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EXISTENCE AND UNIQUENESS OF WEAK SOLUTION FOR NONLINEAR WEIGHTED (p,q)-LAPLACIAN SYSTEM WITH APPLICATION ON AN OPTIMAL CONTROL PROBLEM

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Abstract. In this paper, we prove the existence and uniqueness results of weak solution for weighted (p,q)-Laplacian system with Dirichlet boundary condition. The proof of the results is made by Browder theorem method. Also, the optimal control of the weighted (p,q)-Laplacian system will be study as an application.

1. Introduction

Existence results of weak solutions for nonlinear systems involving weighted (p,q)-Laplacian operators have been studied using the sub-supersolutions method (see [1, 4, 9, 10, 11, 14]), the theory of nonlinear monotone operators method (see [18, 19]), the Browder theorem method (see [3]) and the mountain pass theorem with the saddle point theorem (see [20]).

In [7], under some certain conditions on $a(x)$ and $f(x,u)$, the author proved the existence and uniqueness of weak solution for the following weighted p-Laplacian boundary value problem

$$\left\{\begin{array}{ll} -\Delta_{P,p}u+\lambda a(x)|u|^{p-2}u=f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.1}$$

using the method of the Browder theorem. Existence and uniqueness of weak solutions for nonlinear systems involving p-Laplacian or weighted p -Laplacian operators using the Browder theorem method have been the subject of much recent interest (we refer only to [2, 8]).

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Here, by the Browder theorem method, we are concerned with the existence and uniqueness of weak solution for a quasilinear weighted (p,q)-Laplacian system

$$(1.2)\qquad \begin{cases} -\Delta_{P,p}u + \lambda a(x)|u|^{p-2}u = f(x,u,v) & \text{in } \Omega, \\ -\Delta_{Q,q}v + \lambda b(x)|v|^{q-2}v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{P,p}$ with $p > 2$ and $P = P(x)$ is a weighted function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[|P(x)\nabla u|^{p-2}\nabla u]$, λ is a positive parameter, $0 < \alpha \le a(x) \le \beta < +\infty$, $0 < \gamma \le b(x) \le \delta < +\infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Let $f : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a caratheodory (CAR) function which is decreasing with respect to the second variable and $g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be also a caratheodory (CAR) function which is decreasing with respect to the third variable, $i.e.$,

$$(1.3)\qquad \begin{cases} f(x,s_1,t) \le f(x,s_2,t) & \text{for } a.e.\ x \in \Omega \quad \text{and } s_1, s_2 \in \mathbb{R}, \quad s_1 \ge s_2, \\ g(x,s,t_1) \le g(x,s,t_2) & \text{for } a.e.\ x \in \Omega \quad \text{and } t_1, t_2 \in \mathbb{R}, \quad t_1 \ge t_2. \end{cases}$$

Assume, moreover, that there exists $f_0 \in L^{p'}(\Omega)$, $g_0 \in L^{q'}(\Omega)$, $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$ and $c_1 > 0$, $c_2 > 0$ such that

$$(1.4)\qquad \begin{cases} |f(x,s,t)| \le c_1[f_0(x) + |s|^{p-1} + |t|^{q/p'}] \\ |g(x,s,t)| \le c_2[g_0(x) + |s|^{p/q'} + |t|^{q-1}]. \end{cases}$$

Also, the optimal control of the weighted (p,q)-Laplacian system will be studied as an application. In the present work, besides the Browder theorem method as in the most of the papers, we consider the case of a system and, instead of the Laplacian, we work with the weighted $p-$Laplacian.

Our paper is organized as follows: section 2 contains some basic definitions concerning the nonlinear operators that will be used throughout the paper. Also, the space setting of the problem and some basic characteristics, as the equivalent norm and imbedding results, will be introduced. In section 3, we state the main results of existence and uniqueness of weak solutions of the problem (1.2). Section 4 is devoted for the study of optimal control as an application.

2. Preliminaries and Space Setting

First, the aim of this section is to recall some basic definitions concerning the operators which we use extensively in this paper (see [6]).

Definition 2.1. Let $A : V \to V^*$ be an operator on a real Banach space V, where V^* is the dual space of V. We say that the operator A is:

(i) bounded iff it maps bounded sets into bounded sets i.e. for each $r > 0$ there exists $M > 0$ (M depending on r) such that

$$\|u\| \leq r \implies \|A(u)\| \leq M, \qquad \forall u \in V;$$

(ii) *coercive*: iff $\lim\limits_{\|u\|\to\infty} \frac{\langle A(u),u\rangle}{\|u\|} = \infty$;

(iii) *monotone* iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0$ *for all* $u_1, u_2 \in V$;

(iv) *strictly monotone* iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0$ *for all* $u_1, u_2 \in V$, $u_1 \neq u_2$;

(v) *strongly monotone* iff $\langle A(u_1)-A(u_2), u_1-u_2\rangle > k\|u_1-u_2\|$ *for all* $u_1, u_2 \in V$, $u_1 \neq u_2$;

(vi) *continuous* iff $u_n \longrightarrow u$ implies $A(u_n) \longrightarrow A(u)$, *for all* $u_n, u \in V$;

(vii) *strongly continuous* iff $u_n \xrightarrow{w} u$ implies $A(u_n) \longrightarrow A(u)$, *for all* $u_n, u \in V$;

(viii) *demicontinuous* iff $u_n \longrightarrow u$ implies $A(u_n) \xrightarrow{w} A(u)$, *for all* $u_n, u \in V$;

Consequently, the following remarks are satisfied:

Remark 1. *Every strongly continuous operator is continuous which is demicontinuous operator.*

Remark 2. *Every strongly continuous operator is bounded operator.*

Remark 3. *Every strictly monotone operator is monotone operator.*

Theorem 2.1. (***Browder Theorem*** [13]) *Let A be a reflexive real Banach space. Moreover let $A : V \to V^*$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space V. Then, the equation $A(u) = f$ has at least one solution $u \in V$ for each $f \in V^*$. If moreover, A is strictly monotone operator, then the equation $A(u) = f$ has precisely one solution $u \in V$ for every $f \in V^*$.*

Now, we recall the precise definition of the weighted Sobolev space $W^{1,p}(P,\Omega)$ which is the set of all real valued functions u defined in Ω for which

$$\|u\|_{W^{1,p}(P,\Omega)} = \left(\int_\Omega |u|^p + \int_\Omega P(x)|\nabla u|^p \right)^{1/p} < \infty, \tag{2.1}$$

where $P(x)$ is a weight function, *i.e.*, a measurable function which is strictly positive *a.e.* in Ω, satisfying the conditions

$$P(x) \in L^1_{Loc}(\Omega),\ (P(x))^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega),\ \text{with } p > 1, \tag{2.2}$$

and

$$(P(x))^{-s} \in L^1(\Omega), \quad \text{with } \ s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty). \tag{2.3}$$

Since we are dealing with the Dirichlet problem, we introduce the space $W_0^{1,p}(P,\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(P,\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(P,\Omega)} = \left(\int_\Omega P(x)|\nabla u|^p \right)^{1/p} < \infty, \tag{2.4}$$

which is equivalent to the norm given by (2.1). Due to (2.2), both spaces $W^{1,p}(P,\Omega)$ and $W_0^{1,p}(P,\Omega)$ are well defined reflexive Banach Spaces. Also, the space $W_0^{1,p}(P,\Omega)$ is compactly imbedded into the space $L^p(\Omega)$, i.e.

$$W_0^{1,p}(P,\Omega) \hookrightarrow L^p(\Omega). \tag{2.5}$$

For a discussion about the space setting we refer the reader to [5] and the references therein.

The space setting of the problem under study is the Banach space $W = W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)$ and the norm of $z = (u,v) \in W$ is given by $\|z\|_W = \|u\|_{W_0^{1,p}(P,\Omega)} + \|v\|_{W_0^{1,q}(Q,\Omega)}$, where $\|u\|_{W_0^{1,p}(P,\Omega)} = (\int_\Omega P(x)|\nabla u|^p)^{1/p}$ and $\|v\|_{W_0^{1,q}(\Omega)} = (\int_\Omega Q(x)|\nabla v|^q)^{1/q}$. By the continuity of the embedding $W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega) \hookrightarrow L^p(\Omega)\times L^q(\Omega)$, there exists positive constants c_{pemb}, c_{qemb} such that

$$\|u\|_{L^p(\Omega)} \le c_{pemb}\|u\|_{W_0^{1,p}(P,\Omega)},\ \|v\|_{L^q(\Omega)} \le c_{qemb}\|v\|_{W_0^{1,q}(Q,\Omega)} \text{ for all } (u,v) \in W. \tag{2.6}$$

where c_{pemb} is the constant of the embedding of $W_0^{1,p}(P,\Omega)$ into $L^p(\Omega)$ and c_{qemb} is the constant of the embedding of $W_0^{1,q}(Q,\Omega)$ into $L^q(\Omega)$.

Throughout this paper we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between W and W^*.

3. Existence and Uniqueness Results

In this section, using the method of Browder theorem, the existence and uniqueness of weak solution for equation (1.2) will be proved.

Definition 3.1. we say that $(u, v) \in W = W_0^{1,p}(P, \Omega) \times W_0^{1,q}(Q, \Omega)$ is a weak solution of system (1.2) if

$$\begin{aligned} \int_\Omega P(x)|\nabla u|^{p-2}\nabla u \ \nabla\phi + \lambda \int_\Omega a(x)|u|^{p-2}u\phi &= \int_\Omega f(x,u,v)\phi \text{ for all } \phi \in W_0^{1,p}(P,\Omega), \\ \int_\Omega Q(x)|\nabla v|^{q-2}\nabla v \ \nabla\psi + \lambda \int_\Omega b(x)|v|^{q-2}v\psi &= \int_\Omega g(x,u,v)\psi \text{ for all } \psi \in W_0^{1,q}(Q,\Omega). \end{aligned}$$

The main result concerning problem (1.2) is the following theorem:

Theorem 3.1. *Let $p, q \geq 2$, $\lambda > 0$ and $f, g \in CAR(\Omega \times \mathbb{R} \times \mathbb{R})$ satisfy (1.3) and (1.4). Then problem (1.2) has a unique weak solution.*

Proof. We define for $\lambda > 0$, the operator $T : \Omega \times W \to W^*$ as

$$T(u, v) := A(u, v) + \lambda B(u, v) - C(u, v),$$

where the operators $A, B : W \to W^*$and $C(x, u, v) : \Omega \times W \to W^*$ are given by

$$\langle A(u,v), (\phi,\psi)\rangle = \int_\Omega P(x)|\nabla u|^{p-2}\nabla u\nabla\phi + \int_\Omega Q(x)|\nabla v|^{q-2}\nabla v\nabla\psi,$$

$$\langle B(u,v), (\phi,\psi)\rangle = \int_\Omega a(x)|u|^{p-2}u\phi + \int_\Omega b(x)|v|^{q-2}v\psi,$$

and

$$\langle C(x,u,v), (\phi,\psi)\rangle = \int_\Omega f(x,u,v)\phi + \int_\Omega g(x,u,v)\psi,$$

for all $(u, v) \in W$.

Now, $(u, v) \in W$ is a weak solution of (1.2) if

$$\langle T(u,v), (\phi,\psi)\rangle = \langle A(u,v), (\phi,\psi)\rangle + \lambda \langle B(u,v), (\phi,\psi)\rangle - \langle C(x,u,v), (\phi,\psi)\rangle = 0,$$

holds for any $(\phi, \psi) \in W$. Thus, to find a weak solution of (1.2) is equivalent to finding $(u, v) \in W$ which satisfies the operator equation $T(u, v) = 0$.

Now, we focus on the operators A, B, and C :

a) A, B and C are well defined. The operator $A(u,v)$ can be written as the sum of the two operators $A_1(u,v)$ and $A_2(u,v)$ where

$$\langle A_1(u,v),(\phi,\psi)\rangle = \int_\Omega P(x)|\nabla u|^{p-2}\nabla u\ \nabla\phi$$

and

$$\langle A_2(u,v),(\phi,\psi)\rangle = \int_\Omega Q(x)|\nabla v|^{q-2}\nabla v\ \nabla\psi.$$

Using Hölder's inequality, it is easy to prove that both A_1 and A_2 are well defined, so their sum, the operator A will be the same. Similarly, the operator $B(u,v)$ may be written as the sum of the two operators $B_1(u,v)$ and $B_2(u,v)$ where

$$\langle B_1(u,v),(\phi,\psi)\rangle = \int_\Omega a(x)|u|^{p-2}u\phi \quad \text{and} \quad \langle B_2(u,v),(\phi,\psi)\rangle = \int_\Omega b(x)|v|^{q-2}v\psi.$$

It is easy to prove that both B_1 and B_2 are well defined, so their sum, the operator B will be the same.

Also, the operator $C(x,u,v)$ can be written as the sum of $C_1(x,u,v)$ and $C_2(x,u,v)$ where

$$\langle C_1(x,u,v),(\phi,\psi)\rangle = \int_\Omega f(x,u,v)\phi \quad \text{and} \quad \langle C_2(x,u,v),(\phi,\psi)\rangle = \int_\Omega g(x,u,v)\psi.$$

For the operator C_1, using Hölder's inequality, we can easily see that

$$\begin{aligned}
|\langle C_1(x,u,v),(\phi,\psi)\rangle| &\leq c_1\left(\int_\Omega (f_0(x)+|u|^{p-1}+|v|^{q/p'})\,|\phi|\right)\\
&\leq c_1\left(\int_\Omega |f_0(x)|^{p'}\right)^{1/p'}\left(\int_\Omega |\phi|^p\right)^{1/p}\\
&\quad +c_1\left(\int_\Omega |u|^p\right)^{1/p'}\left(\int_\Omega |\phi|^p\right)^{1/p}+\left(\int_\Omega |v|^q\right)^{1/p'}\left(\int_\Omega |\phi|^p\right)^{1/p}\\
&= c_1\left[||f_0||_{L^{p'}(\Omega)}+||u||^{p/p'}_{L^p(\Omega)}+||v||^{q/p'}_{L^q(\Omega)}\right]||\phi||_{L^p(\Omega)}<\infty.
\end{aligned}$$

Similarly for the operator C_2,and hence C is well defined.

b) A, B and C are bounded operators. Indeed, for every u, v such that $||u||_{W_0^{1,p}(P,\Omega)} \leq M$ and $||v||_{W_0^{1,q}(Q,\Omega)} \leq N$, one obtains

$$||A_1(u,v)||_{W^*} = \sup_{||\phi||_{W_0^{1,p}(\Omega)}\leq 1} |\langle A_1(u), \phi\rangle| \leq \sup_{||\phi||_{W_0^{1,p}(\Omega)}\leq 1} \int_\Omega P(x)|\nabla u|^{p-2}\nabla u\ \nabla\phi.$$

Using Hölder's inequality, A_1 becomes

$$||A_1(u,v)||_{W^*} = \sup_{||\phi||_{W_0^{1,p}(\Omega)}\leq 1} \left(\int_\Omega P(x)|\nabla u|^p\right)^{1/p'} \left(\int_\Omega P(x)|\nabla \phi|^p\right)^{1/p} \leq M^{p/p'}.$$

Similarly,

$$||A_2(u,v)||_{W^*} = \sup_{||\psi||_{W_0^{1,q}(\Omega)}\leq 1} \left(\int_\Omega Q(x)|\nabla v|^q\right)^{1/q'} \left(\int_\Omega Q(x)|\nabla \psi|^q\right)^{1/q} \leq N^{q/q'}.$$

Hence A is bounded. Also,

$$\begin{aligned} ||B_1(u,v)||_{W^*} &= \sup_{||\phi||_{W_0^{1,p}(\Omega)}\leq 1} |\langle B_1(u), \phi\rangle| \\ &\leq \beta \sup_{||\phi||_{W_0^{1,p}(P,\Omega)}\leq 1} \left(\int_\Omega |u|^p\right)^{1/p'} \left(\int_\Omega |\phi|^p\right)^{1/p} \leq \beta c_{pemb}^{1+(p/p')}\ M^{p/p'}. \end{aligned}$$

Similarly,

$$||B_2(u,v)||_{W^*} \leq \delta \sup_{||\psi||_{W_0^{1,q}(Q,\Omega)}\leq 1} \left(\int_\Omega |v|^q\right)^{1/q'} \left(\int_\Omega |\psi|^p\right)^{1/q} \leq \delta c_{qemb}^{1+(q/q')}\ N^{q/q'}.$$

Then B is bounded.

Finally, for the operator $C_1(x,u,v)$, a simple calculation shows that

$$\begin{aligned} ||C_1(x,u,v)||_{W^*} &= \sup_{||\phi||_{W_0^{1,p}(P,\Omega)}\leq 1} |\langle C_1(x,u,v), \phi\rangle| \\ &\leq c_1 \sup_{||\phi||_{W_0^{1,p}(P,\Omega)}\leq 1} \int_\Omega (f_0(x) + |u|^{p-1} + |v|^{q/p'})\,|\phi| \\ &\leq c_1 \sup_{||\phi||_{W_0^{1,p}(P,\Omega)}\leq 1} \left[||f_0||_{L^{p'}(\Omega)} + ||u||^{p/p'}_{L^p(\Omega)} + ||v||^{q/p'}_{L^q(\Omega)}\right] ||\phi||_{L^p(\Omega)} \\ &\leq c_1 c_{pemb}(\ ||f_0||_{L^{p'}(\Omega)} + c_{pemb}^{p/p'}\,||u||^{p/p'}_{W_0^{1,p}(P,\Omega)} + c_{qemb}^{q/p'}\,||v||^{q/p'}_{W_0^{1,q}(Q,\Omega)}) \\ &\leq c_1 c_{pemb}(\ ||f_0||_{L^{p'}(\Omega)} + c_{pemb}^{p/p'} M^{p/p'} + c_{qemb}^{q/q'}\ N^{q/p'}). \end{aligned}$$

By routine calculations for the operator $C_2(x,u,v)$, one has

$$\begin{aligned} ||C_2(x,u,v)||_{W^*} &= \sup_{||\psi||_{W_0^{1,q}(Q,\Omega)}\leq 1} |\langle C_2(x,u,v),\psi\rangle| \leq \\ &\leq c_2 c_{qemb}(\ ||g_0||_{L^{q'}(\Omega)} + c_{qemb}^{q/q'} N^{q/q'} + c_{pemb}^{p/q'}\ M^{p/q'}). \end{aligned}$$

Hence $C(x,u,v)$ is bounded.

c) A, B and C are continuous operators. If $u_n \longrightarrow u$ in $W_0^{1,p}(P,\Omega), v_n \longrightarrow v$ in $W_0^{1,q}(Q,\Omega)$. Then, we have $\|u_n - u\|_{W_0^{1,p}(P,\Omega)} \longrightarrow 0, \|v_n - v\|_{W_0^{1,q}(Q,\Omega)} \longrightarrow 0$ so that $\|\nabla u_n - \nabla u\|_{L^p(\Omega)} \longrightarrow 0, \|\nabla v_n - \nabla v\|_{L^q(\Omega)} \longrightarrow 0.$

Applying Dominated Convergence Theorem, one obtains

$$\left\|(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\right\|_{L^p(\Omega)} \longrightarrow 0, \quad \left\|(|\nabla v_n|^{q-2}\nabla v_n - |\nabla v|^{q-2}\nabla v)\right\|_{L^q(\Omega)} \longrightarrow 0.$$

Now, for A_1, we deduce that

$$\begin{aligned} \|A_1(u_n) - A_1(u)\|_{W^*} &= \sup_{||\phi||_{W_0^{1,p}(P,\Omega)}\leq 1} |\langle A_1(u_n) - A_1(u),\phi\rangle| \\ &\leq \sup_{||v||_{W_0^{1,p}(P,\Omega)}\leq 1} \left(\int_\Omega P(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u]^{p'}\right)^{1/p'} \times \\ &\quad \left(\int_\Omega P(x)|\phi|^p\right)^{1/p} \to 0 \ \text{ for } n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \|A_2(v_n) - A_2(v)\|_{W^*} &= \sup_{||\psi||_{W_0^{1,q}(Q,\Omega)}\leq 1} |\langle A_2(v_n) - A_2(v),\psi\rangle| \\ &\leq \sup_{||\psi||_{W_0^{1,q}(Q,\Omega)}\leq 1} \left(\int_\Omega Q(x)[|\nabla v_n|^{q-2}\nabla v_n - |\nabla v|^{q-2}\nabla v]^{q'}\right)^{1/q'} \times \\ &\quad \left(\int_\Omega Q(x)|\psi|^q\right)^{1/q} \to 0 \ \text{ for } n \to \infty. \end{aligned}$$

Hence A is continuous.

Also, for B_1, we can show that

$$\begin{aligned}\|B_1(u_n)-B_1(u)\|_{W^*} &= \sup_{||\phi||_{W_0^{1,p}(P,\Omega)}\leq 1} |\langle B_1(u_n)-B_1(u),\phi\rangle| \\ &\leq \beta c_{pemb}\left(\int_\Omega ([|u_n|^{p-2}u_n-|u|^{p-2}u])^{p'}\right)^{1/p'} \to 0 \text{ for } n\to\infty.\end{aligned}$$

Also,

$$\|B_2(v_n)-B_2(v)\|_{W^*}\leq \delta c_{qemb}\left(\int_\Omega ([|v_n|^{q-2}v_n-|v|^{q-2}v])^{q'}\right)^{1/q} \to 0 \text{ for } n\to\infty.$$

Hence B is continuous.

Finally, the continuity of C_1, C_2 follows from the continuity of the Nemytskij operator associated with f, g and acting from W into W.

d) Let $p\geq 2$, $\forall\, x_1,x_2\in\mathbb{R}^N$, we have the following inequality (see [17])

$$|x_2|^p\geq |x_1|^p+p\,|x_1|^{p-2}\,x_1(x_2-x_1)+\frac{|x_2-x_1|^p}{2^{p-1}-1}. \tag{3.1}$$

Now, using (3.1), one obtains

$$\begin{aligned}\langle A_1(u)-A_1(\phi),u-\phi\rangle &= \int_\Omega P(x)[|\nabla u|^{p-2}\nabla u-|\nabla\phi|^{p-2}\nabla\phi](\nabla u-\nabla\phi) \\ &= -\int_\Omega P(x)|\nabla u|^{p-2}\nabla u(\nabla\phi-\nabla u) \\ &\quad -\int_\Omega P(x)|\nabla\phi|^{p-2}\nabla\phi(\nabla u-\nabla\phi) \\ &\geq \frac{1}{p(2^{p-1}-1)}\int_\Omega P(x)\,|\nabla u-\nabla\phi|^p \\ &\quad +\frac{1}{p(2^{p-1}-1)}\int_\Omega P(x)\,|\nabla\phi-\nabla u|^p \\ &\geq \frac{2}{p(2^{p-1}-1)}\int_\Omega P(x)\,|\nabla u-\nabla\phi|^p \\ &= c(p)\,\|u-\phi\|^p_{W_0^{1,p}(P,\Omega)} \quad \text{for} \quad p\geq 2.\end{aligned}$$

Similarly,

$$\langle A_2(v) - A_2(\psi), v-\psi\rangle \geq \frac{2}{q(2^{q-1}-1)}\int_\Omega Q(x)\,|\nabla v - \nabla\psi|^q$$
$$= c(q)\,\|v-\psi\|^q_{W_0^{1,q}(Q,\Omega)} \quad \text{for} \quad q \geq 2.$$

Hence,

$$\langle A(u,v) - A(\phi,\psi), (u,v)-(\phi,\psi)\rangle \geq$$
$$c(p)\,\|u-\phi\|^p_{W_0^{1,p}(P,\Omega)} + c(q)\,\|v-\psi\|^q_{W_0^{1,q}(Q,\Omega)} \quad \text{for} \quad p,q \geq 2. \tag{3.2}$$

Similarly,

$$\begin{aligned}\langle B_1(u) - B_1(\phi), u-\phi\rangle &= \int_\Omega a(x)[|u|^{p-2}u - |\phi|^{p-2}\phi](u-\phi) \\ &\geq \frac{2}{p(2^{p-1}-1)}\int_\Omega a(x)\,|u-\phi|^p \geq \alpha c(p)\,\|u-\phi\|^p_{L^p(\Omega)} \geq 0.\end{aligned}$$

Also,

$$\langle B_2(v) - B_2(\psi), v-\psi\rangle \geq \gamma c(q)\,\|v-\psi\|^q_{L^q(\Omega)} \geq 0.$$

Hence,

$$\langle B(u,v) - B(\phi,\psi), (u,v)-(\phi,\psi)\rangle \geq 0. \tag{3.3}$$

Similarly, we get

$$\langle F(u) - F(\phi), u-\phi\rangle = \int_\Omega [f(x,u,v) - f(x,\phi,v)](u-\phi).$$

Since f is decreasing with respect to u, then

$$[f(x,u,v) - f(x,\phi,v)](u-\phi) \leq 0,$$

consequently,

$$\langle C_1(u) - C_1(\phi), u-\phi\rangle = \int_\Omega [f(x,u,v) - f(x,\phi,v)](u-\phi) \leq 0,$$

and

$$\langle C_2(v) - C_2(\psi), v-\psi\rangle = \int_\Omega [g(x,u,v) - g(x,u,\psi)](v-\psi) \leq 0.$$

Hence,

$$\langle C(u,v) - C(\phi,\psi), (u,v)-(\phi,\psi)\rangle \leq 0. \tag{3.4}$$

Equations (3.2), (3.3) and (3.4) imply that

$$\langle T(u,v)-T(\phi,\psi),(u,v)-(\phi,\psi)\rangle \geq c(p)\,\|u-\phi\|^p_{W_0^{1,p}(P,\Omega)}+c(q)\,\|v-\psi\|^q_{W_0^{1,q}(Q,\Omega)}.$$

Hence,

$$\langle T(u,v)-T(\phi,\psi),(u,v)-(\phi,\psi)\rangle \geq c_{\min}[\|u-\phi\|^p_{W_0^{1,p}(P,\Omega)}+\|v-\psi\|^q_{W_0^{1,q}(Q,\Omega)}], \tag{3.5}$$

for $p,q\geq 2$, where $c_{\min}=\min\{c(p),c(q)\}$.

Now, to apply Browder Theorem, it remains to prove that T is a coercive operator. From (3.5), we find

$$\langle T(u,v)-T(0,0),(u,v)\rangle \geq \langle T(0,0),u-v\rangle + c_{\min}[\|u\|^p_{W_0^{1,p}(P,\Omega)}+\|v\|^q_{W_0^{1,q}(Q,\Omega)}] \text{ for } p,q\geq 2.$$

On the other hand

$$\begin{aligned}
\langle T(0,0),u-v\rangle &= \langle A(0,0),u-v\rangle+\lambda\,\langle B(0,0),u-v\rangle-\langle C(0,0),u-v\rangle\\
&= -\int_\Omega f(x,0,0)u-\int_\Omega g(x,0,0)v \geq -c_1\int_\Omega f_0(x)u-c_2\int_\Omega g_0(x)v\\
&\geq -c_1(\int_\Omega [f_0(x)]^{p'})^{1/p'}(\int_\Omega |u|^p)^{1/p}-c_2(\int_\Omega [g_0(x)]^{q'})^{1/q'}(\int_\Omega |v|^q)^{1/q}\\
&\geq -c_1c_{pemb}\,||f_0||_{L^{p'}(\Omega)}\,||u||_{W_0^{1,p}(P,\Omega)}-c_{qemb}\,||g_0||_{L^{q'}(\Omega)}\,||v||_{W_0^{1,q}(Q,\Omega)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\langle T(u,v),(u,v)\rangle \geq & c_{\min}[\|u\|^p_{W_0^{1,p}(P,\Omega)}+\|v\|^q_{W_0^{1,q}(Q,\Omega)}]-c_{pemb}\,||f_0||_{L^{p'}(\Omega)}\,||u||_{W_0^{1,p}(P,\Omega)}\\
&-c_{qemb}\,||g_0||_{L^{q'}(\Omega)}\,||v||_{W_0^{1,q}(Q\Omega)}.
\end{aligned}$$

Therefore,

$$\lim\frac{\langle T(u,v),(u,v)\rangle}{||(u,v)||_W}\geq c_{\min}\ \lim\frac{[\|u\|^p_{W_0^{1,p}(P,\Omega)}+\|v\|^q_{W_0^{1,q}(Q,\Omega)}]}{\|u\|_{W_0^{1,p}(P,\Omega)}+\|v\|_{W_0^{1,q}(Q,\Omega)}}=\infty\quad\text{as}\quad ||(u,v)||_W\to\infty$$

This proves the coercivity condition and so, the existence of weak solution for (1.2).

The uniqueness of weak solution of (1.2), is a direct consequence of (3.5). Suppose that $(u_1,v_1),(u_2,v_2)$ be a weak solutions of (1.2) such that $(u_1,v_1)\neq(u_2,v_2)$.

Consequently, (3.5) becomes

$$\begin{aligned}
0 &= \langle T(u_1,v_1)-T(u_2,v_2),(u_1,v_1)-(u_2,v_2)\rangle\\
&\geq c_{\min}[\|u_1-u_2\|^p_{W_0^{1,p}(P,\Omega)}+\|v_1-v_2\|^q_{W_0^{1,q}(Q,\Omega)}]\geq 0\quad\text{for}\quad p,q\geq 2.
\end{aligned}$$

therefore $(u_1, v_1) = (u_2, v_2)$. This completes the proof. □

4. Optimal Control

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. Control problems include a cost (energy) functional that is a function of state and control variables. An optimal control is a set of partial differential equations describing the paths of the control variables that minimize the cost or energy functional. So, the optimal control problem requires the following initial data:

- Space of control U (Hilbert or Banach space)
- For a given control ξ, the state $y(\xi)$ of the system is given by the solution of an operator equation $Ay(\xi) =$ given function of control ξ. The operator A is called the model of the system.
- The observation $z(\xi) =$ given function of $y(\xi)$.
- The cost function $J(\varsigma) =$ given function of $z(\xi)$.
- The control problem is then to find $\inf J(\varsigma)$ over a closed convex subset U_{ad} of U.

The model A of the weighted (p, q)-Laplacian system under study (1.2) is given by

$$Ay = A\{y_1, y_2\} = \{-\Delta_{P,p} y_1 + \lambda a(x)|y_1|^{p-2} y_1, -\Delta_{Q,q} y_2 + \lambda b(x)|y_2|^{q-2} y_2\}.$$

Using the theory of Lions [15] , we can formulate the control problem as the following:

The Banach space $U = W_0^{1,p}(P, \Omega) \times W_0^{1,q}(Q, \Omega)$ being the space of controls. For a control $\xi(\xi_1, \xi_2)$, the state $y(\xi) = \{y_1(\xi), y_2(\xi)\}$ of the system, from (1.2), is given by

$$\left\{\begin{array}{ll} -\Delta_{P,p} y_1(\xi) + \lambda a(x)|y_1(\xi)|^{p-2} y_1(\xi) = f + \xi_1 & \text{in } \Omega, \\ -\Delta_{Q,q} y_2(\xi) + \lambda b(x)|y_2(\xi)|^{q-2} y_2(\xi) = g + \xi_2 & \text{in } \Omega, \\ y_1(\xi) = y_2(\xi) = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.1}$$

The non–quadratic cost functional is given by

$$J(\varsigma) = \int_\Omega P(x)\,|y_1(\varsigma) - z_{d1}|^p + \int_\Omega Q(x)\,|y_2(\varsigma) - z_{d2}|^q + M(\|\varsigma_1\|_p^p + \|\varsigma_2\|_q^q), \tag{4.2}$$

where $Z_d = \{z_{d1}, z_{d2}\} \in W_0^{1,p}(P, \Omega) \times W_0^{1,q}(Q, \Omega)$ and M is a positive constant.

The control problem then is to find inf $J(\varsigma)$ over a closed convex subset U_{ad} of $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$. Since the cost function is given by (4.2), there exists a unique optimal control $\xi = \{\xi_1, \xi_2\} \in U_{ad}$ which is characterized by (see [12, 15, 16])

$$J^{'}(\xi)(\varsigma - \xi) \geq 0 \qquad \text{for all } \varsigma \in U_{ad}. \tag{4.3}$$

From (4.2), we have for $w = (w_1, w_2) \in W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$,

$$\begin{aligned} J^{'}(\xi)\cdot w &= \frac{d}{d\zeta}J(\xi+\zeta w)\Big|_{\xi=0} \\ &= p\int_\Omega P(x)\left|y_1(\xi)-z_{d1}\right|^{p-2}(y_1(\xi)-z_{d1})\frac{\partial}{\partial\xi}y_1(\xi)\cdot w_1 \\ &+pM\int_\Omega \left|\xi_1\right|^{p-2}\xi_1\cdot w_1 \\ &+q\int_\Omega Q(x)\left|y_2(\xi)-z_{d2}\right|^{q-2}(y_2(\xi)-z_{d2})\frac{\partial}{\partial\xi}y_2(\xi)\cdot w_2 \\ &+qM\int_\Omega \left|\xi_2\right|^{q-2}\xi_2\cdot w_2. \end{aligned} \tag{4.4}$$

But, if we set $\frac{\partial}{\partial\xi}y_i(\xi)\cdot w_i = \psi_i(w_i)$, $i = 1, 2$, then (4.4) is equivalent to

$$\begin{aligned} 0 \leq p\int_\Omega P(x)\left|y_1(\xi)-z_{d1}\right|^{p-2}(y_1(\xi)-z_{d1})\psi_1(w_1) + pM\int_\Omega \left|\xi_1\right|^{p-2}\xi_1\cdot w_1 \\ +q\int_\Omega Q(x)\left|y_2(u)-z_{d2}\right|^{q-2}(y_2(u)-z_{d2})\psi_2(w_2) + qM\int_\Omega \left|\xi_2\right|^{q-2}\xi_2.\cdot w_2. \end{aligned} \tag{4.5}$$

Let us define the adjoint state $P(\xi) = \{P_1(\xi), P_2(\xi)\}$ as the solution in $(W_0^{1,p}(P,\Omega))^* \times (W_0^{1,q}(Q,\Omega))^*$ of

$$\begin{cases} A^*P(\xi) = \{\int_\Omega P(x)\left|y_1(\xi)-z_{d1}\right|^{p-2}(y_1(\xi)-z_{d1}), & \\ \qquad Q(x)\left|y_2(\xi)-z_{d2}\right|^{q-2}(y_2(\xi)-z_{d2})\} & \text{in } \Omega, \\ \qquad P(\xi) = \{P_1(\xi), P_2(\xi)\} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

then from (4.1) and (4.5), we deduce that

$$p\int_\Omega [P_1(\xi)(\varsigma_1-\xi_1) + M\left|\xi_1\right|^{p-2}\xi_1] + q\int_\Omega [P_2(\xi)(\varsigma_2-\xi_2) + M\left|\xi_2\right|^{q-2}\xi_2] \geq 0. \tag{4.7}$$

So, the main theorem of control problem is the following:

Theorem 4.1. *If the cost function is given by (4.2), the optimal control u is characterized by (4.6) and (4.7) together with (4.1).*

5. Conclusion

Interest in general forms of nonlinear partial differential problems, whose leading operator is of the (p, q)-Laplacian type, has greatly increased over the last few decades. The main reason is that this kind of nonlinear operator appears naturally in the study of nonlocal diffusion with special features. In this work, we are dealing with system involving weighted (p, q)-Laplacian operator. Existence and uniqueness of weak solution through the Browder theorem method are established. As an application, an optimal control problem related to the system under study has been formulated and analyzed. There are several interesting subjects for future work. The first one is to generalize the results obtained in this paper from systems involving weighted p-Laplacian operator to systems involving singular p-Laplacian or fractional Laplacian ones, that is, $div[|x|^{-ap}|\nabla u|^{p-2}\nabla u]$ under some certain conditions on a and p or $(-\Delta)^s$ with with $s \in (0, 1)$, respectively. The second one is to the use of a method other than the Browder theorem method, such as the sub-super solution method.

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