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CLASS OF ANALYTIC UNIVALENT FUNCTIONS WITH FIXED FINITE NEGATIVE COEFFICIENTS DEFINED BY $q-$ANALOGUE DIFFERENCE OPERATOR

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ABSTRACT. In this paper using a $q-$analogue operator, we define class of univalent functions with fixed finite negative coefficients and determine coefficient estimates and other properties for this class. Various results obtained in this paper are shown to be sharp.

1. Introduction

Let $\mathcal{S}$ denote the class of functions of the form:

$$\mathcal{F}(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in $\mathbb{D} = \{z \in \mathbb{C} :\mid z \mid< 1\}$. We also denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions of the form:

$$\mathcal{F}(z) = z - \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0). \tag{1.2}$$

Given $0 \leq \alpha < 1$, a function $\mathcal{F} \in \mathcal{T}$ is said to be in the class $\mathcal{T}^*(\alpha)$ of starlike functions of order α in $\mathbb{D}$ if

$$Re\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\} > \alpha.$$

For a function $\mathcal{F} \in \mathcal{T}$, the Jackson's $q-$derivative [12] $(0 < q < 1)$, which is already introduced in several earlier investigations (see, for example [2, 3, 4, 5],[8],[10],[17,

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18, 19]);

$$\nabla_q\mathcal{F}(z)=\begin{cases}\frac{\mathcal{F}(z)-\mathcal{F}(qz)}{(1-q)z} & ,z\neq 0\\ \mathcal{F}^{'}(0) & ,z=0\end{cases},$$

that is

$$\nabla_q\mathcal{F}(z)=1-\sum_{k=2}^{\infty}[k]_q a_k z^{k-1}, \tag{1.3}$$

where

$$[j]_q=\frac{1-q^j}{1-q},\ [0]_q=0. \tag{1.4}$$

As $q\to 1^-$, $[j]_q=j$ and $\nabla_q\mathcal{F}(z)=\mathcal{F}^{'}(z)$.

Mostafa and Saleh [13] defined the $\mathcal{H}^m_{\lambda,\mu,q}$ operator for $\lambda\geq\mu\geq 0$, $0<q<1$, by

$$\mathcal{H}^0_{\lambda,\mu,q}\mathcal{F}(z)=\mathcal{F}(z),$$

$$\mathcal{H}^1_{\lambda,\mu,q}\mathcal{F}(z)=\mathcal{H}_{\lambda,\mu,q}\mathcal{F}(z)=(1-\lambda+\mu)\mathcal{F}(z)+(\lambda-\mu)z\nabla_q\mathcal{F}(z)+\lambda\mu z^2\nabla_q^2\mathcal{F}(z),$$

$$\mathcal{H}^2_{\lambda,\mu,q}\mathcal{F}(z)=\mathcal{H}_{\lambda,\mu,q}(\mathcal{H}_{\lambda,\mu,q}\mathcal{F}(z)),$$

and

$$\begin{aligned}\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) &= \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}^{m-1}_{\lambda,\mu,q}\mathcal{F}(z))\\ &= z-\sum_{k=2}^{\infty}\chi^m_{q,k}(\lambda,\mu)a_k z^k, m\in\mathbb{N}_0.\end{aligned} \tag{1.5}$$

where

$$\chi^m_{q,k}(\lambda,\mu)=[1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m. \tag{1.6}$$

Note that

(i) $\lim_{q\to 1-}\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)=\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(z)$ see Orhan et al. [15] (see also [9], [14] and Răducanu and Orhan [16]);

(ii) $\mathcal{H}^m_{1,0,q}\mathcal{F}(z)=\mathcal{D}^m_q\mathcal{F}(z)$(see [11], [20] and [6]);

(iii) $\mathcal{H}^m_{\lambda,0,q}\mathcal{F}(z)=\mathcal{D}^m_{\lambda,q}\mathcal{F}(z)$ (see Aouf et al. [7]);

(iv) $\lim_{q\to 1-}\mathcal{H}^m_{\lambda,0,q}\mathcal{F}(z)=\mathcal{D}^m_{\lambda}\mathcal{F}(z)$ (see Al-Oboudi [1]).

Now, by making use of the operator $\mathcal{H}^m_{\lambda,\mu,q}$, we have

Definition 1.1. Let $0 \leq \gamma < 1$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$, and $\mathcal{F} \in \mathcal{T}$, such that $\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) \neq 0$ for $z \in \mathbb{D}$. We say that $\mathcal{F} \in \mathbb{Y}^m_q(\lambda, \mu, \gamma)$ if

$$Re\left\{\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)}\right\} > \gamma, \tag{1.7}$$

which is equivalent to

$$\left|\frac{\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)} - 1}{\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)} + (1-2\gamma)}\right| < 1. \tag{1.8}$$

Note: for different values of q, λ, μ, γ, we have:
(i) $\lim_{q\to 1-} \mathbb{Y}^m_q(\lambda, \mu, \gamma) = \mathbb{Y}^m(\lambda, \mu, \gamma) = \left\{\mathcal{F}(z) : Re\left\{\frac{z(\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(z))'}{\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(z)}\right\} > \gamma\right\}$;
(ii) $\mathbb{Y}^m_q(\lambda, 0, \gamma) = \mathbb{Y}^m_{\lambda,q}(\gamma) = \left\{\mathcal{F}(z) : Re\left\{\frac{z\nabla_q(\mathcal{D}^m_{\lambda,q}\mathcal{F}(z))}{D^m_{\lambda,q}\mathcal{F}(z)}\right\} > \gamma\right\}$;
(iii) $\mathbb{Y}^m_q(1, 0, \gamma) = \mathbb{Y}^m_q(\gamma) = \left\{\mathcal{F}(z) : Re\left\{\frac{z\nabla_q(\mathcal{D}^m_q\mathcal{F}(z))}{\mathcal{D}^m_q\mathcal{F}(z)}\right\} > \gamma\right\}$.

2. Main Results

Unless indicated, let $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$, $0 \leq \gamma < 1$ and $\mathcal{F}(z)$ given by (1.2).

Theorem 2.1. *The function $\mathcal{F} \in \mathbb{Y}^m_q(\lambda, \mu, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty}([k]_q - \gamma)\chi^m_{q,k}(\lambda, \mu)a_k \leq 1 - \gamma. \tag{2.1}$$

Proof. Assume that (2.1) holds true. It is sufficient to show that

$$\left|\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)} - 1\right| \leq 1 - \gamma.$$

We have

$$\begin{aligned}\left|\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)} - 1\right| &= \left|\frac{\sum_{k=2}^{\infty}(1-[k]_q)\chi^m_{q,k}(\lambda, \mu)a_k z^k}{z - \sum_{k=2}^{\infty}\chi^m_{q,k}(\lambda, \mu)a_k z^k}\right| \\ &\leq \frac{\sum_{k=2}^{\infty}([k]_q - 1)\chi^m_{q,k}(\lambda, \mu)a_k}{1 - \sum_{k=2}^{\infty}\chi^m_{q,k}(\lambda, \mu)a_k}.\end{aligned}$$

This last expression is bounded above by $1-\gamma$. Then $\mathcal{F} \in \mathbb{Y}^m_q(\lambda, \mu, \gamma)$. Now, let $\mathcal{F} \in \mathbb{Y}^m_q(\lambda, \mu, \gamma)$, then

$$Re\left\{\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)}\right\} = Re\left\{\frac{z - \sum_{k=2}^{\infty}[k]_q\chi^m_{q,k}(\lambda, \mu)a_k z^k}{z - \sum_{k=2}^{\infty}\chi^m_{q,k}(\lambda, \mu)a_k z^k}\right\} > \gamma.$$

Choose values of z on real axis so that $\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)}$ is real. Letting $z \to 1^-$ through real values, we have

$$1-\sum_{k=2}^{\infty}[k]_q\chi^m_{q,k}(\lambda,\mu)a_kz^k \geq \gamma-\sum_{k=2}^{\infty}\gamma\chi^m_{q,k}(\lambda,\mu)a_kz^k.$$

Thus we obtain

$$\sum_{k=2}^{\infty}([k]_q-\gamma)\chi^m_{q,k}(\lambda,\mu)a_k \leq 1-\gamma,$$

which is (2.1). Hence the theorem. □

Corollary 2.1. *If $\mathcal{F} \in \mathbb{Y}^m_q(\lambda,\mu,\gamma)$, then we have*

$$a_k \leq \frac{1-\gamma}{([k]_q-\gamma)\chi^m_{q,k}(\lambda,\mu)}. \tag{2.2}$$

This equality is attained for the function $\mathcal{F}$ given by

$$\mathcal{F}(z)=z-\frac{1-\gamma}{([k]_q-\gamma)\chi^m_{q,k}(\lambda,\mu)}z^k \tag{2.3}$$

Let $\mathbb{Y}^m_q(\lambda,\mu,\gamma,c_n)$ be the subclass of $\mathbb{Y}^m_q(\lambda,\mu,\gamma)$ consisting of functions of the form

$$\mathcal{F}(z)=z-\sum_{i=2}^{n}\frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi^m_{q,i}(\lambda,\mu)}z^i-\sum_{k=n+1}^{\infty}a_kz^k, \tag{2.4}$$

where here and in the rest of the paper $0 \leq c_i \leq 1$ and $\sum_{i=2}^{n} c_i \leq 1$.

Theorem 2.2. *Let $\mathcal{F}(z)$ be defined by (2.4). Then $\mathcal{F}(z) \in \mathbb{Y}^m_q(\lambda,\mu,\gamma,c_n)$ if and only if*

$$\sum_{k=n+1}^{\infty}([k]_q-\gamma)\chi^m_{q,k}(\lambda,\mu)a_k \leq (1-\gamma)(1-\sum_{i=2}^{n}c_i). \tag{2.5}$$

The result is sharp.

Proof. Putting

$$a_i=\frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi^m_{q,i}(\lambda,\mu)}\ (i=2,3,4,...,n), \tag{2.6}$$

in (2.1), we have

$$\sum_{i=2}^{n}c_i+\sum_{k=n+1}^{\infty}\frac{([k]_q-\gamma)\chi^m_{q,k}(\lambda,\mu)}{(1-\gamma)}a_k \leq 1, \tag{2.7}$$

which implies 2.5. The result is sharp for the function

$$\mathcal{F}(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)} z^i - \frac{(1-\gamma)(1-\sum_{i=2}^n c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)} z^k, \tag{2.8}$$

for $k \geq n+1$. □

Corollary 2.2. *If $\mathcal{F}(z)$ be defined by (2.4) and $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$, then*

$$a_k \leq \frac{(1-\gamma)(1-\sum_{i=2}^n c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)}, \quad (k \geq n+1). \tag{2.9}$$

The result is sharp for the function $\mathcal{F}(z)$ given by (2.8).

Theorem 2.3. *If $\mathcal{F} \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$, then*

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{(1-\gamma)(1-\sum_{i=2}^n c_i)}{([n+1]_q-\gamma)\chi_{q,n+1}^m(\lambda,\mu)}, \tag{2.10}$$

and

$$\sum_{k=n+1}^{\infty} [k]_q a_k \leq \frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^n c_i)}{([n+1]_q-\gamma)\chi_{q,n+1}^m(\lambda,\mu)}. \tag{2.11}$$

Proof. Let $\mathcal{F} \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$. Then, in view of (2.5), we have

$$([n+1]_q-\gamma)\chi_{q,n+1}^m(\lambda,\mu) \sum_{k=n+1}^{\infty} a_k \leq (1-\gamma)(1-\sum_{i=2}^{n} c_i), \tag{2.12}$$

which immediately yields the first assertion.

For the proof of the second assertion, by appealing to (2.5), we have

$$\chi_{q,n+1}^m(\lambda,\mu) \sum_{k=n+1}^{\infty} [k]_q a_k \leq (1-\gamma)(1-\sum_{i=2}^{n} c_i) + \gamma\chi_{q,n+1}^m(\lambda,\mu) \sum_{k=n+1}^{\infty} a_k, \tag{2.13}$$

which in view of (2.10), can be putten in the form:

$$\chi_{q,n+1}^m(\lambda,\mu) \sum_{k=n+1}^{\infty} [k]_q a_k \leq (1-\gamma)(1-\sum_{i=2}^{n} c_i) + \gamma\frac{(1-\gamma)(1-\sum_{i=2}^n c_i)}{([n+1]_q-\gamma)}. \tag{2.14}$$

Upon simplifying the right hand side of (2.14), we have the assertion (2.11). □

Theorem 2.4. *Let the function $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda, \mu, \gamma, c_n)$. Then*

$$|z| - |z|^2 \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} - \frac{(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([k]_q - \gamma)\chi_{q,k}^m(\lambda,\mu)} |z|^{n+1}$$

$$\leq |\mathcal{F}(z)| \leq |z| + |z|^2 \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} + \frac{(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([k]_q - \gamma)\chi_{q,k}^m(\lambda,\mu)} |z|^{n+1},$$

(2.15)

with equality for

$$\mathcal{F}(z) = z - z^2 \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} - \frac{(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([k]_q - \gamma)\chi_{q,k}^m(\lambda,\mu)} z^{n+1}.$$

Proof. For $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda, \mu, \gamma, c_n)$. Then

$$\begin{aligned} |\mathcal{F}(z)| &= \left| z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} a_k z^k \right| \\ &\leq |z| + |z|^2 \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k, \end{aligned}$$

and

$$\begin{aligned} |\mathcal{F}(z)| &= \left| z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} a_k z^k \right| \\ &\geq |z| - |z|^2 \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k, \end{aligned}$$

which in view of (2.10), we have (2.15). □

Theorem 2.5. *Let $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda, \mu, \gamma, c_n)$, then*

$$1 - |z| \sum_{i=2}^{n} \frac{[i]_q c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} - \frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([n+1]_q - \gamma)\chi_{q,n+1}^m(\lambda,\mu)} |z|^n$$

$$\leq |\nabla_q \mathcal{F}(z)| \leq 1 + |z| \sum_{i=2}^{n} \frac{[i]_q c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} + \frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([n+1]_q - \gamma)\chi_{q,n+1}^m(\lambda,\mu)} |z|^n,$$

(2.16)

with equality for

$$\nabla_q \mathcal{F}(z) = 1 - z \sum_{i=2}^{n} \frac{[i]_q c_i(1-\gamma)}{([i]_q - \gamma)\chi_{q,i}^m(\lambda,\mu)} - \frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([n+1]_q - \gamma)\chi_{q,n+1}^m(\lambda,\mu)} z^n.$$

Proof. For $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$. Then

$$\begin{aligned}|\nabla_q \mathcal{F}(z)| &= \left|1-\sum_{i=2}^{n}\frac{[i]_q c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}z^{i-1}-\sum_{k=n+1}^{\infty}[k]_q a_k z^{k-1}\right|\\ &\leq 1+|z|\sum_{i=2}^{n}\frac{[i]_q c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}+|z|^n\sum_{k=n+1}^{\infty}[k]_q a_k,\end{aligned}$$

and

$$\begin{aligned}|\nabla_q \mathcal{F}(z)| &= \left|1-\sum_{i=2}^{n}\frac{[i]_q c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}z^{i-1}-\sum_{k=n+1}^{\infty}[k]_q a_k z^{k-1}\right|\\ &\geq 1-|z|\sum_{i=2}^{n}\frac{[i]_q c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}-|z|^n\sum_{k=n+1}^{\infty}[k]_q a_k,\end{aligned}$$

which in view of (2.11), we have (2.16). □

Theorem 2.6. *Let $\mathcal{F}(z)$ be defined by (2.4), $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$. Then $\mathcal{F}(z)$ is covex of order ρ $(0 \leq \rho < 1)$ in $0<|z|<r_0$, where r_0 is the largest value for which*

$$\sum_{i=2}^{n}\frac{[i]_q c_i([i-1]_q+1-\rho)(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}r_0^{i-1}+\frac{[k]_q([k-1]_q+1-\rho)(1-\gamma)(1-\sum_{i=2}^{n}c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)}r_0^{k-1}\leq 1-\rho, \tag{2.17}$$

for $k \geq n+1$. The result is sharp for the function $\mathcal{F}(z)$ given by (2.8).

Proof. It is sufficent to show that

$$\left|\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right| \leq 1-\rho, \quad (|z|<r_0). \tag{2.18}$$

We have

$$\left|\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right| \leq \frac{\sum_{i=2}^{n}\frac{[i]_q[i-1]_q c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}|z|^{i-1}+\sum_{k=n+1}^{\infty}[k]_q[k-1]_q a_k|z|^{k-1}}{1-\sum_{i=2}^{n}\frac{[i]_q c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}|z|^{i-1}-\sum_{k=n+1}^{\infty}[k]_q a_k|z|^{k-1}}. \tag{2.19}$$

Then

$$\sum_{i=2}^{n}\frac{[i]_q c_i([i-1]_q+1-\rho)(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}|z|^{i-1}+\sum_{k=n+1}^{\infty}[k]_q([k-1]_q+1-\rho)a_k|z|^{k-1}\leq 1-\rho. \tag{2.20}$$

Hence by Theorem 2.2 and (2.20) we have

$$\sum_{i=2}^{n}\frac{[i]_q c_i([i-1]_q+1-\rho)(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)}|z|^{i-1}+\frac{[k]_q([k-1]_q+1-\rho)(1-\gamma)(1-\sum_{i=2}^{n}c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)}|z|^{k-1}\leq 1-\rho. \tag{2.21}$$

Theorem 2.6 follows easily from (2.21). □

Theorem 2.7. *The class $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$ is closed under convex linear compination.*

Proof. Let $\mathcal{F}(z)$ be defined by (2.4). Define the function $h(z)$ by

$$h(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^{m}(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} b_k z^k. \tag{2.22}$$

Suppose that $\mathcal{F}(z)$ and $h(z)$ are in the class $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$, we only need to prove that

$$G(z) = \zeta\mathcal{F}(z) + (1-\zeta)h(z) \quad (0 \le \zeta \le 1), \tag{2.23}$$

also be in the class. Since

$$G(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^{m}(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} \left\{\zeta a_k + (1-\zeta)b_k\right\} z^k, \tag{2.24}$$

then

$$\sum_{k=n+1}^{\infty} \frac{([k]_q-\gamma)\chi_{q,k}^{m}(\lambda,\mu)}{(1-\gamma)} \left\{\zeta a_k + (1-\zeta)b_k\right\} \le (1 - \sum_{i=2}^{n} c_i), \tag{2.25}$$

with the aid of Theorem 2.2. Hence $G(z) \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$. This clearly completes the proof of the Theorem. □

The Theorems 2.8, 2.9 generalize Theorem 2.7, so we omit the proofs.

Theorem 2.8. *Let the functions*

$$\mathcal{F}_j(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^{m}(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} a_{k,j} z^k \tag{2.26}$$

be in the class $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$ for every $j = 1,2,...,x$. Then the function $\mathbb{F}(z)$ defined by

$$\mathbb{F}(z) = \sum_{j=1}^{x} d_j \mathcal{F}_j(z) \quad (d_j \ge 0), \tag{2.27}$$

is also in the class $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$, where

$$\sum_{j=1}^{x} d_j = 1. \tag{2.28}$$

Theorem 2.9. *Let the function $\mathcal{F}_j(z)$ defined by (2.26) be in the class $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$, for each $j = 1,2,...,x$, then the function $\phi(z)$ defined by*

$$\phi(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^{m}(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} b_k z^k, \tag{2.29}$$

also be in the class $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$, *where*

$$b_k = \frac{1}{x}\sum_{j=1}^{x} a_{k,j}. \tag{2.30}$$

Theorem 2.10. *Let*

$$\mathcal{F}_n(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)} z^i, \tag{2.31}$$

and

$$\mathcal{F}_k(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} \frac{(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)} z^k, \tag{2.32}$$

for $k \geq n+1$. *Then the function* $\mathcal{F}(z)$ *is in the class* $\mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$ *if and only if it can be expressed in the form*

$$\mathcal{F}(z) = \sum_{k=n}^{\infty} \eta_k \mathcal{F}_k(z), \tag{2.33}$$

where $\eta_k \geq 0$ $(k \geq n)$ *and*

$$\sum_{k=n}^{\infty} \eta_k = 1. \tag{2.34}$$

Proof. We suppose that the function $\mathcal{F}(z)$ can be expressed in the form (2.33). Then from (2.31), (2.32) and (2.34) we have

$$\mathcal{F}(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\gamma)}{([i]_q-\gamma)\chi_{q,i}^m(\lambda,\mu)} z^i - \sum_{k=n+1}^{\infty} \frac{\eta_k(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)} z^k. \tag{2.35}$$

Since

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{\eta_k(1-\gamma)(1-\sum_{i=2}^{n} c_i)}{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)} \frac{([k]_q-\gamma)\chi_{q,k}^m(\lambda,\mu)}{(1-\gamma)} \\
&= (1-\sum_{i=2}^{n} c_i)\sum_{k=n+1}^{\infty} \eta_k \\
&= (1-\sum_{i=2}^{n} c_i)(1-\eta_n) \\
&\leq (1-\sum_{i=2}^{n} c_i). \qquad (2.36)
\end{aligned}$$

Then $\mathcal{F}(z) \in \mathbb{Y}_q^m(\lambda,\mu,\gamma,c_n)$.

Conversely, assuming that $\mathcal{F}(z)$ defined by (2.4) be in the class $\mathbb{Y}_q^m(\lambda, \mu, \gamma, c_n)$ which satisfies (2.9) for $k \geq n+1$, we obtain

$$\eta_k = \frac{([k]_q - \gamma)\chi_{q,k}^m(\lambda, \mu)}{(1-\gamma)(1-\sum_{i=2}^n c_i)} a_k \leq 1, \tag{2.37}$$

and

$$\eta_k = 1 - \sum_{k=n+1}^{\infty} \lambda_k. \tag{2.38}$$

This compelets the proof of the Theorem 2.10. □

Corollary 2.3. *The extreme points of the class* $\mathbb{Y}_q^m(\lambda, \mu, \gamma, c_n)$ *are the functions* $\mathcal{F}_k(z)$ $(k \geq n)$ *given by (2.31) and (2.32) in Theorem 2.10.*

Remark 1. *For different values of* λ, μ, q *and* γ *in our results, we have results for the special classes defined in the introduction.*

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