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FURTHER RESULTS ON $\mathcal{I}$ AND $\mathcal{I}^*-$CONVERGENCE OF SEQUENCES IN GRADUAL NORMED LINEAR SPACES

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Abstract. In this paper, following a very recent and new approach, we introduce the notion of gradual $\mathcal{I}-$limit point, gradual $\mathcal{I}-$cluster point, and prove certain properties of both the notions. We also investigate some new properties of gradual $\mathcal{I}-$Cauchy and gradual $\mathcal{I}^*-$Cauchy sequences and show that the condition (AP) plays a crucial role to relate both the notions. Finally, we investigate the notion of $\mathcal{I}$ and $\mathcal{I}^*-$divergence of sequences in gradual normed linear spaces and prove the essence of the condition (AP) again to establish the relationship between the notions.

1. Introduction and Background

In 1965, the concept of fuzzy sets [33] was introduced by Zadeh as one of the extensions of the classical set-theoretical concept. Nowadays it has wide applications in different branches of science and engineering. The term "fuzzy number" plays a vital role in the study of fuzzy set theory. Fuzzy numbers were essentially the generalization of intervals, not numbers. Indeed fuzzy numbers do not obey a couple of algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many researchers due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et.al. [11] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function which is defined in the interval $(0, 1]$. So in

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some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

In 2011, Sadeqi and Azari [26] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological points of view. Further investigation in this direction has been occurred due to Ettefagh et. al. [9, 10] and many others. For an extensive study on gradual real numbers, one may refer to [1, 5, 18, 31].

On the other hand in 2001, the concept of ideal convergence was first introduced by Kostyrko et. al. [16] mainly as an extension of statistical convergence. They also showed that the ideal convergence was also a generalized form of some other known convergence concepts. Later on, several investigations in this direction have been carried out. In [3, 24], the notion of $\mathcal{I}-$Cauchy sequences was investigated. In [6, 7], Esi studied ideal convergence of strongly almost summable single and double sequences in $2-$normed and $n-$normed settings. In [21, 22], Mursaleen and Mohiuddine investigated ideal convergence of single and double sequences respectively in probabilistic normed spaces. In [23], Mursaleen et. al. investigated ideal convergence in intuitionistic fuzzy normed spaces. In [32], Tripathy and Hazarika investigated ideal convergence in paranormed spaces. For an extensive view, [4, 8, 12, 13, 14, 15, 17, 19, 20, 25, 27, 28, 29, 30] can be addressed where many more references can be found.

Research on the convergence of sequences in gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of convergence of sequences in gradual normed linear spaces. Recently, Choudhury and Debnath [2] have introduced the notion of $\mathcal{I}-$convergence of sequences in gradual normed linear spaces. This paper is the continuation of the above work.

2. Definitions and Preliminaries

Definition 2.1. [11] A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual number is said to be non-negative if for every $\xi \in (0, 1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

Definition 2.2. [11] Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi)$, $\forall \xi \in (0, 1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$A_{\tilde{r}_1 + \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi) \quad and \quad A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi), \ \forall \xi \in (0, 1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $A_{\tilde{p}}(\xi) = p$ for any $\xi \in (0, 1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and multiplication forms a real vector space [11].

Definition 2.3. [26] Let X be a real vector space. The function $|| \cdot ||_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\xi \in (0, 1]$, following three conditions are true for any $x, y \in X$
(G_1) $A_{||x||_G}(\xi) = A_{\tilde{0}}(\xi)$ if and only if $x = 0$;
(G_2) $A_{||\lambda x||_G}(\xi) = |\lambda| A_{||x||_G}(\xi)$ for any $\lambda \in \mathbb{R}$;
(G_3) $A_{||x+y||_G}(\xi) \leq A_{||x||_G}(\xi) + A_{||y||_G}(\xi)$.

The pair $(X, || \cdot ||_G)$ is called a gradual normed linear space (GNLS).

Example 2.1. [26] *Let* $X = \mathbb{R}^n$ *and for* $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \xi \in (0, 1]$, *define* $||\cdot||_G$ *by* $A_{||x||_G}(\xi) = e^{\xi} \sum_{i=1}^{n} |x_i|$. *Then,* $||\cdot||_G$ *is a gradual norm on* $\mathbb{R}^n$ *and* $(\mathbb{R}^n, ||\cdot||_G)$ *is a GNLS.*

Definition 2.4. [26] Let (x_k) be a sequence in the GNLS $(X, || \cdot ||_G)$. Then (x_k) is said to be gradual convergent to $x \in X$, if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_{\varepsilon}(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x||_G}(\xi) < \varepsilon$, $\forall n \geq N$.

Definition 2.5. [26] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual Cauchy, if for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x_j||_G}(\xi) < \varepsilon$, $\forall k, j \geq N$.

Theorem 2.1. *([26], Theorem 3.6) Let $(X, ||\cdot||_G)$ be a GNLS, then every gradual convergent sequence in X is also a gradual Cauchy sequence.*

Definition 2.6. [16] Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if

(i) $\emptyset \in \mathcal{I}$;

(ii) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(iii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

Some standard examples of ideal are given below:

(i) The set $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathbb{N}$. Here $\mathbb{N}$ denotes the set of all natural numbers.

(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an admissible ideal in $\mathbb{N}$.

(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ is an admissible ideal in $\mathbb{N}$.

(iv) Suppose $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ be a decomposition of $\mathbb{N}$ (for $i \neq j$, $D_i \cap D_j = \emptyset$). Then the set $\mathcal{I}$ of all subsets of $\mathbb{N}$ which intersects finitely many D_p's forms an ideal in $\mathbb{N}$.

More important examples can be found in [12] and [15].

Definition 2.7. [16] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Definition 2.8. [16] Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$. A real-valued sequence (x_k) is said to be $\mathcal{I}$-convergent to l if for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} :\mid x_k - l \mid\geq \varepsilon\}$ belongs to $\mathcal{I}$. l is called the $\mathcal{I}$-limit of the sequence (x_k) and is written as $\mathcal{I}-\lim_k x_k = l$.

Definition 2.9. [16] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A real valued sequence (x_k) is said to be $\mathcal{I}^*-$convergent to l, if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\}$ in the associated filter $\mathcal{F}(\mathcal{I})$ such that $\lim_{k\in M} x_k = l$. Symbolically, $\mathcal{I}^* - \lim_k x_k = l$.

Definition 2.10. [16] A real number x_0 is said to be $\mathcal{I}-$limit point of a real-valued sequence (x_k) provided that there is a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_k x_{m_k} = x_0$.

The set of all $\mathcal{I}-$limit points of the sequence (x_k) is denoted by $\mathcal{I}(\Lambda_x)$.

Definition 2.11. [16] A real number x_0 is said to be $\mathcal{I}-$cluster point of a real-valued sequence (x_k) provided that for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - x_0| < \varepsilon\} \notin \mathcal{I}$.

The set of all $\mathcal{I}-$cluster points of the sequence (x_k) is denoted by $\mathcal{I}(\Gamma_x)$.

Definition 2.12. [24] A real-valued sequence (x_k) is said to be $\mathcal{I}-$Cauchy, if for every $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |x_k - x_N| \geq \varepsilon\} \in \mathcal{I}$.

Definition 2.13. [24] A real-valued sequence (x_k) is said to be $\mathcal{I}^*-$Cauchy if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is a ordinary Cauchy sequence.

Definition 2.14. [3] A real-valued sequence (x_k) is said to be $\mathcal{I}-$divergent if there exists some $l \in \mathbb{R}$ such that for any $B > 0$, $\{k \in \mathbb{N} : |x_k - l| \leq B\} \in \mathcal{I}$.

Definition 2.15. [3] A real-valued sequence (x_k) is said to be $\mathcal{I}^*-$divergent if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I})$ and x_{m_k} is divergent i.e., there exists a real number $l \in \mathbb{R}$ satisfying $|x_{m_k} - l| \to \infty$ as $k \to \infty$.

Definition 2.16. [16] An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP), if for every countable family of mutually disjoint sets $\{C_n\}_{n\in\mathbb{N}}$ from $\mathcal{I}$, there exists a countable family of sets $\{B_n\}_{n\in\mathbb{N}}$ such that the symmetric difference $C_j \triangle B_j$ is finite for every $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Definition 2.17. [2] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}-$convergent to $x \in X$ if for every $\xi \in (0,1]$ and $\varepsilon > 0$, the set $C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}$. Symbolically, $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$.

Definition 2.18. [2] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*-$convergent to $x \in X$ if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is gradual convergent to x. Symbolically, $x_k \xrightarrow{\mathcal{I}^*-||\cdot||_G} x$.

Definition 2.19. [2] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}-$Cauchy if for every $\varepsilon > 0$ and $\xi \in (0,1]$, there exists a natural number $N (= N_\varepsilon(\xi))$ such that the set $C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x_N||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}$.

Theorem 2.2. [2] *Let $(X, ||\cdot||_G)$ be a GNLS. Then every gradually $\mathcal{I}-$convergent sequence in X is gradually $\mathcal{I}-$Cauchy sequence.*

Definition 2.20. [2] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*-$Cauchy if there exists a set $M = \{m_1 < m_2 < ... < m_k < ..\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is gradual Cauchy sequence.

Theorem 2.3. [2] *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. If (x_k) is gradually $\mathcal{I}^*-$Cauchy then it is gradually $\mathcal{I}-$Cauchy.*

Throughout the article, $\mathcal{I}$ will denote the non-trivial admissible ideal in $\mathbb{N}$.

3. Gradual $\mathcal{I}-$limit points and Gradual $\mathcal{I}-$cluster points

Definition 3.1. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then $x_0 \in X$ is said to be gradual $\mathcal{I}-$limit point of (x_k), if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \notin \mathcal{I}$ such that $x_{m_k} \xrightarrow{||\cdot||_G} x_0$.

For any sequence (x_k), the set of all gradual $\mathcal{I}-$limit points is denoted by

$$\mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)}).$$

Definition 3.2. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then $x_0 \in X$ is said to be gradual $\mathcal{I}-$cluster point of (x_k) if for any $\varepsilon > 0$ and $\xi \in (0,1]$, the set $\{k \in \mathbb{N} : A_{||x_k - x_0||_G}(\xi) < \varepsilon\} \notin \mathcal{I}$.

For any sequence (x_k), the set of all gradual $\mathcal{I}-$cluster points is denoted by

$$\mathcal{I} - ||\cdot||_G(\Gamma_{(x_k)}).$$

Example 3.1. *Let $X = \mathbb{R}^n$ and $||\cdot||_G$ be the norm defined in Example 2.1. Consider the decomposition of $\mathbb{N}$ given by $\mathbb{N} = \bigcup_{r=1}^{\infty} D_r$, where $D_r = \{2^{r-1}(2s-1) : s \in \mathbb{N}\}, r \in \mathbb{N}$. Clearly, $D_i \cap D_j = \emptyset$ for $i \neq j$. Consider the sequence (x_k) in $\mathbb{R}^n$ defined by $x_k = (0, 0, ..., 0, 1 - \frac{1}{r})$ whenever $k \in D_r$. Then,*
(i) $(0, 0, ..., 0, 1 - \frac{1}{r}) \in \mathcal{I}_d - ||\cdot||_G(\Lambda_{(x_k)})$, $r = 1, 2, ...$
(ii) $(0, 0, ..., 0, 1) \in \big(\mathcal{I}_d - ||\cdot||_G(\Gamma_{(x_k)})\big) \setminus \big(\mathcal{I}_d - ||\cdot||_G(\Lambda_{(x_k)})\big)$.

Justification. (i) For every $n-$tuple of the form $(0, 0, ..., 0, 1 - \frac{1}{r})$ $(r = 1, 2, ...)$, there exists a set $D_r \notin \mathcal{I}_{\mathrm{d}}$ (because $d(D_r) = 2^{-r} \neq 0$) such that $(x_k)_{k \in D_r} \xrightarrow{||\cdot||_G} (0, 0, ..., 0, 1 - \frac{1}{r})$. Therefore, $(0, 0, ..., 0, 1 - \frac{1}{r}) \in \mathcal{I}_d - ||\cdot||_G(\Lambda_{(x_k)})$ for $r = 1, 2, ..$.
(ii) Now to prove $(0, 0, ..., 0, 1) \notin \mathcal{I}_d - ||\cdot||_G(\Lambda_{(x_k)})$, we assume the contrary. Then by definition, there exists $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $d(M) > 0$ and $x_{m_k} \xrightarrow{||\cdot||_G} (0, 0, ..., 0, 1)$. As $\mathbb{N} = \bigcup_{r=1}^{\infty} D_r$, so the following relation holds for a $j \in \mathbb{N}$:
(3.1)
$$M = M\cap\mathbb{N} = M\cap\left(\bigcup_{r=1}^{\infty} D_r\right) = \bigcup_{r=1}^{\infty}(M\cap D_r) = \left(\bigcup_{r=1}^{j}(M \cap D_r)\right)\cup\left(\bigcup_{r=j+1}^{\infty} (M \cap D_r)\right).$$

Now it is easy to verify that the set $M \cap D_r$ is finite for $r = 1, 2, ..$ and so the set $\bigcup_{r=1}^{j}(M \cap D_r)$ is also finite and eventually (3.1) yields the following

$$d(M) \leq d\left(\bigcup_{r=j+1}^{\infty} (M \cap D_r)\right). \tag{3.2}$$

Again, as $\bigcup_{r=j+1}^{\infty} (M \cap D_r) \subseteq \{1 \cdot 2^j, 2 \cdot 2^j, 3 \cdot 2^j, ...\}$, so from (3.2) we have for any $j = 1, 2, 3, ...,$

$$d(M) \leq d(\{1 \cdot 2^j, 2 \cdot 2^j, 3 \cdot 2^j, ...\}) = 2^{-j}.$$

This implies that $d(M) = 0$, which is a contradiction. Hence our assumption was wrong and we must have $(0, 0, ..., 0, 1) \notin \mathcal{I}_d - ||\cdot||_G(\Lambda_{(x_k)})$.

Next we will show that $(0,0,...,0,1) \in \mathcal{I}_d - ||\cdot||_G(\Gamma_{(x_k)})$. For any $\xi \in (0,1]$ and $j \in \mathbb{N}$, we have

$$d(\{k \in \mathbb{N} : e^{\xi}(1-\frac{1}{j}) < A_{||x_k||_G}(\xi) < e^{\xi}\}) = d\left(\mathbb{N} \setminus \left(\bigcup_{r=1}^{j} D_r\right)\right) = 2^{-j} \neq 0.$$

Therefore,

$$\{k \in \mathbb{N} : e^{\xi}(1-\frac{1}{j}) < A_{||x_k||_G}(\xi) < e^{\xi}\} \notin \mathcal{I}_d \text{ for } r = 1, 2, ... \tag{3.3}$$

Let $\varepsilon > 0$ be arbitrary. Then, by Archimedean property there exists some $n_0 \in \mathbb{N}$ such that $\frac{e^{\xi}}{n_0} < \varepsilon$ which results the following inclusion

$$\begin{aligned}\{k \in \mathbb{N} : e^{\xi}(1-\frac{1}{j}) < A_{||x_k||_G}(\xi) < e^{\xi}\} &= \{k \in \mathbb{N} : A_{||x_k-(0,0,...,0,1)||_G}(\xi) < \frac{e^{\xi}}{n_0}\} \\ &\subset \{k \in \mathbb{N} : A_{||x_k-(0,0,...,0,1)||_G}(\xi) < \varepsilon\}.\end{aligned}$$

Now from (3.3), it is clear that the set in the left-hand side of the above inclusion does not belong to $\mathcal{I}_d$, so the set in the right-hand side also does not belong to $\mathcal{I}_d$. Hence $(0,0,...,0,1) \in \mathcal{I}_d - ||\cdot||_G(\Gamma_{(x_k)})$.

Theorem 3.1. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x_0$. Then $\mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)}) = \{x_0\}$.*

Proof. The proof is easy, so omitted. □

Theorem 3.2. *For any sequence (x_k) in the GNLS $(X, ||\cdot||_G)$, $\mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)}) \subset \mathcal{I} - ||\cdot||_G(\Gamma_{(x_k)})$.*

Proof. Let $x_0 \in \mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)})$. Then there exist a set $M = \{m_1 < m_2 < ... < m_k < ...\} \notin \mathcal{I}$ such that $x_{m_k} \xrightarrow{||\cdot||_G} x_0$. This means that for any $\varepsilon > 0$ and $\xi \in (0,1]$, there exists $k_0 \in \mathbb{N}$ such that $A_{||x_{m_k}-x_0||_G}(\xi) < \varepsilon \, \forall k \geq k_0$. Let $B = \{k \in \mathbb{N} : A_{||x_k-x_0||_G}(\xi) < \varepsilon\}$. Then $B \supseteq M \setminus \{m_1, m_2, ..., m_{k_0}\}$ and since $\mathcal{I}$ is an admissible with $M \notin \mathcal{I}$, so $B \notin \mathcal{I}$. Hence $x_0 \in \mathcal{I} - ||\cdot||_G(\Gamma_{(x_k)})$ and the inclusion is true.

To prove that the inclusion is strict, consider Example 3.1. It was shown that $(0,0,...,0,1) \in \mathcal{I}_d - ||\cdot||_G(\Gamma_{(x_k)})$ whereas $(0,0,...,0,1) \notin \mathcal{I}_d - ||\cdot||_G(\Lambda_{(x_k)})$. □

Theorem 3.3. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that $\{k \in \mathbb{N} : x_k \neq y_k\} \in \mathcal{I}$. Then,*
(i) $\mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)}) = \mathcal{I} - ||\cdot||_G(\Lambda_{(y_k)})$ and (ii) $\mathcal{I} - ||\cdot||_G(\Gamma_{(x_k)}) = \mathcal{I} - ||\cdot||_G(\Gamma_{(y_k)})$.

Proof. (i) Let $x_0 \in \mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)})$. Then, there exists a set $M \subset \mathbb{N}$ with $M = \{m_1 < m_2 < ... < m_k < ...\} \notin \mathcal{I}$ such that $x_{m_k} \xrightarrow{||\cdot||_G} x_0$. Now since the inclusion $\{k \in M : x_k \neq y_k\} \subseteq \{k \in \mathbb{N} : x_k \neq y_k\}$ holds, so $N = \{k \in M : x_k = y_k\} \notin \mathcal{I}$ and $N \subseteq M$. Let $N = \{n_1 < n_2 < ... < n_k < ...\}$ Then, $y_{n_k} \xrightarrow{||\cdot||_G} x_0$ holds and eventually we have $\mathcal{I} - ||\cdot||_G(\Lambda_{(x_k)}) \subseteq \mathcal{I} - ||\cdot||_G(\Lambda_{(y_k)})$. By symmetry, $\mathcal{I}-||\cdot||_G(\Lambda_{(y_k)}) \subseteq \mathcal{I}-||\cdot||_G(\Lambda_{(x_k)})$. Hence we have, $\mathcal{I}-||\cdot||_G(\Lambda_{(x_k)}) = \mathcal{I}-||\cdot||_G(\Lambda_{(y_k)})$.

(ii) Suppose $x_0 \in \mathcal{I}-||\cdot||_G(\Gamma_{(x_k)})$. Then by definition, for any $\varepsilon > 0$, and $\xi \in (0, 1]$, the set $B = B(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x_0||_G}(\xi) < \varepsilon\} \notin \mathcal{I}$. Let $C = C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||y_k - x_0||_G}(\xi) < \varepsilon\}$. We claim that $C \notin \mathcal{I}$. Because if $C \in \mathcal{I}$, then $\mathbb{N} \setminus C \in \mathcal{F}(\mathcal{I})$ and then by the hypothesis we obtain, $(\mathbb{N} \setminus C) \cap \{k \in \mathbb{N} : x_k = y_k\} \in \mathcal{F}(\mathcal{I})$. Consequently, the inclusion $(\mathbb{N} \setminus B) \supset (\mathbb{N} \setminus C) \cap \{k \in \mathbb{N} : x_k = y_k\}$ leads us to the contradiction that $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$. Therefore, we must have, $C \notin \mathcal{I}$ i.e., $x_0 \in \mathcal{I} - ||\cdot||_G(\Gamma_{(y_k)})$. Thus, $\mathcal{I}-||\cdot||_G(\Gamma_{(x_k)}) \subseteq \mathcal{I}-||\cdot||_G(\Gamma_{(y_k)})$. By symmetry, $\mathcal{I}-||\cdot||_G(\Gamma_{(y_k)}) \subseteq \mathcal{I}-||\cdot||_G(\Gamma_{(x_k)})$. Hence we have, $\mathcal{I} - ||\cdot||_G(\Gamma_{(x_k)}) = \mathcal{I} - ||\cdot||_G(\Gamma_{(y_k)})$. □

4. Gradual $\mathcal{I}$ and $\mathcal{I}^*-$Cauchy sequences

For $\varepsilon > 0$, and a sequence (x_k) in the GNLS $(X, ||\cdot||_G)$, we denote $E_n(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x_n||_G}(\xi) \geq \varepsilon\}$, $n \in \mathbb{N}$.

Theorem 4.1. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, the following conditions are equivalent:*
(i) (x_k) is a gradually $\mathcal{I}-$Cauchy sequence,
(ii) For any $\varepsilon > 0$, there exists $D \in \mathcal{I}$ such that for all $p, q \notin D$, $A_{||x_p - x_q||_G}(\xi) < \varepsilon$,
(iii) For any $\varepsilon > 0$, $\{n \in \mathbb{N} : E_n(\xi, \varepsilon) \notin \mathcal{I}\} \in \mathcal{I}$.

Proof. (i) $\Rightarrow$ (ii) Let (x_k) be a gradually $\mathcal{I}-$Cauchy sequence. Then by definition, for any $\varepsilon > 0$ and $\xi \in (0, 1]$, there exists a $N \in \mathbb{N}$ such that $E_N(\xi, \frac{\varepsilon}{2}) \in \mathcal{I}$. Put $D = E_N(\xi, \frac{\varepsilon}{2})$. Then, for any $p, q \notin D$ we have $A_{||x_p - x_N||_G}(\xi) < \frac{\varepsilon}{2}$ and $A_{||x_q - x_N||_G}(\xi) < \frac{\varepsilon}{2}$ which consequently implies

$$A_{||x_p - x_q||_G}(\xi) = A_{||x_p - x_N + x_N - x_q||_G}(\xi) \leq A_{||x_p - x_N||_G}(\xi) + A_{||x_q - x_N||_G}(\xi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(ii) $\Rightarrow$ (iii) To prove (iii) from (ii), it is sufficient to show that $\{n \in \mathbb{N} : E_n(\xi, \varepsilon) \notin \mathcal{I}\} \subset D$. If possible suppose there is a natural number $s \in \{n \in \mathbb{N} : E_n(\xi, \varepsilon) \notin \mathcal{I}\}$ such that $s \notin D$. Then, $E_s(\xi, \varepsilon) \notin \mathcal{I}$. Take $t \in E_s(\xi, \varepsilon) \setminus D$. Then we have $t \notin D$ and $A_{||x_t - x_s||_G}(\xi) \geq \varepsilon$. But this is a contradiction because $s, t \notin D$ implies $A_{||x_t - x_s||_G}(\xi) < \varepsilon$ by our assumption. Hence, $\{n \in \mathbb{N} : E_n(\xi, \varepsilon) \notin \mathcal{I}\} \subset D$.

(iii) $\Rightarrow$ (i) Suppose for any $\varepsilon > 0$, $\{n \in \mathbb{N} : E_n(\xi, \varepsilon) \notin \mathcal{I}\} \in \mathcal{I}$. Then, $\{n \in \mathbb{N} : E_n(\xi, \varepsilon) \in \mathcal{I}\} \in \mathcal{F}(\mathcal{I})$ and so is non-empty. Let $N \in \{n \in \mathbb{N} : E_n(\xi, \varepsilon) \in \mathcal{I}\}$. Then, for any $\varepsilon > 0$ and $\xi \in (0, 1]$, $\{k \in \mathbb{N} : A_{||x_k - x_N||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}$ holds and hence (x_k) is gradually $\mathcal{I}-$Cauchy sequence. □

Corollary 4.1. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Then a gradually Cauchy sequence (x_k) in the GNLS $(X, || \cdot ||_G)$ is also gradually $\mathcal{I}-$Cauchy.*

Now, we give an example to show that the reverse of Theorem 2.3 is not necessarily true.

Example 4.1. *Let $X = \mathbb{R}^n$ and $|| \cdot ||_G$ be the norm defined in Example 2.1. Consider the ideal $\mathcal{I}$ consisting of all subsets of $\mathbb{N}$ which intersects finitely many D_r's where $D_r = \{2^{r-1}(2s - 1) : s \in \mathbb{N}\}, r \in \mathbb{N}$ is the decomposition of $\mathbb{N}$ into disjoint subsets i.e., $\mathbb{N} = \bigcup_{r=1}^{\infty} D_r$ and $D_i \cap D_j = \emptyset$ for $i \neq j$. Consider the sequence (x_k) in $\mathbb{R}^n$ defined by $x_k = (0, 0, ..., 0, \frac{1}{r})$, if $k \in D_r$. Now since the sequence $(\frac{1}{n})$ is a Cauchy sequence in $\mathbb{R}$, so as a consequence for given $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that the following inequality,*

$$A_{||(0,0,...,0,\frac{1}{m})-(0,0,...,0,\frac{1}{n})||_G}(\xi) = e^{\xi} \left| \frac{1}{m} - \frac{1}{n} \right| < e^{\xi} \cdot \frac{\varepsilon}{2e^{\xi}} = \frac{\varepsilon}{2}$$

holds for all $m, n \geq N$. Now take $D = \bigcup_{r=1}^{N} D_r$. Then clearly $D \in \mathcal{I}$ and for any $p, q \notin D$, $A_{||x_p - x_q||_G}(\xi) < \varepsilon$. Therefore by Theorem 4.1, (x_k) is gradually $\mathcal{I}-$Cauchy. But we claim that (x_k) is not gradually $\mathcal{I}^*-$Cauchy.

To prove our claim, we assume the contrary. Then by definition, there exists a set $M = \{m_1 < m_2 < ... < m_k < ..\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is gradual Cauchy sequence i.e., for any $\varepsilon > 0$ and $\xi \in (0, 1]$, there exists some $k_0 \in \mathbb{N}$ such that

$$A_{||x_{m_i} - x_{m_j}||_G}(\xi) < \varepsilon \; \forall i, j \geq k_0. \tag{4.1}$$

Now, $M \in \mathcal{F}(\mathcal{I})$ implies $\mathbb{N} \setminus M \in \mathcal{I}$. By the construction of $\mathcal{I}$, there exists $m \in \mathbb{N}$ such that $\mathbb{N} \setminus M \subset \bigcup_{r=1}^{m} D_r$ which subsequently implies that $D_{m+1}, D_{m+2} \in M$. Now by the construction of D_r's, we can say that for the obtained k_0 in (4.1), there exists $m_i \in D_{m+1}$ and $m_j \in D_{m+2}$ such that $m_i, m_j \geq k_0$. But then the following equality

$$A_{||x_{m_i}-x_{m_j}||_G}(\xi) = e^{\xi}\left|\frac{1}{m+1} - \frac{1}{m+2}\right| = \frac{e^{\xi}}{(m+1)(m+2)}$$

contradicts (4.1) for any particular choice of ε with $\varepsilon < \frac{e^{\xi}}{(m+1)(m+2)}$. Thus our assumption was wrong and hence (x_k) is not gradually $\mathcal{I}^*-$Cauchy.

Theorem 4.2. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ which satisfies the condition (AP). Then, every gradually $\mathcal{I}-$Cauchy sequence in the GNLS $(X, ||\cdot||_G)$ is also gradually $\mathcal{I}^*-$Cauchy sequence.*

Proof. Suppose (x_k) be an $\mathcal{I}-$Cauchy sequence in the GNLS $(X, ||\cdot||_G)$. Then for any $\varepsilon > 0$ and $\xi \in (0,1]$, there exists a natural number $N(= N_{\varepsilon}(\xi))$ such that $\{k \in \mathbb{N} : A_{||x_k-x_N||_G}(\xi) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. In particular, for $\varepsilon = \frac{1}{p}$, $p \in \mathbb{N}$ we have $M_p = \{k \in \mathbb{N} : A_{||x_k-x_{n_p}||_G}(\xi) < \frac{1}{p}\} \in \mathcal{F}(\mathcal{I})$ where $n_p = N_{\frac{1}{p}}(\xi)$. Now as $\mathcal{I}$ satisfies (AP), so by Lemma (4) of [24], there exists $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that $M \setminus M_p$ is a finite set for all $p \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then by Archimedean property, we choose $p_0 \in \mathbb{N}$ such that $\frac{2}{p_0} < \varepsilon$. Clearly, the set $M \setminus M_{p_0}$ is a finite set and eventually there exists $k_0 \in \mathbb{N}$ such that $m_i, m_j \in M_{p_0}$ for all $i, j \geq k_0$

i.e., $A_{||x_{m_i}-x_{n_p}||_G}(\xi) < \frac{1}{p_0}$ and $A_{||x_{m_j}-x_{n_p}||_G}(\xi) < \frac{1}{p_0}$ for all $i, j \geq k_0$.

Now

$$\begin{aligned}
A_{||x_{m_i}-x_{m_j}||_G}(\xi) &= A_{||x_{m_i}-x_{n_p}+x_{n_p}-x_{m_j}||_G}(\xi) \\
&\leq A_{||x_{m_i}-x_{n_p}||_G}(\xi) + A_{||x_{m_j}-x_{n_p}||_G}(\xi) \\
&< \frac{1}{p_0} + \frac{1}{p_0} = \frac{2}{p_0} < \varepsilon \ \forall i, j \geq k_0.
\end{aligned}$$

Hence, (x_k) is gradually $\mathcal{I}^*-$Cauchy sequence. □

5. $\mathcal{I}$ and $\mathcal{I}^*-$divergence in GNLS

Definition 5.1. Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually divergent if there exists an element $x \in X$ such that for any $\xi \in (0,1]$, $A_{||x_k-x||_G}(\xi) \to \infty$ as $k \to \infty$.

Definition 5.2. Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}-$divergent if there exists an element $x \in X$ such that for any positive real number B and $\xi \in (0,1]$, $\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \leq B\} \in \mathcal{I}$.

It is easy to observe that if $\mathcal{I}$ is admissible, then a gradually divergent sequence is gradually $\mathcal{I}-$divergent.

Definition 5.3. Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*-$divergent if there exists $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that (x_{m_k}) is gradually divergent i.e., for any $\xi \in (0,1]$, there exists atleast one element $x \in X$ such that $A_{||x_{m_k} - x||_G}(\xi) \to \infty$ as $k \to \infty$.

Theorem 5.1. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, if (x_k) is gradually $\mathcal{I}^*-$divergent then (x_k) is gradually $\mathcal{I}-$divergent.*

Proof. From the assumption, it is clear that there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that for any $\xi \in (0,1]$, there exists at least one $x \in X$ such that $A_{||x_{m_k} - x||_G}(\xi) \to \infty$ as $k \to \infty$. This means that for any positive real number B, there exists $k_0 \in \mathbb{N}$ such that $A_{||x_{m_k} - x||_G}(\xi) > B$ for all $k > k_0$. Thus, we have

$$\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \leq B\} \subseteq (\mathbb{N} \setminus M) \cup \{1, 2, 3, ..., k_0\} \in \mathcal{I}, \text{ as } \mathcal{I} \text{ is admissible.}$$

Hence, the sequence (x_k) is gradually $\mathcal{I}-$divergent. □

The following example shows that the reverse of the above theorem is not always true.

Example 5.1. *Consider the ideal $\mathcal{I}$ of Example 4.1. Define a sequence (x_k) in $\mathbb{R}^n$ as follows $x_k = (0, 0, ..., 0, r)$, if $k \in D_r$. Then (x_k) is $\mathcal{I}-$divergent but not $\mathcal{I}^*-$divergent.*

Justification. Clearly, $A_{||x_k - \mathbf{0}||_G}(\xi) = re^{\xi}$ for $k \in D_r$. Then for any $B > 0$ and $\xi \in (0,1]$, there exists $m \in \mathbb{N}$ such that $B < me^{\xi}$ and eventually the following inclusion holds

$$\{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \leq B\} \subseteq \{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \leq me^{\xi}\}. \tag{5.1}$$

Also, $A_{||x_k-\mathbf{0}||_G}(\xi) = re^{\xi}$ for $k \in D_r$ implies

$$\{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \leq me^{\xi}\} = \bigcup_{r=1}^{m} D_r \in \mathcal{I}. \tag{5.2}$$

From (5.1) and (5.2), we obtain $\{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \leq B\} \in \mathcal{I}$. Hence (x_k) is $\mathcal{I}-$divergent.

But the above sequence is not gradually $\mathcal{I}^*-$divergent. Because for any $H \in \mathcal{I}$ there exists $m \in \mathbb{N}$ such that $H \subseteq \bigcup_{r=1}^{m} D_r$ and as a consequence $D_{m+1} \subseteq \mathbb{N} \setminus H$. Let $M = \{m_1 < m_2 < ... < m_k < ...\}$ denote the set $\mathbb{N} \setminus H$, then $M \in \mathcal{F}(\mathcal{I})$ and (x_{m_k}) is gradual convergent to $(0, 0, .., 0, m+1)$, a contradiction. Hence (x_k) is not gradually $\mathcal{I}^*-$divergent.

We end up by providing a theorem that gives a condition under which gradual $\mathcal{I}-$divergence of a sequence in the GNLS $(X, || \cdot ||_G)$ implies gradual $\mathcal{I}^*-$divergence.

Theorem 5.2. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, || \cdot ||_G)$. Then, gradually $\mathcal{I}-$divergence of (x_k) implies gradually $\mathcal{I}^*-$divergence if $\mathcal{I}$ satisfies the condition (AP).*

Proof. By hypothesis, for every $\xi \in (0, 1]$, there exists a $x \in X$ such that for any $B > 0$, $C(\xi, B) = \{k \in \mathbb{N} : A_{||x_k-x||_G}(\xi) \leq B\} \in \mathcal{I}$. This enables us to construct a countable family of mutually disjoint sets $\{C_m(\xi)\}_{m\in\mathbb{N}}$ in $\mathcal{I}$ by considering

$$C_1(\xi) = \{k \in \mathbb{N} : A_{||x_k-x||_G}(\xi) \leq 1\}$$

and

$$C_m(\xi) = \{k \in \mathbb{N} : m - 1 < A_{||x_k-x||_G}(\xi) \leq m\} = C(\xi, m) \setminus C(\xi, m - 1), \text{ for } m \geq 2.$$

Now since $\mathcal{I}$ satisfies the condition (AP), so for the above countable collection $\{C_m(\xi)\}_{m\in\mathbb{N}}$, there exists another countable family of subsets $\{B_m(\xi)\}_{m\in\mathbb{N}}$ of $\mathbb{N}$ satisfying

$$C_r(\xi) \triangle B_r(\xi) \text{ is finite } \forall r \in \mathbb{N} \text{ and } B(\xi) = \bigcup_{r=1}^{\infty} B_r(\xi) \in \mathcal{I}. \tag{5.3}$$

Let $B > 0$ be arbitrary. Now choose $m \in \mathbb{N}$ such that $B < m+1$. Then, the following inclusion holds

$$\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \leq B\} \subseteq \bigcup_{r=1}^{m+1} C_r(\xi) \in \mathcal{I}.$$

Using (5.3) we can say that there exists an integer $k_0 \in \mathbb{N}$, such that

$$\bigcup_{r=1}^{m+1} B_r(\xi) \cap (k_0, \infty) = \bigcup_{r=1}^{m+1} C_r(\xi) \cap (k_0, \infty).$$

Choose $k \in \mathbb{N} \setminus B(\xi) \in \mathcal{F}(\mathcal{I})$ such that $k > k_0$. Then we must have $k \notin \bigcup_{r=1}^{m+1} B_r(\xi)$ and eventually $k \notin \bigcup_{r=1}^{m+1} C_r(\xi)$. But then $A_{||x_k - x||_G}(\xi) > B$ for all $k > k_0$ in $M(= \mathbb{N} \setminus B(\xi))$, which means that (x_k) is gradually $\mathcal{I}^*-$divergent. This completes the proof. □

6. Conclusion

In this paper, we firstly investigated the notion of gradual $\mathcal{I}$ limit and gradual $\mathcal{I}$ cluster points and proved Theorem 3.2 to describe the relationship between the notions. Then, we moved our attention to $\mathcal{I}$ and $\mathcal{I}^*-$Cauchy sequences in the gradual normed linear spaces and established Theorem 4.2 which revealed the significance of the condition (AP) under which an $\mathcal{I}-$Cauchy sequence becomes $\mathcal{I}^*-$Cauchy. At the end, we have investigated the concept of $\mathcal{I}$ and $\mathcal{I}^*-$divergence and established Theorem 5.1 and Theorem 5.2 to represent their interrelationship.

Summability theory and the convergence of sequences have wide applications in various branches of mathematics particularly, in mathematical analysis. The obtained results may be useful for future researchers to explore various notions of convergences in the gradual normed linear spaces in more detail.

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References

[1] F. Aiche, D. Dubois, Possibility and gradual number approaches to ranking methods for random fuzzy intervals, *Commun. Comput. Inf. Sci.*, **299** (2012), 9–18.

[2] C. Choudhury, S. Debnath, On $\mathcal{I}-$convergence of sequences in gradual normed linear spaces, *Facta Univ. Ser. Math. Inform.*, **36**(3) (2021), 595–604.

[3] P. Das, S. K. Ghosal, Some further results on $I-$Cauchy sequences and condition (AP), *Comput. Math. Appl.*, **59**(8) (2010), 2597–2600.

[4] K. Dems, On $\mathcal{I}-$Cauchy sequences, *Real Anal. Exchange*, **30**(1) (2004-2005), 123–128.

[5] D. Dubois, H. Prade, Gradual elements in a fuzzy set, *Soft Comput.*, **12**(2) (2007), 165–175.

[6] A. Esi, Strongly almost summable sequence spaces in $2-$normed spaces defined by ideal convergence and an Orlicz function, *Stud. Univ. Babes-Bolyai Math.*, **57**(1) (2012), 75–82.

[7] A. Esi, Strongly summable double sequence spaces in $n-$normed spaces defined by ideal convergence and an Orlicz function, *Kyungpook Math. J.*, **52**(2) (2012), 137–147.

[8] A. Esi, M. K. Ozdemir, $\Lambda-$strongly summable sequence spaces in $n-$normed spaces defined by ideal convergence and an Orlicz function, *Math. Slovaca*, **63**(4) (2013), 829–838.

[9] M. Ettefagh, F. Y. Azari, S. Etemad, On some topological properties in gradual normed spaces, *Facta Univ. Ser. Math.*, **35**(3) (2020), 549–559.

[10] M. Ettefagh, S. Etemad, F. Y. Azari, Some properties of sequences in gradual normed spaces, *Asian-Eur. J. Math.*, **13**(4) (2020), 2050085.

[11] J. Fortin, D. Dubois, H. Fargier, Gradual numbers and their application to fuzzy interval analysis, *IEEE Trans. Fuzzy Syst.*, **16**(2) (2008), 388–402.

[12] J. Gogola, M. Macaj, T. Visnyai, On $I_c^{(q)}-$convergence, *Ann. Math. Inform.*, **38** (2011), 27–36.

[13] B. Hazarika, A. Esi, On asymptotically Wijsman lacunary statistical convergence of set sequences in ideal context, *Filomat*, **31**(9) (2017), 2691–2703.

[14] B. Hazarika, A. Esi, Lacunary ideal summability and its applications to approximation theorem, *J. Anal.*, **27** (2019), 997–1006.

[15] P. Kostyrko, M. Macaj, T. Salat, M. Sleziak, I-convergence and extremal I-limit points, *Math. Slovaca*, **55**(4) (2005), 443–464.

[16] P. Kostyrko, T. Salat, W. Wilczynski, I-convergence, *Real Anal. Exch.*, **26**(2) (2000-2001), 669–686.

[17] M. Kucukaslan, M. Altinok, Ideal limit superior-inferior, *Gazi Univ. J. Sci.*, **30**(1) (2017), 401–411.

[18] L. Lietard, D. Rocacher, Conditions with aggregates evaluated using gradual numbers, *Control Cybernet.*, **38**(2) (2009), 395–417.

[19] M. Mursaleen, A. Alotaibi, On $\mathcal{I}-$convergence in random 2-normed spaces, *Math. Slovaca* ,**61**(6) (2011), 933–940.

[20] M. Mursaleen, S. Debnath, D. Rakshit, $I-$statistical limit superior and $I-$statistical limit inferior, *Filomat*, **31**(7) (2017), 2103–2108.

[21] M. Mursaleen, S. A. Mohiuddine, On ideal convergence of double sequences in probabilistic normed spaces, *Math. Rep. (Bucur.)*, **12(62)**(4) (2010), 359–371.

[22] M. Mursaleen, S. A. Mohiuddine, On ideal convergence in probabilistic normed spaces, *Math. Slovaca*, **62**(1) (2012), 49–62.

[23] M. Mursaleen, S. A. Mohiuddine, O. H. H. Edely, On ideal convergence of double sequences in intuitionistic fuzzy normed spaces, *Comput. Math. Appl.*, **59**(2) (2010), 603–611.

[24] A. Nabiev, S. Pehlivan, M. Gurdal, On $I-$Cauchy sequences, *Taiwanese J. Math.*, **11**(2) (2007), 569–576.

[25] F. Nuray, H. Gok, U. Ulusu, $I_\sigma-$convergence, *Math. Commun.*, **16**(2) (2011), 531–538.

[26] I. Sadeqi, F. Y. Azari, Gradual normed linear space, *Iran. J. Fuzzy Syst.*, **8**(5) (2011), 131–139.

[27] T. Salat, B. C. Tripathy, M. Ziman, On some properties of $I-$convergence, *Tatra Mt. Math. Publ.*, **28**(2) (2004), 274–286.

[28] E. Savas, Generalized asymptotically $\mathcal{I}-$lacunary statistical equivalent of order α for sequences of sets, *Filomat*, **31**(6) (2017), 1507–1514.

[29] E. Savas, M. Gurdal, Ideal convergent function sequences in random 2-normed spaces, *Filomat*, **30**(3) (2016), 557–567.

[30] H. Sengul, M. Et, On $I-$lacunary statistical convergence of order α of sequences of sets, *Filomat*, **31**(8) (2017), 2403–2412.

[31] E. A. Stock, Gradual numbers and fuzzy optimization, Ph. D. Thesis, University of Colorado Denver, Denver, America, 2010.

[32] B. C. Tripathy, B. Hazarika, Paranorm I-convergent sequence spaces, *Math. Slovaca*, **59**(4) (2009), 485—494.

[33] L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8**(3) (1965), 338–353.

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