-disk dynamo system with delays. It is also our future work.

Acknowledgement

The author would like to thank the editor and the anonymous reviewers for their useful comments and advice, which are vital for improving the quality of this paper.

References

[1] T. Rikitake, Oscillators of a system of disk dynamos, *Math. Proc. Camb. Philos. Soc.,* **54(1)** (1958), 89–105.

[2] A.E. Cook, P.H. Roberts, The Rikitake two-disc dynamo system, *Math. Proc. Camb. Philos. Soc.,* **68(2)** (1970), 547–569.

[3] A.E. Cook, Two disc dynamos with viscous friction and time delay, *Math. Proc. Camb. Philos. Soc.,* **17** (1972), 135–153.

[4] C.S. Feng, L.J. Li, Y.J. Liu, Z.C. Wei, Global Dynamics of the chaotic disk dynamo system driven by noise, *Complexity,* **2020** (2020), 8375324.

[5] Z.C. Wei, B. Zhu, J. Yang, M. Perc, M. Slavine, Bifurcation analysis of two disc dynamos with viscous friction and multiple time delays, *Appl. Math. Comput.,* **347** (2019), 265–281.

[6] S.N. Deng, J.C. Ji, G.L. Wen, H.D. Xu, A comparative study of the dynamics of a three-disk dynamo system with and without time delay, *Appl. Math. Comput.,* **399** (2021), 126016.

[7] P. Muthukumar, P. Balasubramaniam, K. Ratnavelu, Detecting chaos in a system of four disk dynamos and its control, *Nonlinear Dynamics,* **83** (2016), 2419–2426.

[8] K. Ito, Chaos in the Rikitake two-disc dynamo system, *Earth Planet. Sci. Lett.,* **51** (1980), 451–456.

[9] T. Miura, T. Kai, Chaotic behaviours of a system of three disk dynamos, *Phys. Lett. A,* **101** (1984), 450–454.

[10] J. Llibre, M. Messias, Global dynamics of the Rikitake system, *Physica D*, **238** (2009), 241–252.

[11] V. Vembarasan, P. Balasubramaniam, Chaotic synchronization of Rikitake system based on T-S fuzzy control techniques, *Nonlinear Dyn.,* **74** (2013), 31–44.

[12] Z. Wei, A. Akgul, U.E. Kocamaz, I. Moroz, W. Zhang, Control, electronic circuit application and fractional-order analysis of hidden chaotic attractors in the self-exciting homopolar disc dynamo, *Chaos Solitons Fractals,* **111** (2018), 157–168.

[13] S. Vaidyanathan, Adaptive control of Rikitake two-disk dynamo system, *Int. J. ChemTech Research*, **8** (2015), 121–133.

[14] J. Jian, Z. Zhao, New estimations for ultimate boundary and synchronization control for a disk dynamo system, *Nonlinear Analysis: Hybrid Systems,* 9 (2013), 56–66.

[15] R.A. Tudoran, On asymptotically stabilizing the Rikitake two-disk dynamo dynamics, *Nonlinear Anal. RWA*, **12** (2011), 2505–2510.

[16] V.B. Kolmanovskii, A.D. Myshkis, Introduction to the theory and applications of functional-differential equations Mathematics and its Applications, 463, Kluwer Academic Publishers, Dordrecht (1999).

(915 S. Jackson Street, Montgomery, AL, USA, 36104) Department of Mathematics and Computer Science, Alabama State University

Email address, Chunhua Feng: `cfeng@alasu.edu`