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NONLINEAR IMPLICIT CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATION WITH FRACTIONAL BOUNDARY CONDITIONS

NEDJEMEDDINE DERDAR

Abstract. In this paper, we establish the existence and uniqueness of solutions for a class of problem for nonlinear implicit fractional differential equations of Caputo-Hadamard type with fractional boundary conditions. The results are obtained by using Banach fixed point theorem and Schauder's fixed point theorem. An example is included to show the applicability of our results.

1. Introduction

The origins of fractional calculation go back to the late 17th century. In fact, some mathematicians (L'Hopsital, Leibnitz(1695) began to consider how to define the fractional derivative. But it is only during the last three decades that fractional calculation has been the most interesting and the applications of fractional derivatives have become more diversified. There are several definitions of fractional derivatives, the definitions of Riemann-Liouville (1832), Riemann (1849), Caputo (1997), Grünwald-Letnikov (1867), we refer the reader to [[27], [33]].

Fractional calculus is widely and efficiently used to describe many phenomena arising in engineering, physics, control theory, bioengineering and biomedical sciences, viscoelasticity, finance, stochastic processes and economy. Recently, fractional differential equations have attracted many authors (see for instance [[1], [2], [10], [11], [18], [30]] and references therein).

The Caputo-Hadamard derivative is a new approach obtained from the Hadamard

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derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo-Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo-Hadamard derivative of a constant is zero [25]. And the most important advantage of Caputo-Hadamard derivative is that it provided a new definition through which the integer order initial conditions can be defined for fractional. Details and properties of Hadamard fractional derivatives, integrals and Caputo-Hadamard derivative can be found in [[15], [28], [30]].

The implicit fractional differential equations (IFDEs) are a very important class of fractional differential equations. This type of equation is derived from the implicit ordinary differential equation (IODE) of the form

$$f(t, x(t), x'(t), \ldots, x^{(n-1)}) = 0,$$

with different kind of initial or boundary conditions, for more details see [[5], [6], [19], [20], [22], [32]].

Benchohra et al. [[7], [12], [13]] and Nieto et al. [[26],[31]] have initiated the study of of implicit fractional differential equations (IFDEs) of the form

$$D^{\alpha}x(t) = f(t, x(t), D^{\alpha}x(t)).$$

with different kind of initial or boundary conditions. This kind of equation is important in many disciplines in different fields of science and engineering [35].

In [34], Vivek, Elsayed and Kanagarajan proved the existence and stability of solution for a class of boundary value problem (BVP) for nonlinear fractional implicit differential equations (FIDEs) with complex order

$$\begin{cases} {}^{c}D^{\theta}y(t) = f(t, y(t), {}^{c}D^{\theta}y(t)), \ \theta = m + i\alpha \ t \in J := [0, T] \\ ay(0) + by(T) = c, \end{cases}$$

where ${}^{c}D^{\theta}$ is the Caputo fractional derivative of order $\alpha \in \mathbb{C}$. Let $\alpha \in \mathbb{R}_{+}$, $0 < \alpha < 1$, $m \in (0, 1]$, and $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given continuous function. Here a, b, c are real constants with $a + b \neq 0$. The results are based upon the Banach contraction principle and Schaefer's fixed point theorem.

In [17], Derbazi and Hammouche proved the existence and uniqueness of solutions to the boundary-value problem of the fractional differential equations

$$\begin{cases} {}^cD^{\alpha}_{0+}y(t) = f(t, y(t), {}^cD^{\beta}_{0+}y(t)), \quad 0 < t < 1 \\ y(0) = 0,\ y'(0) = aI^{\sigma_1}_{0+}y(\eta_1), {}^cD^{\beta_1}_{0+}y(1) = bI^{\sigma_2}_{0+}y(\eta_2) \end{cases}$$

where ${}^cD^{\nu}$ is the Caputo fractional derivative of order $\nu \in \{\alpha, \beta, \beta_1\}$ such that $2 < \alpha \leq 3$, $0 < \beta$, $\beta_1 \leq 1$, I^{θ}_{0+} is the Riemann-Liouville fractional integral of order $\theta > 1$, $\theta \in \{\sigma_1, \sigma_2\}$, $J := [0, 1]$, and $f : T \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, a, b are suitably chosen real constants. The results are based upon the Banach contraction principle, Schauder fixed point theorem and Krasnoselskii's fixed point theorem.
In [29] Karthikeyan and Arul we examine the existence and uniqueness of integral boundary value problem for implicit fractional differential equations (IFDE's) involving Caputo-Hadamard fractional derivative differential equations

$$\begin{cases} {}^{\mathcal{CH}}D^{\vartheta}x(t) = f(t, x(t), {}^{\mathcal{CH}}D^{\vartheta}x(t)),\ t \in \mathcal{J} := [b, \mathcal{T}], \\ x(b) = 0, x(\mathcal{T}) = \lambda \int_b^{\sigma} x(s)ds,\ b < \sigma < \mathcal{T},\ \lambda \in \mathbb{R}, \end{cases}$$

where ${}^{\mathcal{CH}}D^{\alpha}$ is the Caputo-Hadamard fractional derivative, $1 < \vartheta \leq 2$, $g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. We prove the existence and uniqueness results by utilizing Banach and Schauder's fixed point theorem.
In this paper, we establish the existence and uniqueness of solutions for a class of problem for nonlinear implicit fractional differential equations of Caputo-Hadamard type with fractional boundary conditions

$$\begin{cases} {}^C_HD^rx(t) = f(t, x(t), {}^C_HD^rx(t)),\ t \in J := [1, T],\ 1 < r \leq 2, \\ x(1) = 0,\ \alpha_H I^qx(\eta) + \beta^C_HD^{\gamma}x(T) = \lambda, \quad q, \gamma \in (0, 1], \end{cases} \tag{1.1}$$

where ${}^C_HD^{(.)}$ is the Caputo-Hadamard fractional derivative, ${}_HI^q$ is the standard Hadamard fractional integral, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, α, β, λ are real constants, and $\eta \in (1, T)$.
This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section 2. In Section 3, we give two results, one based on Banach fixed point theorem (3.1) and another one based on Schauder's fixed point theorem (3.2). In Section 4, we illustrate the results obtained with an example.

2. Preliminaries

In this section we present some basic definitions, notations and preliminaries facts which are used throughout this paper.

Let $J := [1, T]$ be an interval in $\mathbb{R}$ where $T > 1$. we denote by $C(J, \mathbb{R})$ the space of continuous functions $x : J \longrightarrow \mathbb{R}$. The space $C(J, \mathbb{R})$ is a Banach space with supremum norm $\|.\|$ defined by

$$\|x\|_\infty = \sup\{|x(t)| : \ t \in J\}.$$

Let now $[a, b]$, $(-\infty < a < b < +\infty)$ be interval finite and we let $AC([a, b],\ \mathbb{R})$ be the space of functions $g : [a, b] \longrightarrow \mathbb{R}$ that are absolutely continuous.

Let $\delta = t\frac{d}{dt}$ is the Hadamard derivative, $\delta^n = \delta(\delta^{n-1})$, we consider the set of functions:

$$AC^n_\delta([a, b],\ \mathbb{R}) = \{g : [a, b] \longrightarrow \mathbb{R} : \delta^{n-1}g(t) \in AC([a, b],\ \mathbb{R})\}.$$

Definition 2.1. (See [30]) The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $g : [1, +\infty) \longrightarrow \mathbb{R}$ is defined as

$$ {}_H I^\alpha_1 g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} g(s) \frac{ds}{s}. \tag{2.1}$$

where Γ is the Euler gamma function, and $\log(.) = \log_e(.)$.

Definition 2.2. (See [30], [33]) For a function $g \in AC^n_\delta([a, b],\ \mathbb{R})$, the Caputo-Hadamard fractional derivative of order α is defined as

$$ {}^C_H D^\alpha_1 g(t) = \frac{1}{\Gamma(n-\alpha)} (t\frac{d}{dt})^n \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} g(s) \frac{ds}{s},\ n-1 < \alpha < n, \tag{2.2}$$

where $\delta^n = (t\frac{d}{dt})^n$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1. ([25, Lemma 2.5, p.6]) *Let $g \in AC^n_\delta[a, b]$ or $g \in C^n_\delta[a, b]$ and $\alpha \in \mathbb{C}$. Then*

$$ {}_H I^\alpha_a ({}^C_H D^\alpha_a g)(t) = g(t) - \sum_{k=0}^{n-1} \frac{\delta^{(k)} g(a)}{k!} (\log \frac{t}{a})^k. \tag{2.3}$$

Proposition 2.1. ([30, Property 2.16, p.112], [25]) *Let $\alpha > 0$, $\beta > 0$, $n = [\alpha] + 1$ and $a > 0$, then*

$$\left({}_H I^\alpha_{a^+} (\log \frac{x}{a} \right)^{\beta-1} \Big)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\log \frac{x}{a} \right)^{\beta+\alpha-1}, \tag{2.4}$$

$$\Big({}_H^C D_{a^+}^{\alpha}(\log\frac{x}{a})^{\beta-1}\Big)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\Big(\log\frac{x}{a}\Big)^{\beta-\alpha-1},\ \beta > \alpha. \tag{2.5}$$

Theorem 2.1. *((See [23]) Let $x(t) \in AC_\delta^n[a,b]$, $0 < a < b < \infty$ and $\alpha \geq 0$, $\beta \geq 0$, then*

$$^C D_a^{\alpha}(I^{\beta}x)(t) = (I^{\beta-\alpha}x)(t), \tag{2.6}$$

$$^C D^{\alpha}(^C D^{\beta}x)(t) = (^C D^{\alpha+\beta}x)(t). \tag{2.7}$$

Theorem 2.2 (Banach's fixed point theorem)**.** ([36, Theorem 1.41, p.16]) *Let (X,d) be a complete metric space, and $\mathcal{F} : \Omega \longrightarrow \Omega$ a contraction mapping:*

$$d(\mathcal{F}(x), \mathcal{F}(y)) \leq kd(x,y), \tag{2.8}$$

where $0 < k < 1$, for each $x, y \in \Omega$. Then, there exists a unique fixed point x of $\mathcal{F}$ in Ω: $\mathcal{F}(x) = x$.

Theorem 2.3 (Schauder's fixed point theorem)**.** ([36, Theorem 1.42, p. 16]) *Let X be a Banach space and $\Omega \subset X$ a convex, closed and bounded set. If $\mathcal{F} : \Omega \longrightarrow \Omega$ is a continuous operator such that $\mathcal{F}(\Omega) \subset X$, $\mathcal{F}(\Omega)$ is relatively compact, then $\mathcal{F}$ has at least one fixed point in Ω.*

3. Main results

Definition 3.1. A function $x \in AC_\delta^2(J,\ \mathbb{R})$ is said to be a solution of the problem (1.1) if x satisfies the equation ${}_H^C D^r x(t) = f(t, x(t), {}_H^C D^r x(t))$, and satisfies the conditions $x(1) = 0,\ \alpha_H I^q x(\eta) + \beta_H^C D^{\gamma} x(T) = \lambda$.

To prove the existence of solutions to the problem (1.1), we need the following auxiliary lemma.

Lemma 3.1. *Let $h : [0,\ +\infty) \longrightarrow \mathbb{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}h(s)\frac{ds}{s} + \frac{\log t}{\Lambda}\Bigg[\lambda - \frac{\alpha}{\Gamma(r+q)}\int_1^{\eta}(\log\frac{\eta}{s})^{r+q-1}h(s)\frac{ds}{s} - \frac{\beta}{\Gamma(r-\gamma)}\int_1^T(\log\frac{T}{s})^{r-\gamma-1}h(s)\frac{ds}{s}\Bigg], \tag{3.1}$$

where

$$\Lambda = \frac{\alpha(\log\eta)^{q+1}}{\Gamma(q+2)} + \frac{\beta(\log T)^{1-\gamma}}{\Gamma(2-\gamma)},$$

if and only if x is a solution of the fractional boundary value problem,

$$ {}_H^C D^r x(t) = h(t), \qquad 1 < r \le 2, \tag{3.2}$$

$$ x(1) = 0,\ \alpha_H I^q x(\eta) + \beta_H^C D^\gamma x(T) = \lambda, \qquad q,\gamma \in (0,1]. \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2) and then using Lemma 2.1, we obtain

$$ x(t) = c_0 + c_1 \log t + {}_H I^r h(t), \tag{3.4}$$

Applying the first boundary condition (3.3) in (3.4), we find that

$$ x(1) = c_0 = 0. \tag{3.5}$$

Using Proposition 2.1, we can write

$$ {}_H^C D^\gamma x(t) = \frac{c_1}{\Gamma(2-\gamma)}(\log t)^{1-\gamma} + {}_H I^{r-\gamma} h(t), \tag{3.6}$$

and

$$ {}_H I^q x(t) = \frac{c_1}{\Gamma(2+q)}(\log t)^{1+q} + {}_H I^{r+q} h(t), \tag{3.7}$$

Using the second boundary condition (3.3), we get

$$ \alpha\Big[\frac{c_1}{\Gamma(2+q)}(\log\eta)^{q+1} + {}_H I^{r+q} h(\eta)\Big] + \beta\Big[\frac{c_1}{\Gamma(2-\gamma)}(\log T)^{1-\gamma} + {}_H I^{r-\gamma} h(T)\Big] = \lambda, \tag{3.8}$$

thus,

$$ \Big[\frac{\alpha(\log\eta)^{q+1}}{\Gamma(2+q)} + \frac{\beta(\log T)^{1-\gamma}}{\Gamma(2-\gamma)}\Big] c_1 + \alpha_H I^{r+q} h(\eta) + \beta_H I^{r-\gamma} h(T) = \lambda. \tag{3.9}$$

Consequently,

$$ c_1 = \frac{\lambda}{\Lambda} - \frac{\alpha}{\Lambda} {}_H I^{r+q} h(\eta) - \frac{\beta}{\Lambda} {}_H I^{r-\gamma} h(T), \tag{3.10}$$

where

$$\Lambda = \frac{\alpha(\log\eta)^{q+1}}{\Gamma(2+q)} + \frac{\beta(\log T)^{1-\gamma}}{\Gamma(2-\gamma)}.$$

Substituting into (3.4), we obtain (3.1). □

Our first result is based on Banach fixed point theorem. We assume the following conditions to prove the existence of a solution of problem (1.1):

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H2) There exist constants $L_1 > 0$, $0 < L_2 < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq L_1|u-\bar{u}| + L_2|v-\bar{v}|,$$

for any u, v, $\bar{u}$ and $\bar{v} \in \mathbb{R}$ for a.e., $t \in J$.

Theorem 3.1. *If the hypotheses* (H1)-(H2) *are satisfied, and if*

$$\rho := \frac{L_1}{1-L_2}\left[\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\alpha|(\log T)(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)}\right] < 1, \tag{3.11}$$

then there exists a unique solution $x \in AC^2_\delta(J,\ \mathbb{R})$ for problem (1.1) *on J.*

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator $\mathcal{F} : C(J,\ \mathbb{R}) \longrightarrow C(J,\ \mathbb{R})$ defined by

$$\begin{aligned}\mathcal{F}x(t) = \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}\sigma_x(s)\frac{ds}{s} + \frac{\log t}{\Lambda}\Bigg[\lambda - \frac{\alpha}{\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}\sigma_x(s)\frac{ds}{s} \\ - \frac{\beta}{\Gamma(r-\gamma)}\int_1^T (\log\frac{T}{s})^{r-\gamma-1}\sigma_x(s)\frac{ds}{s}\Bigg],\end{aligned} \tag{3.12}$$

where

$$\sigma_x(s) = f\Big(s, x(s), D^r x(s)\Big).$$

It is clear that the fixed points of $\mathcal{F}$ are solutions of problem (1.1).
Let x, $y \in AC^2_\delta(J,\ \mathbb{R})$. Then for each $t \in J$ we have

$$\Big|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\Big| \tag{3.13}$$

$$= \left| \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \sigma_x(s) \frac{ds}{s} + \frac{\log t}{\Lambda} \left[\lambda - \frac{\alpha}{\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} \sigma_x(s) \frac{ds}{s} \right. \right.$$

$$\left. - \frac{\beta}{\Gamma(r-\gamma)} \int_1^T (\log \frac{T}{s})^{r-\gamma-1} \sigma_x(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \sigma_y(s) \frac{ds}{s}$$

$$\left. - \frac{\log t}{\Lambda} \left[\lambda - \frac{\alpha}{\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} \sigma_y(s) \frac{ds}{s} - \frac{\beta}{\Gamma(r-\gamma)} \int_1^T (\log \frac{T}{s})^{r-\gamma-1} \sigma_y(s) \frac{ds}{s} \right] \right|$$

$$\leq \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} |\sigma_x(s) - \sigma_y(s)| \frac{ds}{s} + \frac{|\alpha| \log t}{|\Lambda| \Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} |\sigma_x(s) - \sigma_y(s)| \frac{ds}{s}$$

$$+ \frac{|\beta| \log t}{|\Lambda| \Gamma(r-\gamma)} \int_1^T (\log \frac{T}{s})^{r-\gamma-1} |\sigma_x(s) - \sigma_y(s)| \frac{ds}{s},$$

where

$$\sigma_x(t) = f(t, x(t), \sigma_x(t)),$$

and

$$\sigma_y(t) = f(t, y(t), \sigma_y(t)).$$

By (H2), we have

$$\begin{aligned} |\sigma_x(t) - \sigma_y(t)| &= |f(t, x(t), \sigma_x(t)) - f(t, y(t), \sigma_y(t))| \\ &\leq L_1 |x(t) - y(t)| + L_2 |\sigma_x(t) - \sigma_y(t)|, \end{aligned} \tag{3.14}$$

so

$$|\sigma_x(t) - \sigma_y(t)| \leq \frac{L_1}{1 - L_2} |x(t) - y(t)|. \tag{3.15}$$

By replacing (3.15) in the inequality (3.13), we get

$$|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \leq \frac{1}{\Gamma(r)} \frac{L_1}{1 - L_2} \int_1^t (\log \frac{t}{s})^{r-1} |x(s) - y(s)| \frac{ds}{s} + \tag{3.16}$$

$$\begin{aligned}
&\frac{|\alpha|\log t}{|\Lambda|\Gamma(r+q)}\frac{L_1}{1-L_2}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}|x(s)-y(s)|\frac{ds}{s}\\
&+\frac{|\beta|\log t}{|\Lambda|\Gamma(r-\gamma)}\frac{L_1}{1-L_2}\int_1^T(\log\frac{T}{s})^{r-\gamma-1}|x(s)-y(s)|\frac{ds}{s}\\
\leq&\Bigg[\frac{1}{\Gamma(r)}\frac{L_1}{1-L_2}\int_1^t(\log\frac{t}{s})^{r-1}\frac{ds}{s}+\frac{|\alpha|\log t}{|\Lambda|\Gamma(r+q)}\frac{L_1}{1-L_2}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}\frac{ds}{s}\\
&+\frac{|\beta|\log t}{|\Lambda|\Gamma(r-\gamma)}\frac{L_1}{1-L_2}\int_1^T(\log\frac{T}{s})^{r-\gamma-1}\frac{ds}{s}\Bigg]|x(s)-y(s)|\\
\leq&\frac{L_1}{1-L_2}\left[\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\alpha|(\log T)(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)}\right]|x(s)-y(s)|.
\end{aligned}$$

Thus

$$\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\|_\infty\leq\rho\|x-y\|_\infty, \tag{3.17}$$

for $x,\ y\in AC^2_\delta(J,\ \mathbb{R})$, where

$$\rho:=\frac{L_1}{1-L_2}\left[\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\alpha|(\log T)(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)}\right]. \tag{3.18}$$

Consequently by (3.11), $\mathcal{F}$ is a contraction. As a consequence of Banach fixed point theorem, we deduce that $\mathcal{F}$ has a fixed point which is a solution of the problem (1.1). □

The second result is based on Schauder's fixed point theorem.

(H3) There exist $p,\ \nu,\ \omega\in C(J,\mathbb{R}_+)$ with $\omega^*=\sup_{t\in J}\omega(t)<1$, $\nu^*=\sup_{t\in J}\nu(t)$ and $p^*=\sup_{t\in J}p(t)$, such that

$$|f(t,u,v)|\leq p(t)+\nu(t)|u|+\omega(t)|v|,$$

for any $u,\ v\in\mathbb{R}$ for a.e., $t\in J$.

Theorem 3.2. *Assume that conditions* $(H1),(H3)$ *hold. If*

$$\omega^*+M\nu^*<1, \tag{3.19}$$

with

$$M:=\frac{(\log T)^r}{\Gamma(\alpha+1)}+\frac{|\alpha|(\log T)(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)},$$

then the problem (1.1) *has at least one solution.*

Proof. Let

$$R \geq \frac{Mp^* + \frac{|\lambda|(1-\omega^*)\log T}{|\Lambda|}}{1-(\omega^* + M\nu^*)}, \tag{3.20}$$

and consider

$$D_R = \{x \in C(J,\mathbb{R}) : \ \|x\|_\infty \leq R\}. \tag{3.21}$$

Clearly, the subset D_R is closed, bounded and convex. We shall use Schander's fixed point theorem to prove that the operator $\mathcal{F}$ defined by (3.12) has a fixed point. The proof will be given in three steps.

Step 1: $\mathcal{F}$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $AC^2_\delta(J,\ \mathbb{R})$, then for each $t \in J$.

$$\left|\mathcal{F}(x_n)(t) - \mathcal{F}(x)(t)\right| \tag{3.22}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}|g_n(s)-g(s)|\frac{ds}{s} + \frac{|\alpha|\log t}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}|g_n(s)-g(s)|\frac{ds}{s}\\
&+ \frac{|\beta|\log t}{|\Lambda|\Gamma(r-\gamma)}\int_1^T (\log\frac{T}{s})^{r-\gamma-1}|g_n(s)-g(s)|\frac{ds}{s}\\
&\leq \left[\frac{(\log T)^r}{\Gamma(\alpha+1)} + \frac{|\alpha|(\log T)(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)}\right]|g_n(s)-g(s)|,
\end{aligned}$$

where $g,\ g_n \in C(J,\mathbb{R})$ such that

$$g(t) = f(t, x(t), g(t)), \text{ and } g_n(t) = f(t, x_n(t), g_n(t)).$$

Since g is a continuous functions (i.e., f is continuous), then by the Lebesgue dominated convergence theorem, we have

$$\|\mathcal{F}(x_n)(t) - \mathcal{F}(x)(t)\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.23}$$

Therefore $\mathcal{F}(x_n)(t) \longrightarrow \mathcal{F}(x)(t)$ as $n \longrightarrow \infty$ which implies that $\mathcal{F}$ is continuous.

Step 2: $\mathcal{F}(D_R) \subset D_R$

Let $x \in D_R$. We show that $\mathcal{F}(x) \in D_R$. For each $t \in J$, we have

$$\left|\mathcal{F}(x)(t)\right| \tag{3.24}$$

$$\leq \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}|g(s)|\frac{ds}{s} + \frac{|\alpha|\log t}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}|g(s)|\frac{ds}{s}$$
$$+ \frac{|\beta|\log t}{|\Lambda|\Gamma(r-\gamma)}\int_1^T (\log\frac{T}{s})^{r-\gamma-1}|g(s)|\frac{ds}{s} + \frac{|\lambda|\log t}{|\Lambda|},$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f(t, x(t), g(t))$$

From (H3), for $t \in J$ we have

$$\begin{aligned} |g(t)| &= |f(t,x(t),g(t))| \\ &\leq p(t) + \nu(t)|x(t)| + \omega(t)|g(t)| \\ &\leq p^* + \nu^*\|x\|_\infty + \omega^*|g(t)|. \end{aligned} \tag{3.25}$$

Hence,

$$|g(t)| \leq \frac{p^* + \nu^* R}{1-\omega^*}. \tag{3.26}$$

By replacing (3.26) in the inequality (3.24), we get

$$\left|\mathcal{F}(x)(t)\right| \tag{3.27}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}|g(s)|\frac{ds}{s} + \frac{|\alpha|\log t}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}|g(s)|\frac{ds}{s} \\ &+ \frac{|\beta|\log t}{|\Lambda|\Gamma(r-\gamma)}\int_1^T (\log\frac{T}{s})^{r-\gamma-1}|g(s)|\frac{ds}{s} + \frac{|\lambda|\log t}{|\Lambda|} \\ &\leq \Big(\frac{p^*+\nu^*R}{1-\omega^*}\Big)\left[\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\alpha|(\log T)(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)}\right] + \frac{|\lambda|\log T}{|\Lambda|} \\ &= \frac{M(p^*+\nu^*R)}{(1-\omega^*)} + \frac{|\lambda|\log T}{|\Lambda|} \\ &\leq R. \end{aligned}$$

Step 3: we show that $\mathcal{F}(D_R)$ is equicontinuous.

By step 2, it is obvious that $\mathcal{F}(D_R) \subset D_R$ is bounded. For the equicontinuity of $\mathcal{F}(D_R)$.

Let $t_1,\, t_2 \in (1,T],\;\; t_1 < t_2$ and let $x \in D_R$. Then

$$\Big|\mathcal{F}(x)(t_2) - \mathcal{F}(x)(t_1)\Big| \tag{3.28}$$

$$\begin{aligned}
&= \left|\frac{1}{\Gamma(r)}\int_1^{t_1}(\log\frac{t_2}{s})^{r-1}g(s)\frac{ds}{s} - \frac{1}{\Gamma(r)}\int_1^{t_2}(\log\frac{t_1}{s})^{r-1}g(s)\frac{ds}{s}\right|\\
&= \left|\frac{1}{\Gamma(r)}\int_1^{t_1}\left[(\log\frac{t_2}{s})^{r-1} - (\log\frac{t_1}{s})^{r-1}\right]g(s)\frac{ds}{s} + \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}g(s)\frac{ds}{s}\right|\\
&\le \frac{|g(s)|}{\Gamma(r)}\Big|\int_1^{t_1}\Big[(\log\frac{t_2}{s})^{r-1} - (\log\frac{t_1}{s})^{r-1}\Big]\frac{ds}{s}\Big| + \frac{|g(s)|}{\Gamma(r)}\Big|\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}\frac{ds}{s}\Big|\\
&\le \frac{p^*+\nu^*R}{(1-\omega^*)\Gamma(r)}\Big|\int_1^{t_1}\Big[(\log\frac{t_2}{s})^{r-1} - (\log\frac{t_1}{s})^{r-1}\Big]\frac{ds}{s}\Big| + \frac{p^*+\nu^*R}{(1-\omega^*)\Gamma(r)}\Big|\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}\frac{ds}{s}\Big|\\
&\le \frac{p^*+\nu^*R}{(1-\omega^*)\Gamma(r+1)}\left[\left|(\log t_1)^r + (\log\frac{t_2}{t_1})^r - (\log t_2)^r\right| + \left|(\log\frac{t_2}{t_1})^r\right|\right]\\
&\le \frac{p^*+\nu^*R}{(1-\omega^*)\Gamma(r+1)}\left[\left|(\log t_1)^r - (\log t_2)^r\right|\right].
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right hand side of the obove inequality tends to zero, proving the equicontinuity. In either case the Arzela-Ascoli theorem yields that $\mathcal{F}(D_R)$ is relatively compact, and hence Schauder's fixed point theorem asserts that $\mathcal{F}$ has a fixed point. By construction, a fixed point of $\mathcal{F}$ is a solution of problem (1.1). □

4. Example

Consider the following nonlinear problem

$$\begin{cases} {}^C_H D^{\frac{3}{2}}x(t) = \dfrac{t}{10}\cos(x(t)) + \dfrac{t^{-2}}{5}\sin({}^C_H D^{\frac{3}{2}}x(t)) + \dfrac{3t^{-3}}{5},\;\; t \in [1,e],\\ x(1) = 0,\;\; \dfrac{1}{2}{}_H I^{\frac{1}{2}}x(2) + 2{}^C_H D^{\frac{1}{3}}x(e) = \dfrac{3}{4}, \end{cases} \tag{4.1}$$

we see that $r = \dfrac{3}{2}$, $q = \dfrac{1}{2}$, $\gamma = \dfrac{1}{3}$, $\eta = 2$, $\alpha = \dfrac{1}{2}$, $\beta = 2$, $\lambda = \frac{3}{4}$, $T = e$
and

$$f(t,u,v) = \frac{t}{10}\cos u + \frac{1}{5t^2}\sin v + \frac{3}{5t^3}, t \in [1,e], u, v \in \mathbb{R}. \tag{4.2}$$

Clearly, the function f is continuous, and for u, v, $\overline{u}$, $\overline{v} \in \mathbb{R}$ and $t \in [1, e]$.
We have

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{t}{10}|\cos u - \cos \overline{u}| + \frac{1}{5t^2}|\sin v - \sin \overline{v}| \\ \leq \frac{e}{10}|u - \bar{u}| + \frac{1}{5}|v - \bar{v}|, \tag{4.3}$$

Hence, condition (H2) is satisfied with $L_1 = \dfrac{e}{10}$, $L_2 = \dfrac{1}{5}$.
Thus,

$$\rho := \frac{L_1}{1-L_2}\left[\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\alpha|(\log T)(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^{r-\gamma+1}}{|\Lambda|\Gamma(r-\gamma+1)}\right] = 0.53050 < 1, \tag{4.4}$$

so it follows from Theorem 3.1 that the problem (4.1) has a unique solution $x \in C^2_\delta([1,e],\ \mathbb{R})$.

$$|f(t,u,v)| \leq \frac{3}{5t^3} + \frac{t}{10}|\cos(u)| + \frac{1}{5t^2}|\sin(v)|, \tag{4.5}$$

so condition (H3) is satisfied with $p(t) = \dfrac{3}{5t^3}$, $\nu(t) = \dfrac{t}{10}$, $\omega(t) = \dfrac{1}{5t^2}$,
and $\nu^* = \dfrac{e}{10}$, $\omega^* = \dfrac{1}{5}$.
We shall show that condition (3.19) holds with $T = e$. Indeed,

$$\omega^* + M\nu^* = 0.62440 < 1. \tag{4.6}$$

Simple computations show that all conditions of Theorem 3.2 are satified. It follows that the problem (4.1) has at least solution defined on $[1, e]$.

5. Conclusions

In this paper, we have discussed the existence and uniqueness of solutions for a class of nonlinear implicit fractional differential equations with fractional boundary conditions involving the Caputo-Hadamard fractional derivative. The Caputo-Hadamard derivative is a new approach obtained from the Hadamard derivative by changing the order of its differential and integral parts. Our results in action prove that the Caputo-Hadamard approach works perfectly. Banach contraction principle theorem was the key of our analysis to establish existence and uniqueness of solution of our problem by adding suitable conditions on the nonlinear term. We succeeded to establish existence of solutions by using Schauder fixed point theorem. we present an example to demonstrate the consistency of our the theoretical findings.

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Department of Mathematics, Higher Normal School of Technological Education-Skikda, Algeria

Email address: `nedjm1972@yahoo.fr`